A Solution to the 5th & 8th Busemann-Petty Problems near the Unit Ball

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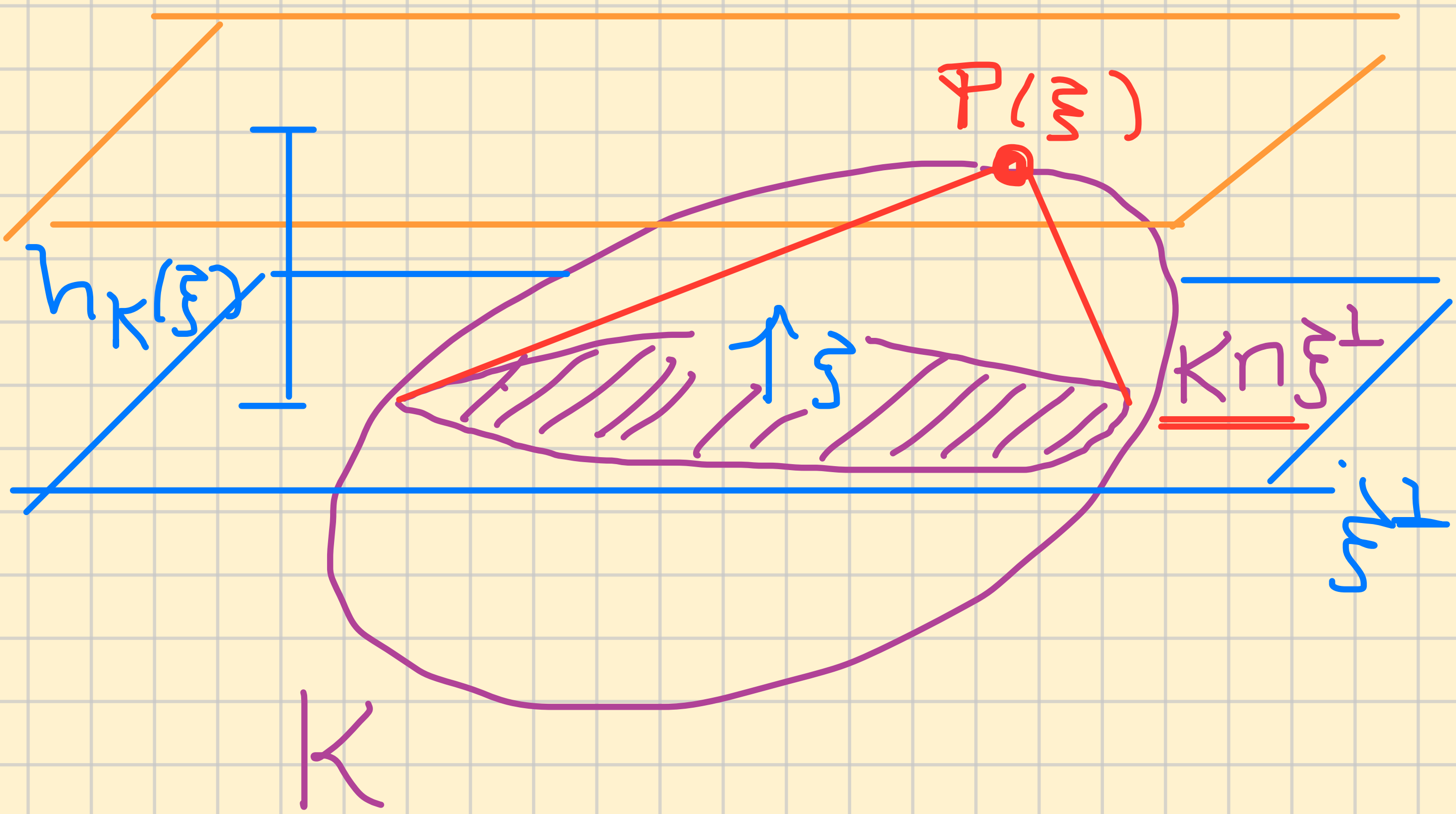
$K \subseteq \mathbb{R}^n$ origin symm convex body

