Isometries of the class of Ball-Bodies

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Joint work with Shiri Artstein-Avidan and Dan Florentin

AGA online seminar

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Outline

1. The metric space $\mathcal{S}_n$ of ball-bodies

2. Surjective isometries and a geodesic appoach

3. General isometries and a more global approach

Ball bodies - definitions

Ball-bodies: convex bodies $K \in \mathcal{K}^n$ which are intersections of closed unit Euclidean balls: $K = \bigcap_{x \in A} (x + B_2^n)$ for some $A \subseteq \mathbb{R}^n$.

$\mathcal{S}_n :=$ the class of all ball-bodies.

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Note: $\mathcal{S}_n$ is closed under Minkowski averages:
If $K, L \in \mathcal{S}_n$ with $K + K' = B_2^n$, $L + L' = B_2^n$:
$\frac{1}{2}(K + L) + \frac{1}{2}(K' + L') = B_2^n \quad \Rightarrow \quad \frac{1}{2}(K + L) \in \mathcal{S}_n$.

Equivalent definitions

Bodies which are "spindle convex": for any $x, y \in K$ the spindle

$$[x, y]_s := \bigcap_{z:\ \{x,y\} \subseteq z + B_2^n} z + B_2^n.$$

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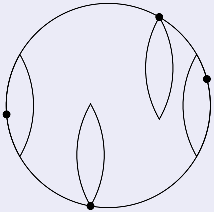
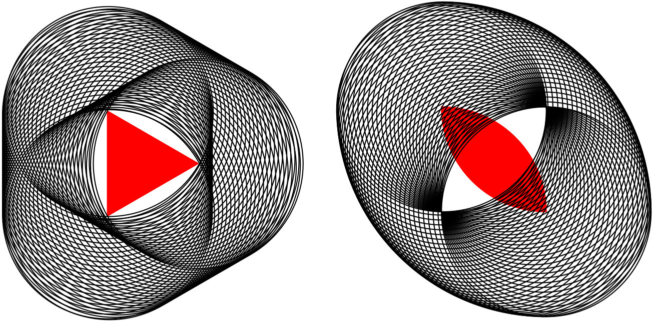
Bodies which slide freely in B_2^n: for any $x \in \partial B_2^n$ there is some $y \in \mathbb{R}^n$ such that $x \in y + K \subseteq B_2^n$. (We won't use this definition.)



A natural duality map on $\mathcal{S}_n$

For $K \in \mathcal{K}^n$, define

$$K^c = \bigcap_{x \in K} x + B_2^n.$$



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Note: for any $K, L \in \mathcal{K}^n$:

1. $K \subseteq L \Rightarrow L^c \subseteq K^c$,
2. $K \subseteq K^{cc}$,
3. $K = K^{cc}$ if and only if $K \in \mathcal{S}_n$,

i.e. c-duality is an "order-reversing quasi-involution".

For $K, L \in \mathcal{S}_n$: $\left(\frac{1}{2}(K + L)\right)^c = \frac{1}{2}(K^c + L^c)$ by the summands definition.

Ball bodies appear naturally - cost duality

In '22 Artstein-Avidan, Sadovsky and Wyczesany gave a characterization of all order-reversing quasi-involutions of a set X: every such $T : \mathcal{P}(X) \to \mathcal{P}(X)$ is given by

$$T(K) = \bigcap_{x \in K} \{y \in X \mid c(x, y) \geq 0\}$$

for some symmetric $c : X \times X \to \{\pm 1\}$.

Specifically, the map $K \mapsto K^c$ is a cost-duality with the cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \{\pm 1\}$ given by

$$c(x, y) = \begin{cases} 1 & |x - y| \leq 1 \\ -1 & |x - y| > 1 \end{cases}.$$

Ball bodies appear naturally - Kneser-Poulsen conjecture

Conjecture (Gromov '87)

A set $P = \{p_1, ..., p_N\} \subseteq \mathbb{R}^n$ *is called a contraction of another set* $Q = \{q_1, ..., q_N\} \subseteq \mathbb{R}^n$ *if* $\forall i, j : |p_i - p_j| \leq |q_i - q_j|$.

Conjecture: If P *is a contraction of* Q *then* $Vol_n(P^c) \geq Vol_n(Q^c)$.

Some known special cases:

1. Bounded N: $N \leq n + 1$ (Gromov '87), $N \leq n + 3$ (Bezdek, Connelly '01).
2. Dimension 2 (Bezdek, Connelly '01).
3. Piecewise-smooth contractions (Bezdek, Connely '01).
4. Uniform contraction with $N \geq (1 + \sqrt{2})^n$ (Bezdek, Naszódi '17).
5. inradius instead of Vol_n (Bezdek, Lángi, Naszódi, Papez '07).

