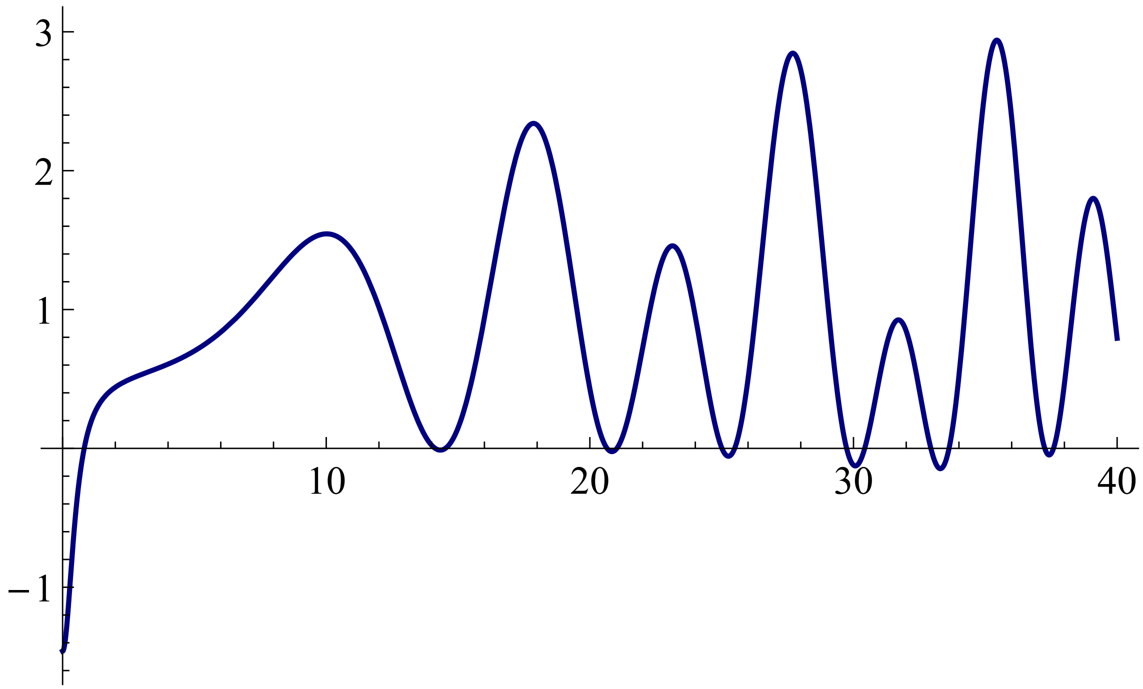
Rational approximations to ζ

Keith Ball



RH The Riemann zeta function has no zeroes to the *right* of the critical line $\{s : \Re s = 1/2\}$.

The customary formulation is equivalent by virtue of the functional equation proved by Riemann.

To show that a holomorphic function has no zeroes in a half-plane it suffices to express the function as a locally uniform limit of holomorphic functions with no zeroes there.

Lindelöf Hypothesis $\zeta(1/2+it)$ does not grow as fast as t^{ϵ}.