S^{n-1} unit sphere

$\xi \in S^{n-1}$, $\quad \xi^\perp = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$

$h_K(\xi) = \max\{x \cdot \xi \,,\ x \in K\}$



BP5: If $h_K(\xi)\, \mathrm{vol}_{n-1}(K \cap \xi^\perp) = C$

is K an ellipsoid?

indep of $\xi \in S^{n-1}$

- True if $K = B_2^n$, $C = \mathrm{vol}_{n-1}(B_2^{n-1})$
- True for $K \Rightarrow$ True for TK

$T \in Gl(n)$, $\qquad C \cdot |\det T|$

- Not true $n = 2$

BP8:

If $G(p(\xi))\left(\mathrm{vol}_{n-1}(K\cap\xi^{\perp})\right)^{n+1} = C$

indep of $\xi\in S^{n-1}$

is K an ellipsoid?

- True $K = B_2^n$
- True for $K \Rightarrow$ True for TK

  $T\in Gl(n)$ $\qquad C\,|\det T|^{n-1}$

- True for $n=2$

Analytic Reformulation

$$\rho_K(\xi) = \max\{t>0 : t\xi \in K\}$$

$$\mathrm{vol}_{n-1}(K\cap\xi^\perp) = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)\,d\sigma(\theta)$$

$$= c_n \,\mathcal{R}(\rho_K^{n-1})(\xi), \quad \mathcal{R}1 = 1$$

Spherical Radon Transf.

BPS: $h_K(\xi)\,\mathrm{vol}_{n-1}(K\cap\xi^\perp) = c$

$$\downarrow$$

$$h_K(\xi)\,\mathcal{R}(\rho_K^{n-1})(\xi) = 1$$

Solving BP5

If $h_K = \underbrace{\left(\mathcal{R}(\rho_K^{n-1})\right)^{-1}}$ $:= \underline{T}\rho_K$

is K an ellipsoid?

If K is close to the unit ball,

Yes.

<u>Idea</u>: Construct a sequence $(K_m)_{m=1}^{\infty}$, with $K_1 = \underline{K}$

and $h_{K_{m+1}} = T\rho_{K_m}$.

$$\operatorname{dist}(TK, B_2^n) \leq \mu \operatorname{dist}(K, B_2^n)$$

$$0 < \mu < 1$$

If K is close to B_2^n,

- $\log d_{BM}(K, B_2^n) \approx \|\rho_K - 1\|_\infty$

We will show:

Assuming $\|\rho_K - 1\|_\infty < \delta$,

then T is a contraction

in $L^2(S^{n-1})$

$$\|T\rho_{K_m} - 1\|_2 \le \mu \|\rho_{K_m} - 1\|_2$$

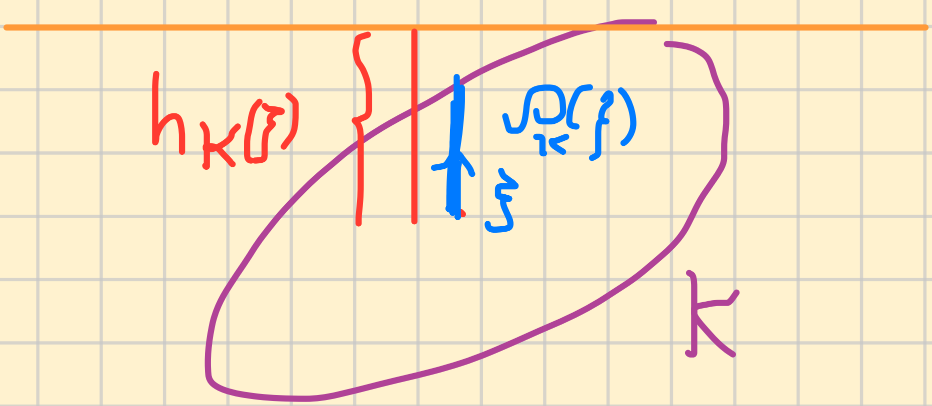
$$\|$$

$$\|h_{K_{m+1}} - 1\|_2$$

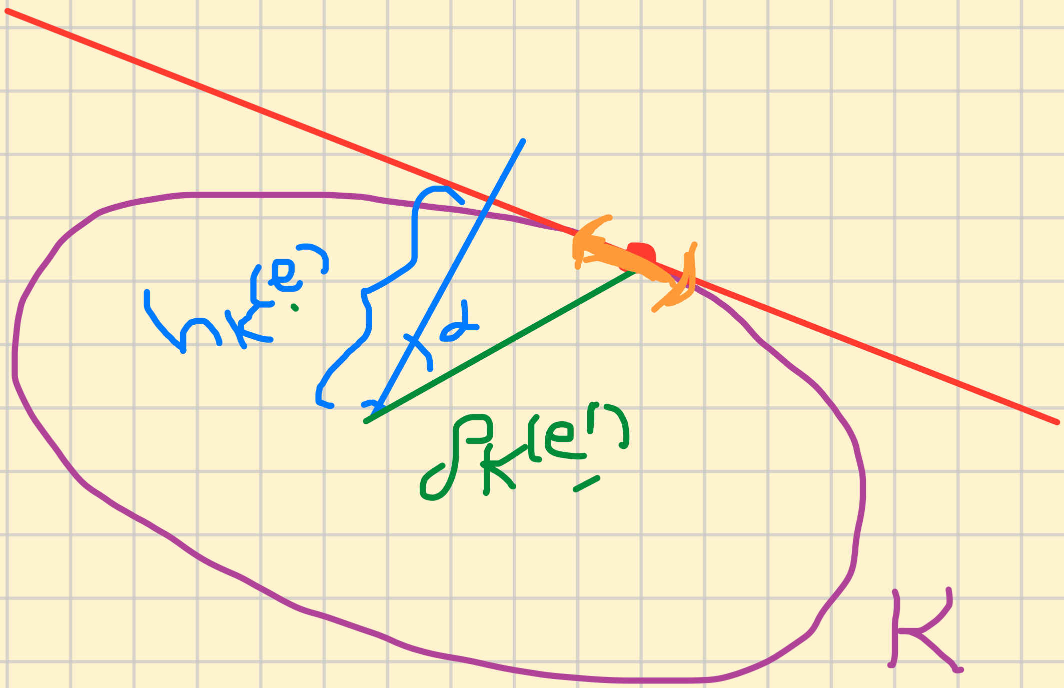
To continue the iteration, we need to estimate