Ball bodies appear naturally - constant width bodies

Constant-width-1 bodies are exactly c-self-dual bodies since $h_K(u) + h_{K^c}(-u) = 1$ on S^{n-1}. For K of constant width 1:

$$\underbrace{\left(\sqrt{3+\frac{2}{n+1}}-1\right)^n}_{\approx 0.73^n} Vol_n\left(\frac{1}{2}B_2^n\right) \overset{\text{Schramm '88}}{\leq} Vol_n(K)$$

$$\overset{\text{Urysohn's ineq.}}{\leq} Vol_n\left(\frac{1}{2}B_2^n\right).$$

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Until recently: no examples asymptotically smaller than $\frac{1}{2}B_2^n$.

Arman, Bondarenko, Nazarov, Prymak, Radchenko '24 give a family $K_n \in \mathcal{S}_n$ of constant width bodies with $Vol_n(K_n) \leq 0.9^n Vol_n\left(\frac{1}{2}B_2^n\right)$.

Ball bodies appear naturally - constant width bodies

Theorem (Blaschke-Lebesgue)

The minimal area constant width body in $\mathbb{R}^2$ is the Reuleaux triangle.

The conjectured minimizers in $\mathbb{R}^3$ are the Meissner tetrahedra:

The two Meissner tetrahedra. Image: Meissner's mysterious bodies, Kawohl & Weber '11

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Open Question: Which constant width 1 bodies in $\mathbb{R}^n$ have minimal volume?

$\mathcal{K}^n$ vs $\mathcal{S}_n$ - Carathéodory's theorem

Theorem (Carathéodory 1911)

Let $X \subseteq \mathbb{R}^n$. *Then for any* $y \in \mathrm{conv}(X) = X^{\circ\circ}$ *there are* $x_0, x_1, ..., x_n \in X$ *such that* $y \in \mathrm{conv}\{x_0, ..., x_n\}$.

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Let $X \subseteq \mathbb{R}^n$. Then for any $y \in \mathrm{conv}(X) = X^{\circ\circ}$ there are $x_0, x_1, \ldots, x_n \in X$ such that $y \in \mathrm{conv}\{x_0, \ldots, x_n\}$.

An analogue for ball-bodies:

Theorem (Bezdek, Lángi, Naszódi, Papez '07)

Let $X \subseteq \mathbb{R}^n$ be closed. Then for any $y \in X^{cc}$ there are $x_0, x_1, \ldots, x_n \in X$ such that $y \in \{x_0, \ldots, x_n\}^{cc}$.

Definition

$I(K) =$ the minimal number of directions which illuminate $K \in \mathcal{K}^n$.

Conjecture (Hadwiger conjecture)

For any convex body $K \in \mathcal{K}^n$, $I(K) \leq 2^n$.

$\mathcal{K}^n$ vs $\mathcal{S}_n$ - Hadwiger-Boltjansky illumination

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Conjecture (Hadwiger conjecture)

For any convex body $K \in \mathcal{K}^n$, $I(K) \leq 2^n$.

Theorem (Schramm '88)

For $K \in \mathcal{S}_n$ of constant width, $I(K) \leq 5n\sqrt{n}(4 + \log n)\left(\frac{3}{2}\right)^{\frac{n}{2}} \ll 2^n$.

Theorem (Bezdek, Lángi, Naszódi, Papez '07)

If $A \subseteq \mathbb{R}^3$ with $\operatorname{diam} A \leq 1$ then $I(A^c) \leq 6 < 2^3$.

In '08 Bezdek raised the question: is there $c > 0$ such that for any K of the form $K = A^c$ with A finite, $I(K) \leq (2-c)^n$?

From order preserving/reversing...

Theorem (Böröczky, Schneider '08; Gruber '91)

Any order preserving (reversing) bijection $\mathcal{K}_0^n \to \mathcal{K}_0^n$ is induced by a linear transformation (a linear transformation composed with polarity).

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Theorem (Artstein-Avidan, Florentin '25)

Any order preserving (reversing) bijection $\mathcal{S}_n \to \mathcal{S}_n$ is induced by a rigid motion (a rigid motion composed with c-duality).

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...to isometries!

Equipping $\mathcal{K}^n$ (and $\mathcal{S}_n$) with the Hausdorff metric δ:

$$\delta(K, L) = \min\left\{t \geq 0 \mid K + tB_2^n \supseteq L,\ L + tB_2^n \supseteq K\right\}$$

Theorem (Schneider '75)

Any bijective isometry $(\mathcal{K}^n, \delta) \to (\mathcal{K}^n, \delta)$ is induced by a rigid motion.

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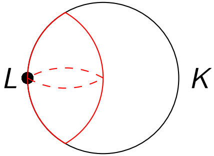
Remark

It can be seen that $K \mapsto K^c$ is an isometry ($\Rightarrow$ continuous) on $\mathcal{S}_n$ by noting that $\delta(K, L) = \left\| h_K|_{S^{n-1}} - h_L|_{S^{n-1}} \right\|_\infty$, and $h_{K^c} = h_{B_2^n} - h_K$.

Proof sketch

Let $T : \mathcal{S}_n \to \mathcal{S}_n$ be a bijective isometry.

