Stochastic Localization with Non-Gaussian Tilts and Applications to Tensor Ising Models

Arianna Piana
joint work with Dan Mikulincer (University Washington)

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WEIZMANN INSTITUTE OF SCIENCE

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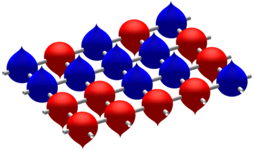
Introduction

The Ising model

We consider the *Ising model* defined on the n-dim. hypercube $\mathcal{C}_n := \{-1, +1\}^n$.



Energy is given by the Hamiltonian

$$-\mathcal{H}(x) = \langle x, Jx \rangle + \langle h, x \rangle,$$

where J is the *interaction matrix* and h the *external field*.
The Gibbs measure is

$$\mu(x) = \frac{1}{Z_\beta} \exp(-\beta \mathcal{H}(x)).$$

Glauber Dynamics (GLD)

Markov chain $(X_t)_{t \geq 0}$ with $Law(X_t) \xrightarrow{t \to \infty} \mu$.

- Start from a configuration $x = (x_1, \ldots, x_n)$.
- Repeat the following at each step
  1. pick a coordinate $i \in [n]$ uniformly at random,
  2. update $(X_t)_i$ according to $\mu | (X_t)_{j \neq i}$.

GLD has an associated reversible transition kernel P_{GLD}.
The mixing time is defined as

$$t_{\mathrm{mix}}(\varepsilon) := \min \left\{ t \geq 0 : \; \forall x \in \mathcal{C}_n \; \| P^t_{\mathrm{GLD}}(x, \cdot) - \mu \|_{\mathrm{TV}} \leq \varepsilon \right\}.$$

Spectral gap and Poincaré inequality

Consider the eigenvalues of P_{GLD}: $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n > -1$. The *spectral gap* is defined as

$$\mathsf{gap} := \lambda_1 - \lambda_2 = 1 - \lambda_2.$$

It holds:

$$t_{\mathrm{mix}}(\varepsilon) \leq \left\lceil \frac{n}{\mathsf{gap}} \left(\log\left(\frac{1}{\min_{x \in \mathcal{C}_n} \mu(x)} \right) + \log\left(\frac{1}{2\varepsilon} \right) \right) \right\rceil .$$

Fix $\varphi : \mathcal{C}_n \to \mathbb{R}$. For GLD, having gap is equivalent to satisfy a Poincaré inequality:

$$\mathrm{Var}_\mu(\varphi) \leq C_{\mathrm{P}}(\mu)\mathcal{E}_\mu(\varphi),$$

where

$$\mathcal{E}_\mu(\varphi) := \frac{1}{2} \sum_{x,y \in \mathcal{C}_n} \big(\varphi(x) - \varphi(y)\big)^2 \mu(x) P_{\mathrm{GLD}}(x,y),$$

and

$$\mathrm{Var}_\mu(\varphi) = \int \varphi^2 \mathrm{d}\mu - \left(\int \varphi \mathrm{d}\mu \right)^2 .$$

Stochastic Localization and two questions

Idea of Stochastic Localization (SL)

SL is a measure-valued process constructed by Eldan in 2013.

Start from a measure on $\mathcal{C}_n$ (or $\mathbb{R}^n$) of the form $\frac{\mathrm{d}m(x)}{\mathrm{d}x} = \exp(-w(x))$.
Perturb it by a Gaussian tilt

$$\exp\left(-w(x)\right)\exp\left(\langle x, \mathsf{d}\rangle - \mathsf{s}\|x\|_2^2\right).$$

w must be "nice", for instance convex or quadratic.

SL definition

Fix μ on $\mathcal{C}_n$. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}^n$.
SL is a process of the form $\mu_t = \mathsf{F}_t\mu$, where F_t is the solution of

$$\mathrm{d}\mathsf{F}_t(x) = \langle x - \mathsf{a}_t, \mathsf{C}_t \mathrm{d}B_t \rangle, \quad \mathsf{F}_0(x) = 1,$$

- a_t is the *barycenter*:

$$\mathsf{a}_t := \int x \mathrm{d}\mu_t(x),$$

- C_t is the *driving matrix*: encodes constraints on the system.