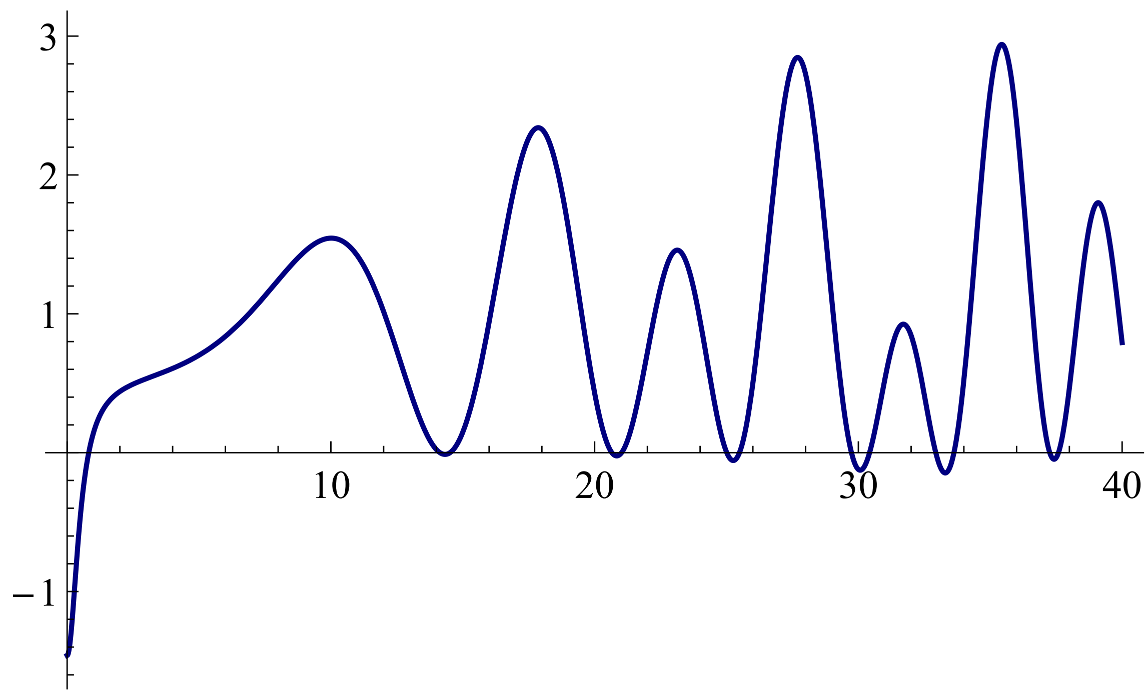
The usual series $\sum \frac{1}{n^s}$ does not converge for $\Re s < 1$ but we can approximate by finite Dirichlet sums.

The terms are

$$\frac{1}{\sqrt{n}} e^{-it\,\log n}.$$

So we are adding up periodic functions whose periods don't match except for trivial cases such as $\log 4 = 2\log 2$.

If you were to "play" the zeta function on a musical instrument it would be a horrible sound with infinite energy.



The small size of $\zeta(1/2 + it)$ depends upon cancellation between the different powers which is extremely difficult to capture.

This talk describes rational functions approximating ζ:

$$\frac{1}{(s-1)},\quad \frac{s+1}{2(s-1)},\quad \frac{4s^2+11s+9}{6(s+3)(s-1)},\quad \frac{(s+2)(3s^2+10s+11)}{4(s^2+6s+11)(s-1)},$$

$$\frac{(s+2)(72s^3+490s^2+1193s+1125)}{30(3s^3+29s^2+106s+150)(s-1)},\ldots$$

Each coefficient in the rational functions depends upon all the Dirichlet terms so the cancellation is built into the coefficients.