1. Note: there exist bodies $K, L \in \mathcal{S}_n$ with multiple midpoints ("discrete geodesics"):



   Pairs (K, L) with a *unique* midpoint $\frac{K+L}{2}$ will be called $\mathcal{S}_n$-*cute*.
2. T maps $\mathcal{S}_n$-cute pairs to $\mathcal{S}_n$-cute pairs (**uses bijectiveness!**), maps midpoint to midpoint.

Proof sketch

3. Characterize cute pairs: assume (K, L) is $\mathcal{S}_n$-cute, $\delta(K, L) \geq 4$. Then (K, L) are both points or both unit balls:
   - i Show the midpoint is a lens (i.e. an intersection of 2 unit balls).
   - ii Show that if the Minkowski average of 2 bodies (in $\mathcal{S}_n$!) is a lens, then they are both translations of the same lens.
   - iii The only $\mathcal{S}_n$-cute pairs of lens are either both points or both unit balls.
4. Either T maps all points to points, or T maps all points to unit balls.
5. If T maps points to points it is a rigid motion, if T maps points to balls it is a composition of a rigid motion with c-duality.

$\square$

What about non-surjective isometries?

Theorem (Gruber, Lettl '80)

Any isometry $(\mathcal{K}^n, \delta) \to (\mathcal{K}^n, \delta)$ is induced by a rigid motion and Minkowski addition of some fixed convex body.

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Corollary

Any isometry of $\mathcal{S}_n$ is bijective.

Proof sketch

Let $T : \mathcal{S}_n \to \mathcal{S}_n$ be an isometry.

Lemma

There is a point in the image of T, and its preimage is either a point or a unit ball.

Lemma $\Rightarrow$ Theorem is similar to Gruber and Lettl's proof: let $K \in \mathcal{S}_n$ be the preimage (a point or a unit ball).

1. Define $M := \{x \in \mathbb{R}^n \mid T(K + x) \neq \mathrm{pt}\}$. Note $0 \notin M$.

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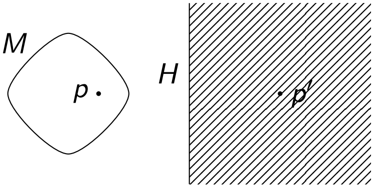
2. Show M is convex. $\Rightarrow$ there is a closed halfspace H: $H \cap M = \emptyset$.
3. Assume $p \in M$, show that $T(K + p) \subseteq [p, p']$:



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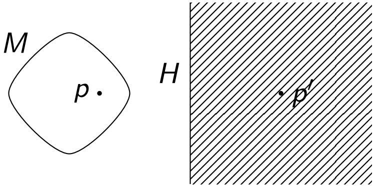

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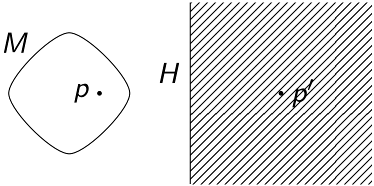

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$\forall x \in \partial H : \delta(T(K+p), x) = \delta(T(K+p), T(K+x)) = \delta(p, x) = |x-p|.$

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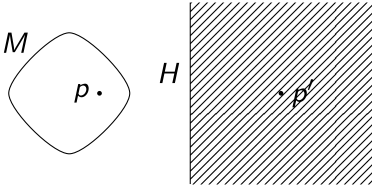

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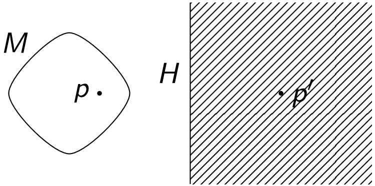

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$$\Rightarrow T(K+p) \subseteq \bigcap_{x \in \partial H} x + |x-p| B_2^n = [p, p'].$$

$\mathrm{pt} \neq T(K+p) \in \mathcal{S}_n$ has no interior, a contradiction $\Rightarrow M = \emptyset$.
Therefore all translates of K map to points. $\square$

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Main claim

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1. Note that any point and any unit ball have Hausdorff distance ≥ 1, but any two bodies in $\mathcal{S}_n$ with the same circumcenter have Hausdorff distance ≤ 1.
2. Moreover, if we have two bodies in $\mathcal{S}_n$ with the same circumcenter and in distance 1 from each other, one of them is a point.

$\square$

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There exist a point and a unit ball, both whose T-images have the same circumcenter.

Proof.

Letting $c(K)$ be the circumcenter of K, define $f_{pt}, f_{ball} : \mathbb{R}^n \to \mathbb{R}^n$:

$$f_{pt}(x) = c(T(x)), \qquad f_{ball}(x) = c(T(x + B_2^n)).$$

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f_{pt}, f_{ball} are continuous, 2-isometries, namely:

$$\Big| |f_{pt}(x) - f_{pt}(y)| - |x-y| \Big| \leq 2, \qquad \Big| |f_{ball}(x) - f_{ball}(y)| - |x-y| \Big| \leq 2.$$

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$\overset{*}{\Rightarrow}$ they are onto. $\square$

* Any continuous ε-isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is onto.

Thank you.