Role of a_t

a_t allows μ_t to be a probability measure. In fact:

$$\mathrm{d}\int \mu_t(x)\mathrm{d}x = \mathrm{d}\int \mathsf{F}_t(x)\mu(x)\mathrm{d}x = \left\langle \int x\mathsf{F}_t(x)\mu(x)\mathrm{d}x - \mathsf{a}_t, \mathsf{C}_t\mathrm{d}B_t \right\rangle = 0.$$

SL is a Gaussian tilt

Apply Itô's formula:

$$\mathrm{d}\log \mathsf{F}_t = \langle x - \mathsf{a}_t, \mathsf{C}_t \mathrm{d}B_t \rangle - \frac{1}{2}\|C_t(x - \mathsf{a}_t)\|_2^2 \mathrm{d}t.$$

We get:

$$\begin{aligned} \mu_t(x) &= \exp\left(\int_0^t \langle x - \mathsf{a}_s, \mathsf{C}_s \mathrm{d}B_s \rangle - \frac{1}{2}\int_0^t \|\mathsf{C}_s(x - \mathsf{a}_s)\|_2^2 \mathrm{d}s \right) \\ &\propto \exp\left(-\langle x, \mathsf{Q}_t x \rangle + \langle \mathsf{L}_t, x \rangle \right) \mu(x), \end{aligned}$$

where

- L_t is some adapted process in $\mathbb{R}^n$,
- $\mathsf{Q}_t = \frac{1}{2}\int_0^t \mathsf{C}_s^2 \mathrm{d}s$ is a matrix-valued process.

Can we localize μ with a non-Gaussian tilt?

Poincaré inequality via SL

Let τ be a suitable stopping time. Suppose we are able to prove

$$\mathsf{Var}_{\mu_\tau}(\varphi) \le C_{\mathsf{P}}(\mu_\tau)\mathcal{E}_{\mu_\tau}(\varphi)\,,$$

Let $\mathsf{M}_t := \int_{\mathcal{C}_n} \varphi \, \mathrm{d}\mu_t$. Then

$$\mathsf{Var}_{\mu}(\varphi) = \mathbb{E}\big[[\mathsf{M}]_t\big] + \mathbb{E}\big[\mathsf{Var}_{\mu_t}(\varphi)\big].$$

Suppose $[\mathsf{M}]_t = 0$ a.s. Then

$$\mathsf{Var}_{\mu}(\varphi) = \mathbb{E}[\mathsf{Var}_{\mu_\tau}(\varphi)] \le C_{\mathsf{P}}(\mu_\tau)\mathbb{E}[\mathcal{E}_{\mu_\tau}(\varphi)] \le C_{\mathsf{P}}(\mu_\tau)\mathbb{E}[\mathcal{E}_{\mu}(\varphi)]\,.$$

Eldan-Koehler-Zeitouni work: role of C_t

Consider $\mu(x) \propto \exp(\langle x, Jx \rangle)$. C_t encodes two constraints

1. $\mathbb{E}_{\mu_t}[\varphi]$ constant in time $\Rightarrow$ $\mathrm{Var}_{\mu_t}(\varphi)$ is a martingale.
2. $J_t = J - \int_0^t \mathsf{C}_s^2 ds$ decreasing, if $\mathsf{rank}(J_t) \geq 2$.

Define $\tau = \min\{t : \mathsf{rank}(J_t) = 1\}$. SL decomposes μ into a mixture of measures of the form

$$w_{u,v}(x) \propto \exp\left(\langle u, x \rangle^2 + \langle v, x \rangle\right).$$

It holds:

1. $\|u\|^2 = \|J_t\|_{\mathrm{op}} \leq \|J\|_{\mathrm{op}}$
2. $C_{\mathrm{P}}(w_{u,v}) \leq (1 - 2\|u\|_2^2)^{-1}$

If $\|J\|_{\mathrm{op}} < \frac{1}{2} \Rightarrow C_{\mathrm{P}}(\mu) \leq (1 - 2\|J\|_{\mathrm{op}})^{-1}$.

What happens if J is a fourth-order tensor?

Naïve answers

About fourth-order tensors

- $T \in (\mathbb{R}^n)^{\otimes 4}$.
- $T : \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$. "$n^2 \times n^2$ matrix"
- T symmetric: $T_{i_1,i_2,i_3,i_4} = T_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)},i_{\sigma(4)}}$ for any permutation σ.
- Injective norm: $\|T\|_{\text{inj}} = \sup_{x\in\mathbb{S}^{n-1}} |T(x,x,x,x)|$.