For each integer $m \geq 0$ we define

$$p_m(t) = (1-t)\left(1-\frac{t}{2}\right)\dots\left(1-\frac{t}{m}\right)$$

and the coefficients $(a_{m,j})$ by

$$p_m(t) = \sum_0^m (-1)^j a_{m,j} t^j.$$

We then set

$$F_m(s) = \sum_0^m \frac{a_{m,j} B_j}{s+j-1}$$

and

$$G_m(s) = \sum_{j=0}^m (-1)^j \frac{a_{m,j}}{s+j-1}.$$

The rational functions in question are the ratios

$$\frac{F_m(s)}{(s-1)G_m(s)}.$$

For example

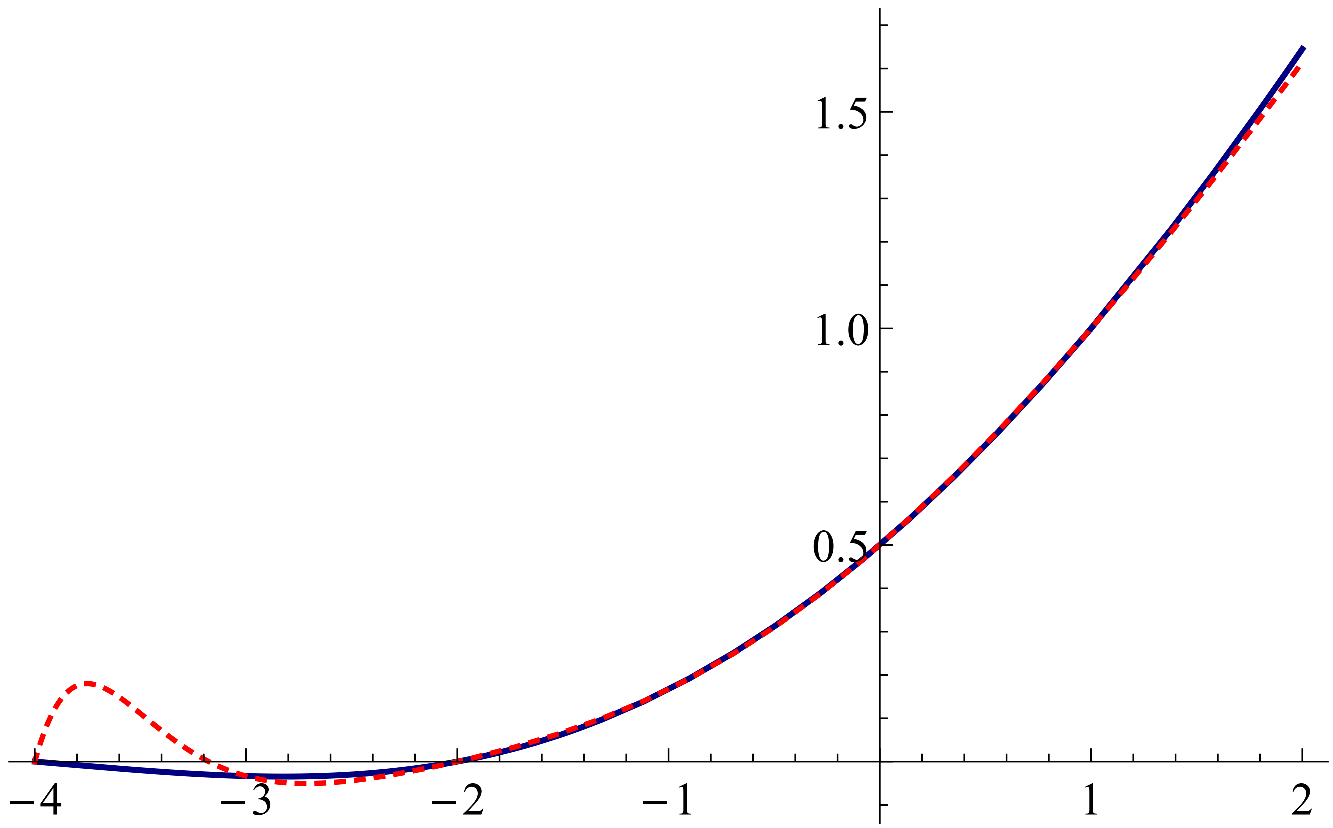
$$F_3(s) = \frac{1}{s-1} - \frac{11}{12s} + \frac{1}{6(s+1)} = \frac{3s^2+10s+11}{12(s-1)s(s+1)}$$

and

$$G_3(s) = \frac{1}{s-1} - \frac{11}{6s} + \frac{1}{s+1} - \frac{1}{6(s+2)} = \frac{s^2+6s+11}{3(s-1)s(s+1)(s+2)}.$$

The m^{th} ratio interpolates ζ at the points $0, -1, -2, \ldots, 1-m$ and has a simple pole with residue 1 at $s = 1$.

The graph shows $(s-1)\zeta(s)$ and the ratio $F_5(s)/G_5(s)$



The sequence converges locally uniformly to ζ, at least to the right of the line $\{s : \Re s = 0\}$.

We shall see that

$$F_m(s) \approx h_m^{1-s}\Gamma(s)\zeta(s)$$

and

$$(s-1)G_m(s) \approx h_m^{1-s}\Gamma(s)$$

where h_m is the partial sum $\sum_{j=1}^{m} 1/j$ of the harmonic series.

The rational functions might still be difficult to analyse: what are the coefficients?

