On the Illumination Conjecture for convex bodies with many symmetries

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joint work with Wen Rui Sun

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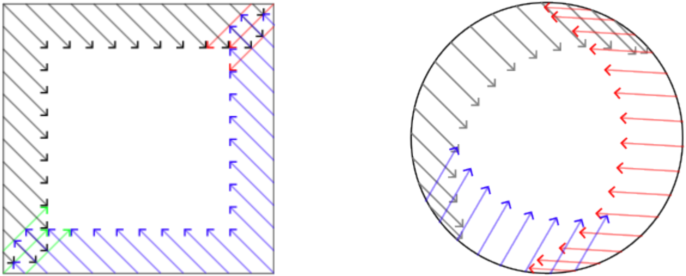
Terminology and the conjecture

- K convex body in $\mathbb{R}^n$, $x \in \partial K$, $d \in \mathbb{R}^n \setminus \{\vec{0}\}$.

  We say that the *direction d illuminates* (or *K-illuminates*) x if there exists $\epsilon > 0$ such that $x + \epsilon d \in \mathrm{int}(K)$.
- A set $\{d_1, d_2, \ldots, d_M\}$ illuminates K if $\forall\ x \in \partial K$ is illuminated by at least one direction d_i in the set (call this an *illuminating set* for K).

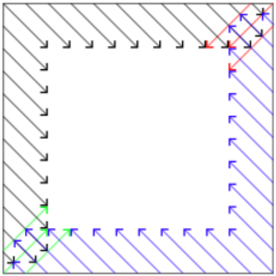


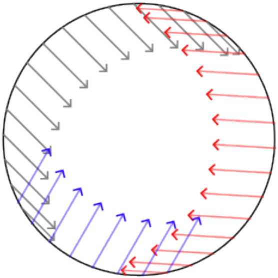
- The minimum size of an illuminating set for K is the *illumination number* $\mathfrak{I}(K)$ of K.

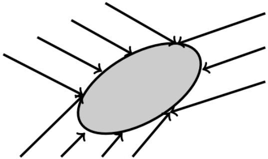
Illumination conjecture (Hadwiger (1957, 1960); Boltyanski (1960))

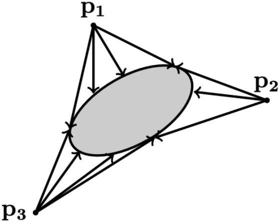
For every convex body K in $\mathbb{R}^n$ we have that $\mathfrak{I}(K) \leqslant 2^n$.

Moreover, 2^n directions are needed <u>only</u> if K is the cube $[-1, 1]^n$ or an affine image of the cube.









Vladimir Boltyanski (left, courtesy Annals of the Moscow University) and Hugo Hadwiger (right, courtesy Oberwolfach Photo Collection).

Figures on this slide, and photographs taken from the survey paper "K. Bezdek and M. A. Khan, *The geometry of homothetic covering and illumination*, in **Discrete Geometry and Symmetry**".

An equivalent conjecture on covering numbers

- Let A, B be bounded subsets of $\mathbb{R}^n$ with non-empty interior. The *covering number of A by B* is given by

$$N(A,B) := \min \left\{ M \in \mathbb{N} : \exists\, x_1, x_2, \ldots, x_M \in \mathbb{R}^n \text{ such that } A \subseteq \bigcup_{i=1}^{M} (x_i + B) \right\}.$$

- Given a convex body K in $\mathbb{R}^n$, we call the quantity

$$N\big(K, \operatorname{int}(K)\big)$$

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- We can check that, for every convex body K, its covering number and its illumination number are equal.

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Theorem (F. W. Levi, 1955)

For every convex body K **in** $\mathbb{R}^2$, $N\big(K, \operatorname{int}(K)\big) = 3$,

unless K is a parallelogram, in which case $N\big(K, \operatorname{int}(K)\big) = 4$.

• In 1957 Hadwiger states as an open problem the analogue of this theorem in higher dimensions.

• In full generality, still open for all $n > 2$.

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In dim 5 and 6: $\mathfrak{I}(K) \leqslant 1002$ and $\mathfrak{I}(K) \leqslant 14140$ respectively (Diao, 2022).