$$\|\rho_{K_{m+1}} - 1\|_2 \text{ by } \|h_{K_{m+1}} - 1\|_2$$

$1+\omega$

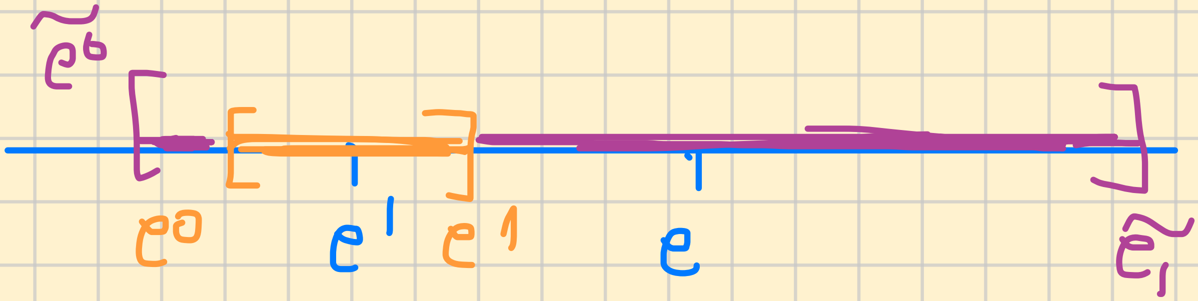


$$\rho_K(\xi) \le h_K(\xi)$$

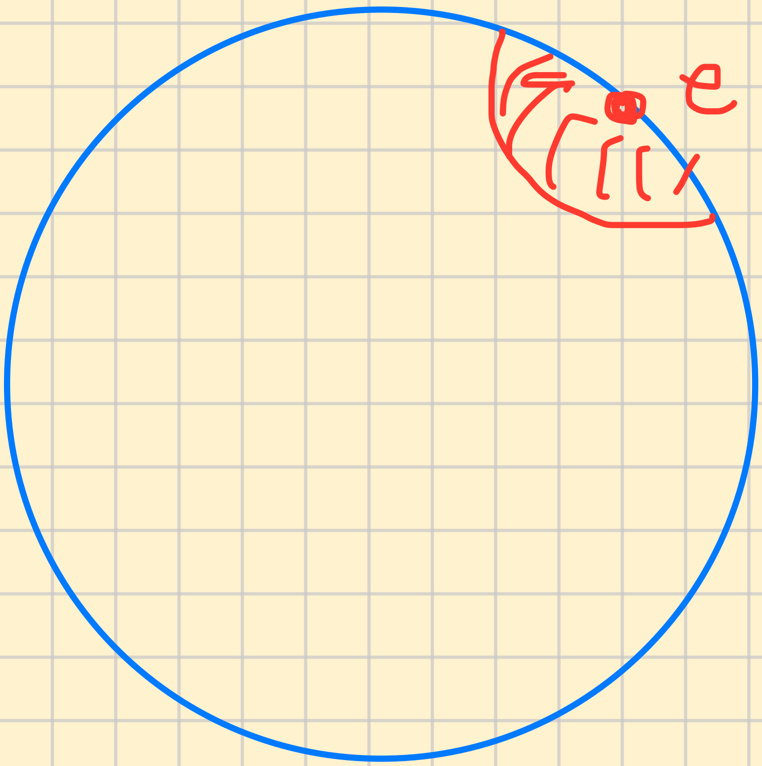


$$h_K(e) = \rho_K(e')\cos\alpha =$$

$$\approx \frac{5}{|[\tilde e_0, \tilde e_1]|}\int_{[\tilde e_0, \tilde e_1]\ni e} \rho_K(e'') \le c\, M \rho_K(e)$$



$$Mf(e) = \sup_{r>0} \frac{1}{|S^{n-1} \cap B(e,r)|} \int_{S^{n-1} \cap \underline{B(e,r)}} |f(y)|\, dy$$



$$h_K = 1 + \sum_{i=2}^{\ell} H^i + \sum_{i=\ell+1}^{\infty} H^i = 1 + \eta + \underline{\nu}$$

Proposition: $\forall \underline{\varepsilon} > 0,\ \forall \underline{\ell} \in \mathbb{N}$

$\exists \delta > 0$: if $\| h_K - 1 \|_2 < \delta$,

$$0 \le h_K - \rho_K \le \varepsilon \|\eta\|_2 + C M \nu$$

$$\|\rho_{K_{m+1}} - 1\|_2 \le \|\underline{h_{K_{m+1}}} - 1\|_2 + \underbrace{\varepsilon \|\eta\|_2}$$

$$+ \underline{C \| \quad \nu \|_2} \le \|h_{K_{m+1}} - 1\|_2 (1 + \hat{\varepsilon})$$

$$= \|T\rho_{K_m} - 1\|_2 (1 + \hat{\varepsilon}) \le \underbrace{(1 + \hat{\varepsilon})\mu}_{\wedge\ 1} \|\underbrace{\rho_{K_m}} - 1\|_2$$

T is a contraction:

$$\rho_K^{n-1} = (1+\omega)^{n-1} = 1 + (n-1)\omega + \cdots$$

$$\left(R(\rho_K^{n-1})\right)^{-1} = 1 - (n-1)R\omega + \cdots$$

$$(n-1) R H^i = \boxed{\lambda_i (n-1)} H^i$$

$$\lambda_i (n-1) < 1 \qquad i \geq 4$$

$$\lambda_2 (n-1) = 1$$

Isotropic position

$$\int_K \langle x, y\rangle^2 \, dy = c\,|x|^2$$

$$\longrightarrow \int_K \underline{y_i y_j \, dy} = 0 \qquad i \neq j$$

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p harmonic poly of degree 2

$$p(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \qquad \sum_{i=1}^{n} \underline{a_{ii}} = 0$$

$$0 = \int_K p(x)\,dx = \frac{1}{n+2} \int_{S^{n-1}} \rho_K^{n+2}(\theta)\, p(\theta)\, d\sigma(\theta)$$