Focus on the F_m:

$$
\begin{array}{cccc}
F_0(s), & F_1(s), & F_2(s), & F_3(s) \\
\dfrac{1}{(s-1)}, & \dfrac{s+1}{2(s-1)s}, & \dfrac{4s^2+11s+9}{12(s-1)s(s+1)}, & \dfrac{(s+2)(3s^2+10s+11)}{12(s-1)s(s+1)(s+2)}
\end{array}
$$

We have a recurrence relation: for each m

$$(s+m-1)\,F_m(s) = \frac{1}{(m+1)} + (m+1)\sum_{j=1}^{m} \frac{F_{m-j}(s)}{j(j+1)}.$$

Equivalently

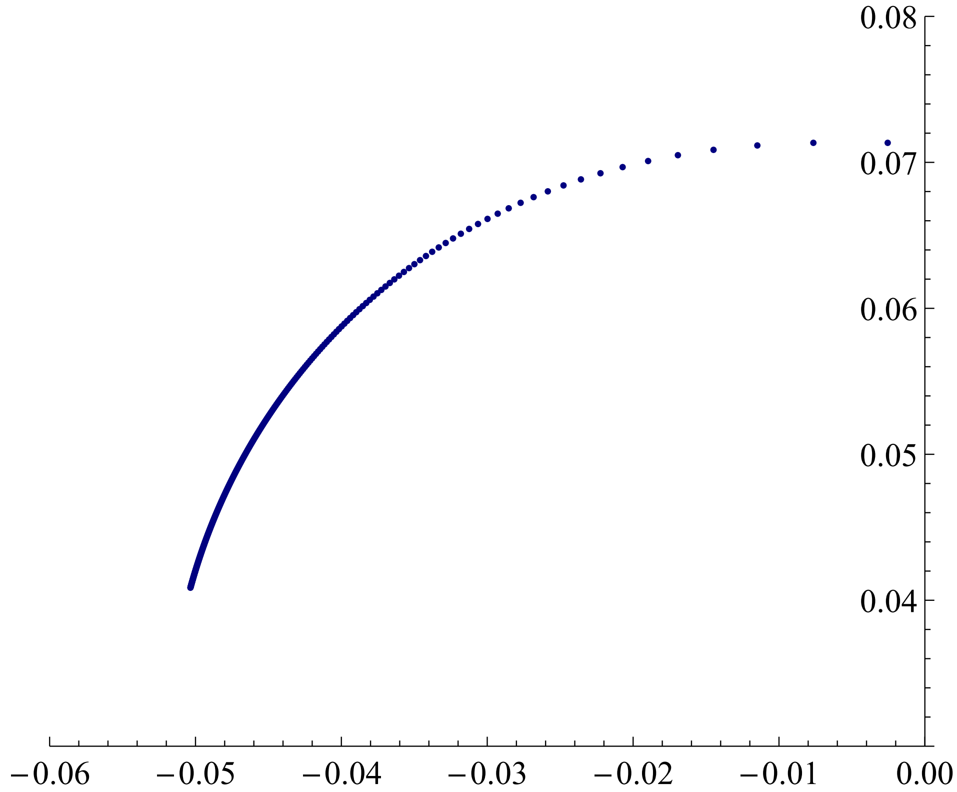
$$\left(1+\frac{s-1}{m}\right)F_m(s)=\frac{1}{m(m+1)}+\frac{m+1}{m}\sum_{j=1}^{m}\frac{F_{m-j}(s)}{j(j+1)}.$$

At each stage we take a weighted average of the previous terms, add a small bit and rotate slightly.

This is a very stable dynamical system.

The dependence of the end result ζ on s can be very sensitive because s rotates at each step. But for each fixed s we have a very smooth way of getting to $\zeta(s)$.

Here are the first few hundred values of $(n+1)F_n(1/2-14i)$.



If we treat the first $m+1$ of these relations as a linear system for the values $F_0(s), F_1(s), \ldots, F_m(s)$ we can express the fact that $F_m(s) = 0$ by the vanishing of a certain determinant.

The numerator of the m^{th} function is the determinant of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{2} & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \ldots & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \ldots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ \frac{1}{m+1} & \frac{1}{m} & \frac{1}{m-1} & \ldots & \frac{1}{2} & 1 \end{pmatrix} + (1-s) \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{m} \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{m} \\ 0 & 0 & 0 & \frac{1}{3} & \ldots & \frac{1}{m} \\ 0 & 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \frac{1}{m} \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

So RH can be restated as what looks like a rather conventional spectral problem.

Connes and Berry & Keating reformulated RH as a statement about the spectrum of an operator acting on an infinite-dimensional function space.

There is a connection between their infinite-dimensional operator and these finite-dimensional ones.