Naïve approach

Let $T \in (\mathbb{R}^n)^{\otimes 4}$. Consider $\nu(x) \propto \exp(T(x))$ on $\mathcal{C}_n$.
Define the following version of SL: $\nu_t = \mathtt{F}_t \nu$, where $\mathtt{F}_t$ solves

$$\mathrm{d}\mathtt{F}_t(x) = \langle x^{\otimes 2} - \mathtt{a}_t, \mathtt{M}_t \mathrm{d}\mathtt{W}_t \rangle_{\mathrm{HS}}, \quad \mathtt{F}_0(x) = 1,$$

with

$$\mathtt{a}_t = \int x^{\otimes 2} \mathrm{d}\nu_t(x),$$

and where $\mathtt{W}_t$ is a Dyson Brownian motion, and $\mathtt{M}_t$ is a fourth-order tensor operating on matrices.

Problems

Apply Itô's formula:

$$\nu_t(x) = \exp\left(\int_0^t \langle x^{\otimes 2} - \mathtt{a}_s, \mathtt{M}_s \mathrm{d}\mathtt{W}_s\rangle_{\mathrm{HS}} - \frac{1}{2}\int_0^t \|\mathtt{M}_s(x^{\otimes 2} - \mathtt{a}_s)\|_{\mathrm{HS}}^2 \mathrm{d}s\right)\mu(x)$$
$$\propto \exp\left(\left\langle x^{\otimes 2}, \left(T - \frac{1}{2}\int_0^t \mathtt{M}_s^2 \mathrm{d}s\right) x^{\otimes 2}\right\rangle_{\mathrm{HS}} + \left\langle \mathtt{L}_t, x^{\otimes 2}\right\rangle_{\mathrm{HS}}\right),$$

where

$$\mathtt{L}_t = \int_0^t \left(\mathtt{M}_s \mathrm{d}\mathtt{W}_s - \mathtt{M}_s^2 \mathtt{a}_s \mathrm{d}s\right).$$

Problems:

1. $\left\langle \mathtt{L}_t, x^{\otimes 2}\right\rangle_{\mathrm{HS}}$ is not linear in x. Spectral gap can deteriorate.
2. We treat T as a *matrix* $\Rightarrow$ $T - \frac{1}{2}\int_0^t \mathtt{M}_s^2 \mathrm{d}s$ collapses to a rank 1 matrix.

Tensor Stochastic Localization

Solution to problem 1

Problem: $\mathtt{L}_t = \int_0^t \left(\mathtt{M}_s \mathrm{d}\mathtt{W}_s - \mathtt{M}_s^2 \mathtt{a}_s \mathrm{d}s\right)$ not linear in x.
Solution: eliminate it. How?

1. Remove barycenter. Choose v_t such that $\mathtt{L}_t$ is arbitrarily small.
2. Encode in $\mathtt{M}_t$ a new constraint to fix $\int 1 \mathrm{d}\nu_t$

$\Rightarrow$ $\mathtt{M}_t$ encodes mass preservation and variance martingale.

$$\mathrm{d}F_t(x) = \langle x^{\otimes 2} - v_t, C_t \mathrm{d}W_t \rangle_{\mathrm{HS}}, \quad F_0(x) = 1,$$

with $\mu_t = F_t \mu$ and where

- C_t is a 4-tensor operating on matrices such that $C_t \mathrm{d}W_t$ is a matrix.
- v_t satisfies:
  Proposition $\exists$ adapted drift process $v_t^\delta \in \mathrm{Mat}(\mathbb{R}^n, \mathbb{R}^n)$, st if X_t^δ satisfies

$$\mathrm{d}X_t^\delta = C_t \mathrm{d}W_t - C_t^2 v_t^\delta \mathrm{d}t, \quad X_0^\delta = 0,$$

then $\sup_{t \geq 0} \|X_t^\delta\|_{\mathrm{HS}} \leq \delta$ a.s.

Solution to problem 2

Define

$$T_t := T - \frac{1}{2}\int_0^t C_s^2 \mathrm{d}s \quad \text{and} \quad \tau = \inf\{t \geq 0 : \mathrm{rank}(T_t) \leq 2\}.$$

We can decompose the measure

$$\begin{aligned}\mu(x) &= \mathbb{E}[\mu_\tau] \\ &\propto \mathbb{E}[\exp(\langle x^{\otimes 2} \otimes x^{\otimes 2}, M_1 \otimes M_1 + M_2 \otimes M_2\rangle_{\mathrm{HS}})] \\ &= \mathbb{E}[\exp(\langle x, M_1 x\rangle^2 + \langle x, M_2 x\rangle^2)]\end{aligned}$$

with $\|M_1\|_{\mathrm{OP}}^2, \|M_2\|_{\mathrm{OP}}^2 \leq \|T\|_{\mathrm{inj}}$.