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General upper bounds: Erdős-Rogers (1964) (+Rogers, Fejes Tóth, Rogers-Shephard):

$$\mathfrak{I}(K) = N\big(K, \mathrm{int}(K)\big) \leqslant \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \left(n \ln n + n \ln(\ln n) + n + o(n)\right)$$

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Huang-Slomka-Tkocz-V (2018): $\mathfrak{I}(K) \leqslant O(4^n e^{-c\sqrt{n}})$

Campos-van Hintum-Morris-Tiba (2022): $\mathfrak{I}(K) \leqslant 4^n \exp(-cn/L_K^2)$

Galicer-Singer (2024+): Alternative proof of the latter bound

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- symmetric spiky balls/cap bodies ($n \leqslant 4$ and $n \geqslant 20$); and for all n, if + 1-unconditionality (Bezdek-Ivanov-Strachan)

– Tikhomirov (2017): there is C such that,

$$\text{if } n \geqslant C \text{ and } K \subset \mathbb{R}^n \text{ 1-symmetric,}$$
$$\text{not an affine image of the cube,} \quad \text{then } \mathfrak{I}(K) \leqslant 2^n - 1.$$

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Recall: K 1-unconditional: $x \in K \ \Rightarrow \ (\epsilon_1 x_1, \epsilon_2 x_2, \ldots, \epsilon_n x_n) \in K$, where $\epsilon_i \in \{\pm 1\}$

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- Dekster (2000): if $K \subset \mathbb{R}^3$ symmetric about a plane, then $\mathfrak{I}(K) \leqslant 8$.

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Corollary

$K \subset \mathbb{R}^n$ 1-unconditional. Suppose that:

$$\text{if } x \in \mathrm{ext}(K), \text{ then } x_i \neq 0 \text{ for all } i \in [n]. \qquad (\dagger)$$

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$n \geqslant 3$, $K \subset \mathbb{R}^n$ 1-symmetric and **not** an affine image of the cube.
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Theorem 3

K 1-unconditional in $\mathbb{R}^3$ or $\mathbb{R}^4$. Then $\mathfrak{I}(K) \leqslant 2^n - 2$ (except for affine images of the cube).

Also, all illuminating sets consist of pairs of opposite directions.

Some comments on the tools and methods in Tikhomirov's approach and in ours

A useful ("local-to-global") lemma

- $K \subset \mathbb{R}^n$, H affine subspace of $\mathbb{R}^n$. Suppose $H \cap \operatorname{int}(K) \neq \emptyset$.

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- $\mathbf{B} \subset \mathbb{R}^n$ 1-unconditional (or 1-symmetric), $x \in \partial \mathbf{B}$. Assume that $x_{i_1} \cdot x_{i_2} \cdots x_{i_k} \neq 0$, while $x_{j_1} = x_{j_2} = \cdots = x_{j_{n-k}} = 0$.
  Set $H_{x,0} := \{y \in \mathbb{R}^n : y_{j_1} = y_{j_2} = \cdots = y_{j_{n-k}} = 0\}$, and consider the 1-unconditional convex subset $H_{x,0} \cap \mathbf{B}$.

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  then $x + \varepsilon d \in \mathrm{relint}(H_{x,0} \cap \mathbf{B})$ for some $\varepsilon > 0 \Rightarrow x + \varepsilon d \in \mathrm{int}(\mathbf{B})$.

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  then $x + \varepsilon d \in \mathrm{relint}(H_{x,0} \cap \mathbf{B})$ for some $\varepsilon > 0 \Rightarrow \; x + \varepsilon d \in \mathrm{int}(\mathbf{B})$.

Corollary

$\mathbf{B}$ 1-unconditional or 1-symmetric in $\mathbb{R}^n \quad \Rightarrow$
$\{-1, 0, 1\}^n \setminus \{\vec{0}\}$ illuminating set for $\mathbf{B}$.

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- $\mathbf{B}$ 1-unconditional or 1-symmetric. $\exists$ diagonal matrix D_0 such that $\pm e_i \in \partial D_0 \mathbf{B}$ for all $i \in [n]$.

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Parameters to work with:

Distance to the cube (Tikhomirov)

If $\mathbf{B} \in \mathcal{U}^n$ or $\mathbf{B} \in \mathcal{S}^n$, set

$$\mathrm{dist}(\mathbf{B}, [-1,1]^n) = \min\{\lambda \geqslant 1 : \tfrac{1}{\lambda}[-1,1]^n \subset \mathbf{B}\}.$$

In fact, $\mathrm{dist}(\mathbf{B}, [-1,1]^n) = \|e_1 + e_2 + \cdots + e_n\|_{\mathbf{B}}$.

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Largest unit subcube (Sun-V.)