If $\Re s > 1$

$$\begin{aligned} G_m(s) &= \sum_{j=0}^{m} (-1)^j \frac{a_{m,j}}{s+j-1} = \sum_{j=0}^{m} (-1)^j a_{m,j} \int_0^1 x^j x^{s-2}\, dx \\ &= \int_0^1 p_m(x) x^{s-2}\, dx \end{aligned}$$

$$p_m(x) = (1-x)\left(1-\frac{x}{2}\right)\cdots\left(1-\frac{x}{m}\right) \approx e^{-h_m x}$$

so it is no surprise that $G_m(s) \approx h_m^{1-s}\Gamma(s-1)$.

We want to do something similar for F_m.

If $\Re s > 1$

$$\int_0^\infty \frac{y}{1-e^{-y}} e^{-y} y^{s-2}\, dy = \int_0^\infty \left(\sum_{n=1}^\infty e^{-ny}\right) y^{s-1}\, dy$$

$$= \sum_{n=1}^\infty \int_0^\infty e^{-ny} y^{s-1}\, dy = \sum_{n=1}^\infty \frac{1}{n^s}\Gamma(s).$$

So

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{-\log(1-(1-e^{-y}))}{1-e^{-y}} e^{-y} y^{s-2}\, dy$$

$$= \int_0^\infty \sum_{k=0}^\infty \frac{1}{k+1}(1-e^{-y})^k\, e^{-y} y^{s-2}\, dy.$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \sum_{k=0}^\infty \frac{1}{k+1}(1-e^{-y})^k\, e^{-y} y^{s-2}\, dy.$$

Using a standard formula for Bernoulli numbers we get that for $\Re s > 1$

$$F_m(s) \;=\; \int_0^1 \left(\sum_{k=0}^m \frac{1}{k+1} \sum_{r=0}^k \binom{k}{r} (-1)^r p_m((r+1)x) \right) x^{s-2}\, dx$$

If x is close to zero then

$$\begin{aligned} \Delta_{m,k}(x) \;&=\; \sum_{r=0}^k \binom{k}{r} (-1)^r p_m((r+1)x) \\ &\approx\; \sum_{r=0}^k \binom{k}{r} (-1)^r e^{-h_m(r+1)x} = (1-e^{-h_m x})^k\, e^{-h_m x}. \end{aligned}$$

For small values of x the integrand is approximately

$$\left(\sum_{k=0}^{m} \frac{1}{k+1} e^{-h_m x}(1-e^{-h_m x})^k\right) x^{s-2}.$$

If the approximation were good for *all* x between 0 and 1 then $F_m(s)$ would be close to

$$\int_0^1 \sum_{k=0}^{m} \frac{1}{k+1} e^{-h_m x}(1-e^{-h_m x})^k x^{s-2}\, dx$$

$$= h_m^{1-s} \int_0^{h_m} \sum_{k=0}^{m} \frac{1}{k+1} e^{-y}(1-e^{-y})^k y^{s-2}\, dy$$

and the integral converges to $\Gamma(s)\zeta(s)$ as $m \to \infty$.

We want to show that

$$h_m^{s-1} F_m(s) \to \Gamma(s)\zeta(s)$$

locally uniformly for $\Re s > 0$.