Decomposition of the measure $\mu(x) \propto \exp(T(x))$

$$\exp\left(T(x)\right)$$

$$\exp\left(\langle x, M_1 x\rangle^2\right) \quad \exp\left(\langle x, M_2 x\rangle^2\right)$$

$$\exp\Big(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right) \quad \langle x, M_1 x\rangle\Big) \quad \exp\Big(\langle x, M_2 x\rangle \quad \left(\langle v_3, x\rangle^2 + \langle v_4, x\rangle^2\right)\Big)$$

$$\exp\Big(\left(\langle v_1, x\rangle^2 + \langle v_2, x\rangle^2\right) \quad \left(\langle u_1, x\rangle^2 + \langle u_2, x\rangle^2\right)\Big) \quad \exp\Big(\left(\langle u_3, x\rangle^2 + \langle u_4, x\rangle^2\right) \quad \left(\langle v_3, x\rangle^2 + \langle v_4, x\rangle^2\right)\Big)$$

Decomposition theorem

Let $\varphi : \mathcal{C}_n \to \mathbb{R}$ test function, let $\mu \propto \exp(T(x))$. Then, $\exists$ decomposition

$$\mu = \int \mu_{\bar{u},\bar{v},\bar{w},\ell}\,\eta(\mathrm{d}\bar{u}, \mathrm{d}\bar{v}, \mathrm{d}\bar{w}, \mathrm{d}\ell),$$

where $\bar{u} = \{u_1, u_2, u_3, u_4\}$, $\bar{v} = \{v_1, v_2, v_3, v_4\}$, and $\bar{w} = \{w_1, w_2\}$, with $u_i, v_i, w_j, \ell \in \mathbb{R}^n$ for $i \in [4]$ and $j = 1, 2$, and $\mu_{\bar{u},\bar{v},\bar{w},\ell}$ are probability measures

$$\mu_{\bar{u},\bar{v},\bar{w},\ell}(x) \propto \exp\left(\sum_{i,j=1}^{2} \langle u_i, x\rangle^2 \langle v_j, x\rangle^2 + \sum_{i,j=3}^{4} \langle u_i, x\rangle^2 \langle v_j, x\rangle^2 + \sum_{i=1}^{2} \langle x, w_i\rangle^2 + \langle \ell, x\rangle\right).$$

Properties:

- For $i \in [4]$, $\|u_i\|_2^2, \|v_i\|_2^2 \leq 2\sqrt{\|T\|_{\mathrm{inj}}}$, $\|w_i\|_2^2 \leq 4n\|T\|_{\mathrm{inj}}$.
- Variance decomposition: $\mathrm{Var}_\mu(\varphi) = \int \mathrm{Var}_{\mu_{\bar{u},\bar{v},\bar{w},\ell}}(\varphi)\eta(\mathrm{d}\bar{u}, \mathrm{d}\bar{v}, \mathrm{d}\bar{w}, \mathrm{d}\ell)$.

Spectral gap for tensor Ising model

Theorem Let μ be a measure on $\mathcal{C}_n$ given by $\mu(x) \propto \exp(T(x))$. Assume $\|T\|_{\text{inj}} \leq \frac{1}{336n}$. Then

$$C_{\mathrm{P}}(\mu) \leq \frac{1}{1 - 336n\|T\|_{\text{inj}}}.$$

Corollary T has independent $\mathcal{N}(0, n^{-3})$ off-diagonal entries and 0 everywhere else. Let μ be a measure on $\mathcal{C}_n$ given by $\mu(x) \propto \exp(\beta T(x))$. If $\beta \lesssim \frac{1}{1205.568}$, then

$$C_{\mathrm{P}}(\mu) \leq \frac{1}{1 - 1205.568\beta}.$$

Another approach

Trick: Normalized variance

Let ν be an unormalized measure. Consider normalized variance

$$\overline{\mathsf{Var}}_\nu(\varphi) = \mathbb{E}_\nu\left[\varphi^2\right] - \frac{\mathbb{E}_\nu\left[\varphi\right]^2}{\mathbb{E}_\nu\left[1\right]}.$$