Second harmonic of ρ_K^{n+2} is 0

$$\rho_k^{n+2} = 1 + (n+2)(\rho_k - 1) + \underbrace{\cdots}_{}^{(\rho_k-1)^2}$$

$\downarrow$ 2nd Harmonics

$$0 = (n+2)\, H_2(\rho_k) + \cdots$$

$$\| H_2(\rho_k) \|_2 \leq \underset{-}{\delta} \| \rho_k - 1 \|_2$$

Summary

- We create a sequence $(K_n)_{n=1}^{\infty}$ with $K_1 = K$

$$h_{K_{n+1}} = T\rho_{K_n} = \left(\mathcal{R}(\rho_{K_n}^{n-1})\right)^{-1}$$

- T is a contraction on harmonics $n \geq 4$

- The 2nd harmonic is very small if K is in isotrop. position

- Bound $\|\rho_{K_n} - 1\|_2$ by $\|h_{K_n} - 1\|_2$ using Hardy–Littlewood

BP8:

$$G(p(\xi)) \cdot \left(\mathrm{vol}_{n-1}(K \cap \xi^{\perp})\right)^{n+1} = c$$

$f_K(\xi) \leftarrow$ reciprocal of Gaussian curvature

$$A(h_K) = \left(\mathcal{R}\, A_K^{n-1}\right)^{n+1}$$

$$\boxed{h_K = A^{-1}\left(\mathcal{R}(A_K^{n-1})\right)^{n+1}}$$

- $A\,1 = 1$
- $A(1+\varphi) = 1 + \left(\Delta\Phi\right)_{S^{n-1}} + P(\varphi)$

$$\left(\Delta\Phi\right) = \tilde{\Delta}$$

$$h_k = A^{-1}\left(\mathcal{R}\rho_k^{n-1}\right)^{n+1}$$

$$\left(\mathcal{R}\rho_k^{n-1}\right)^{\underline{n+1}} = 1 + \underline{(n+1)}(n-1)\mathcal{R}\rho_k^{-1} + \ldots$$

$$= 1 + \gamma$$

$$\boxed{\|\gamma\|_{C^\alpha} \leq \sqrt{\delta}}$$

Laplace Eq

$g \in C^{\alpha}(S^{n-1})$, $\quad G$ (-1)-hom

$\exists !\, F$ $\quad 1$-hom s.t. $\Delta F = G$

$$F(x) = c_n \int \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} \right) G(y)\, dy$$

$$\| F_{x_i x_j} \|_2 \leq C \| g \|_2$$

$$\| F \|_{C^{2+\alpha}} \leq C \| g \|_{C^{\alpha}}$$

$$f = F \Big|_{S^{n-1}}$$

If $\quad g = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} H_m^g$

then $\quad f = \sum \mu_m H_m^g$

$$\mu_m = \frac{1}{(1-m)(m+n-1)}$$

$$\mu_2 = \frac{-1}{n+1}$$

$$A(1+\varphi) = 1 + \tilde{\Delta}\varphi + \underline{P}(\varphi)$$

- Iterations

$$A f = 1 + r$$

$$(\varphi_\ell)_{\ell=0}^{\infty} \subset C^{2+\alpha}$$

$$\tilde{\Delta}\varphi_0 = r \,; \quad \tilde{\Delta}\varphi_1 = r - \underline{P}(\varphi_0)$$

$$\tilde{\Delta}\varphi_2 = r - \underline{P}(\varphi_1) \quad \cdots$$

$$\varphi_\ell \xrightarrow[C^{2+\alpha}]{\ell\to\infty} \varphi \,, \quad A(1+\varphi) = 1 + r$$