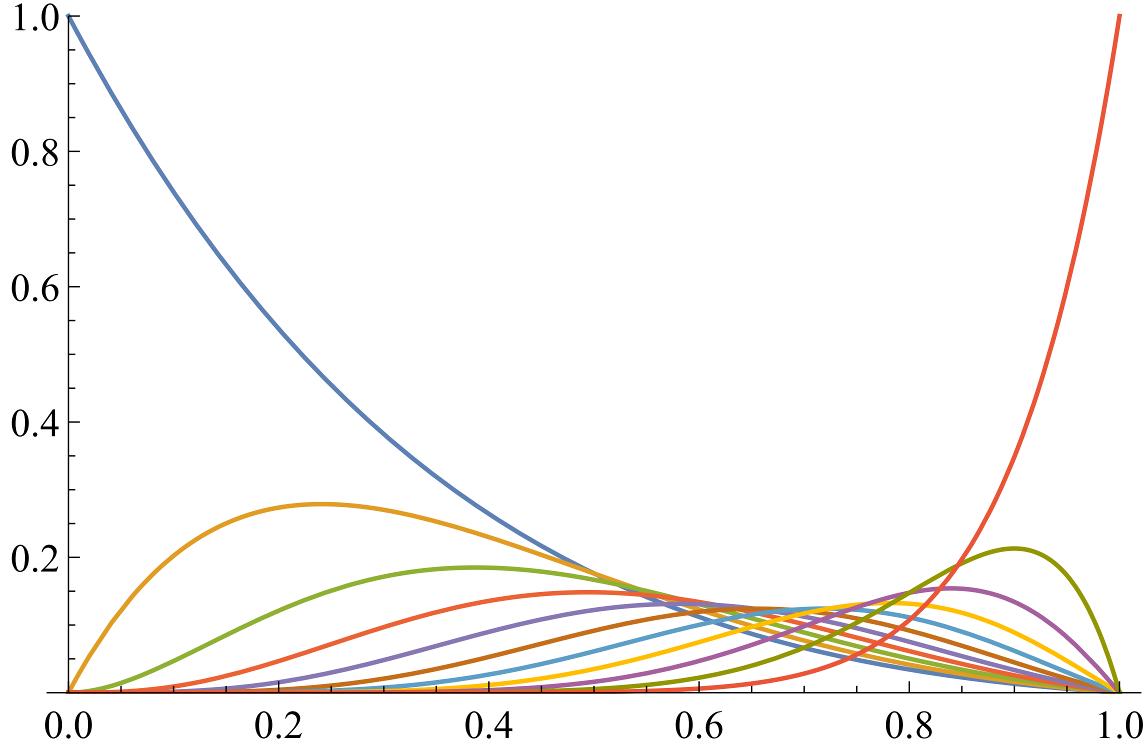
Crossing the pole at $s = 1$ is not the problem.

The difficulty is that unless x is very close to 0, the expressions

$$\Delta_{m,k}(x) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r p_m((r+1)x)$$

involve values of p_m at points well outside the interval $[0, 1]$.

The graph shows the $\Delta_{m,k}$ for $m = 10$.



Lemma 1 (**Key Lemma**). *If* m *is a non-negative integer,* k *is any integer and* $x \in [0, 1]$ *then*

$$\Delta_{m,k}(x) \geq 0.$$

It is trivial to check that

$$\sum_{k=0}^{m} \Delta_{m,k}(x) = 1$$

for all x, so the $\Delta_{m,k}$ form a partition of unity on $[0, 1]$.

After some fairly delicate estimates we get that the ratios

$$\frac{F_m(s)}{(s-1)G_m(s)}$$

converge locally uniformly to $\zeta(s)$ for $\Re s > 0$.

My guess is that they do so on the entire complex plane.

Theorem 2 (**Convergence**)**.**

$$h_m^{s-1}(s-1)F_m(s) \to (s-1)\Gamma(s)\zeta(s)$$

locally uniformly for $\Re s > 0$.

Lemma 1 (**Key Lemma**). *If m is a non-negative integer, k is any integer k and $x \in [0,1]$*

$$\Delta_{m,k}(x) \geq 0.$$

The proof of the key lemma involves the introduction of an additional parameter. For each v define

$$P_m(v,x) = (v+1-x)(v+2-x)\ldots(v+m-x)$$

and

$$\tilde{\Delta}_{m,k}(v,x) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r P_m(v,(r+1)x).$$

$\tilde{\Delta}_{m,k}(0,x) = m!\Delta_{m,k}(x)$ so the key lemma follows from:

Lemma 3. *If m is a non-negative integer, k is an integer, $v \geq 0$ and $0 \leq x \leq 1$ then*

$$\tilde{\Delta}_{m,k}(v, x) \geq 0.$$

Proof We use induction on m. When $m = 0$, $\tilde{\Delta}_{m,k}(v, x)$ is zero unless $k = 0$ in which case it is 1.

We claim that for $m > 0$

$$\tilde{\Delta}_{m,k}(v, x) = (v + 1 - x)\tilde{\Delta}_{m-1,k}(v + 1, x) + kx\tilde{\Delta}_{m-1,k-1}(v + 1 - x, x).$$

Then the inductive step is clear because we can assume that $k \geq 0$ and for the given range of v and x, the number $v + 1 - x$ is also at least 0.