Define $\tilde{\nu} = \frac{\nu}{\mathbb{E}_\nu[1]}$. Then

- $\overline{\mathsf{Var}}_\nu(\varphi) = \mathbb{E}_\nu[1]\mathsf{Var}_{\tilde{\nu}}(\varphi)$.
- If $C_{\mathsf{P}}(\tilde{\nu}) < \infty$. Then, $\forall \varphi : \mathcal{C}_n \to \mathbb{R}$

$$\overline{\mathsf{Var}}_\nu(\varphi) \leq C_{\mathsf{P}}(\tilde{\nu})\mathbb{E}_\nu[1]\mathcal{E}_{\tilde{\nu}}(\varphi) = C_{\mathsf{P}}(\tilde{\nu})\mathcal{E}_\nu(\varphi).$$

Adjustments to TSL

We allow $\mathbb{E}_{\mu_t}[1]$ to vary.
$\Rightarrow$ C_t has one constraint: $\frac{\mathbb{E}_{\nu_t}[\varphi]}{\mathbb{E}_{\nu_t}[1]}$ is constant. This allows to preserve $\overline{\mathrm{Var}}_{\nu_t}$.

$$\exp\left(T(x)\right)$$

$$\exp\left(\langle x, Mx\rangle \; \langle x, Mx\rangle + \langle x, Rx\rangle\right)$$

$$\exp\left(\langle x, Mx\rangle \; \langle v, x\rangle^2 + \langle x, Rx\rangle\right)$$

$$\exp\left(\langle u, x\rangle^2 \; \langle v, x\rangle^2 + \langle x, Rx\rangle\right)$$

$$\exp\left(\langle u, x\rangle^2 \; \langle v, x\rangle^2 + \langle x, w\rangle^2 + \langle \ell, x\rangle\right)$$

Decomposition theorem bis

Let $\varphi : \mathcal{C}_n \to \mathbb{R}$ and $\mu \propto \exp(T(x))$. Then, $\forall \delta > 0$, $\exists$ decomposition

$$\mu = \int \mu_{u,v,w,\ell,\psi} \eta(\mathrm{d}u, \mathrm{d}v, \mathrm{d}w, \mathrm{d}\ell, \mathrm{d}\psi),$$

where with $u, v, w, \ell \in \mathbb{R}^n$ and $\psi : \mathcal{C}_n \to \mathbb{R}$, and $\mu_{u,v,w,\ell,\psi}$ is a non-negative measure of the form

$$\mu_{u,v,w,\ell,\psi}(x) \propto \exp\left(\langle u, x\rangle^2 \langle v, x\rangle^2 + \langle x, w\rangle^2 + \langle \ell, x\rangle + \psi(x)\right).$$

Properties:

- $\|u\|_2^2, \|v\|_2^2 \leq 2\sqrt{\|T\|_{\mathrm{inj}}}$, $\|w\|_2^2 \leq 4n\|T\|_{\mathrm{inj}}$.
- $\max_{x \in \mathcal{C}_n} |\psi(x)| \leq \delta$.
- Variance decomposition:
  $\mathrm{Var}_\mu(\varphi) \leq \int \overline{\mathrm{Var}}_{\mu_{u,v,w,\ell,\psi}}(\varphi) \eta(\mathrm{d}u, \mathrm{d}v, \mathrm{d}w, \mathrm{d}\ell, \mathrm{d}\psi) + \delta.$

Spectral gap for tensor Ising model bis

Theorem Let μ be a measure on $\mathcal{C}_n$ given by $\mu(x) \propto \exp(T(x))$. Assume $\|T\|_{\text{inj}} \leq \frac{1}{56n}$. Then

$$C_{\mathrm{P}}(\mu) \leq \frac{1}{1 - 56n\|T\|_{\text{inj}}}.$$

Corollary T has independent $\mathcal{N}(0, n^{-3})$ off-diagonal entries and 0 everywhere else. Let μ be a measure on $\mathcal{C}_n$ given by $\mu(x) \propto \exp(\beta T(x))$. If $\beta \lesssim \frac{1}{200.928}$, then

$$C_{\mathrm{P}}(\mu) \leq \frac{1}{1 - 200.928\beta}.$$

Thank you!