□

A zero-free region for G_m

The function $p_m(x/h_m)$ differs from e^{-x} by only about $1/h_m^2$ at any point of $[0, 2h_m]$ and so we expect

$$h_m^{s-1} G_m(s)$$

to provide a good approximation to Γ at $s = 1/2 + it$ as long as $\Gamma(s)$ is as large as $1/h_m^2$.

This happens if $|t|$ is at most a bit less than $\frac{2}{\pi} \log \log m$.

In this region we will have $G_m(s) \neq 0$.

We would like to be able to prove a zero-free region without using any properties of Γ.

By expanding the polynomial p_m about 1 instead of 0 we get

$$ms(s-1)G_m(s) = a_{m-1,0} + \frac{2a_{m-1,1}}{s+1} + \frac{6a_{m-1,2}}{(s+1)(s+2)} + \cdots .$$

This doesn't look very promising when s is small since

$$a_{m-1,j} \approx \frac{(\log m)^j}{j!}.$$

However

$$\frac{1}{ms(s-1)G_m(s)} - 1$$

$$= \cfrac{-2a_1}{2a_1 + a_0(s+1) - \cfrac{3a_0a_2(s+1)}{3a_2 + a_1(s+2) - \cfrac{4a_1a_3(s+2)}{4a_3 + a_2(s+3)}}} \ddots$$

Worpitzky: a continued fraction cannot "blow up" (cannot have zero denominator) if the product of two successive denominators has absolute value at least 4 times as large as that of the numerator in between.

The sequence

$$\left(j!a_j\right)_j$$

is log-concave by Newton's Theorem since the a_j are coefficients of a polynomial with negative real roots.

Therefore sequence

$$\left(j\frac{a_j}{a_{j-1}}\right)_j$$

is decreasing

This is enough to prove that the fraction exists for $s = \sigma + it$ with $|t| < \sqrt{\log m}$: much further up than we could have guaranteed by looking at p_m.

A zero-free region for F_m?

The function F_m can be expanded in a continued fraction similar to that for G_m.

$$(m+1)sF_m(s) - 1 = \frac{1}{2}\int_0^1 \sum_{k\le m/2} c_{m,k+1}x^k(1-x)^{(s-1)/2-1}\,dx$$

$$= \frac{c_{m,1}}{s-1} + \frac{2c_{m,2}}{(s-1)(s+1)} + \cdots + \frac{2^{j-1}c_{m,j}(j-1)!}{(s-1)\ldots(s+2j-3)} + \cdots.$$

It is not obvious that these coefficients are non-negative.

However we have a generating function

$$1+\frac{1}{m+1}\sum_{k\geq 1, m\geq 1} c_{m,k} z^{k-1} y^m = (\log(1-y))^2 \frac{\partial}{\partial y}\frac{\sqrt{1-z}}{(1-y)^{\sqrt{1-z}}-1}.$$

and

$$\frac{2\sqrt{1-z}}{(1-y)^{\sqrt{1-z}}-1}$$

$$= -\frac{2}{y} + 1 - \sqrt{1-z} + \cfrac{zy}{3(2-y) - \cfrac{(3+z)y^2}{5(2-y) - \cfrac{(8+z)y^2}{7(2-y)}}} \ddots$$

I believe that for each m the sequence $(c_{m,k})$ is log-concave.

Replace the coefficients in

$$1 + \frac{1}{m+1} \sum_{k \geq 1, m \geq 1} c_{m,k} z^{k-1} y^m$$

by the values at y.

In other words we expand

$$(\log(1-y))^2 \frac{\partial}{\partial y} \frac{\sqrt{1-z}}{(1-y)^{\sqrt{1-z}} - 1}$$

as a series in z.

If we replace y by $1 - e^{-w}$ this is

$$\frac{w^2 e^w (1-z)}{\sinh^2(w/2\sqrt{1-z})}$$

$$\frac{w^2 e^w (1-z)}{\mathrm{sinh}^2(w/2\sqrt{1-z})} = e^w \prod_{n=1}^{\infty} \frac{1}{1 + \frac{w^2(1-z)}{4n^2\pi^2}}.$$

We have log-concavity (by Brunn-Minkowski).