Let $\mathbf{B} \in \mathcal{S}^n$. We set

$$m_{\mathbf{B}} := \max\{k \in [n] : e_1 + e_2 + \cdots + e_k \in \mathbf{B}\}.$$

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- Tikhomirov shows: if $\mathbf{B} \in \mathcal{S}^n$ and $\mathrm{dist}(\mathbf{B}, [-1,1]^n) \geqslant 2$, then $\mathbf{B}$ is illuminated by a set of the form
$$\left(\{-1,1\}^{n-1} \times \{0\}\right) \cup R_0$$
  where R_0 is any subset of $\{-1,0,1\}^n \setminus \{\vec{0}\}$ with the property:

  for every $k \leqslant \lceil \frac{n}{2} \rceil$, and every $y \in \{-1,0,1\}^n$
  with exactly k non-zero coordinates,
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- He then uses a probabilistic argument to show that, if n is sufficiently large, then we can find a set R_0 with this property which satisfies

$$|R_0| \leqslant \frac{2^n}{n} < 2^{n-1}.$$

Here the only need for the assumption $\mathrm{dist}(\mathbf{B}, [-1,1]^n) \geqslant 2$ is to ensure that $m_{\mathbf{B}} \leqslant \frac{n}{2}$.

Thus we can 'simplify' his dichotomy a bit

- $n \geqslant 2$, $\mathbf{B} \in \mathcal{S}^n$ satisfies $1 < \mathrm{dist}(\mathbf{B}, [-1,1]^n) < 2$ and $\|e_1 + e_2\|_{\mathbf{B}} = 1$. Then $\mathbf{B}$ illuminated by

$$T_1 := \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{\pm 1\}^n : \epsilon_i = -1 \text{ for at least one } i \leqslant n-1\} \cup \{e_1 + e_2 + \cdots + e_{n-1}\}.$$

$\rightsquigarrow \mathfrak{I}(\mathbf{B}) \leqslant 2^n - 1$

- $\mathbf{B} \in \mathcal{S}^n$ satisfies $m_{\mathbf{B}} \leqslant \frac{n}{2}$. Then $\mathbf{B}$ illuminated by

$$T_2 = \big(\{-1,1\}^{n-1} \times \{0\}\big) \cup R_0$$

where R_0 a subset of $\{-1,0,1\}^n \setminus \{\vec{0}\}$ with the property stated before:

$\rightsquigarrow \mathfrak{I}(\mathbf{B}) \leqslant 2^{n-1} + \dfrac{2^n}{n}$ for n sufficiently large.

'Obstacles' in low dimensions

'Tricky' convex body 1

Let $n \geqslant 3$, and consider the convex body $\mathbf{B}_1 \in \mathcal{S}^n$ whose vertices are all the coordinate reflections and permutations of e_1 and of $\frac{1}{2}(e_1 + e_2 + \cdots + e_n)$.

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'Tricky' convex body 2

Let $n \geqslant 9$, and consider the convex body $\mathbf{B}_2 \in \mathcal{S}^n$ whose vertices are all coordinate reflections and permutations of e_1 and of $\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$.

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Note: can show that, given any fixed $k \geqslant 1$, we can construct sufficiently high-dimensional convex bodies in $\mathcal{S}^n$ which cannot be illuminated by any set of the form

$$\left(\{-1,1\}^{n-1} \times \{0\}\right) \cup R_k$$

where R_k will contain all $d \in \{-1,0,1\}^n \setminus \{\vec{0}\}$ with support size at most k.

An alternative method

Let $n \geq 2$, $\delta \in (0,1)$. Consider the set

$$G^n(\delta) := \left\{ \epsilon_j e_j + \sum_{i \in [n] \setminus \{j\}} \epsilon_i \delta e_i : \ j \in [n], \epsilon_i \in \{\pm 1\} \text{ for all } i \in [n] \right\}.$$

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Definition: *Deep Illumination*

Let $x \in \mathbb{R}^n \setminus \{\vec{0}\}$. We say that $d \in G^n(\delta)$ *deep illuminates* x if

(i) whenever $x_i \neq 0$, we have $\operatorname{sign}(d_i) = -\operatorname{sign}(x_i)$, AND

(ii) the maximum (in absolute value) coordinate d_{i_0} of d occurs at an index $i_0 \in [n]$ for which $x_{i_0} \neq 0$.

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Note that, a priori, this is more of a 'combinatorial' property, rather than geometric.

For 1-symmetric convex bodies: if it deep illuminates, then it illuminates.

Lemma 1 (Sun-V.)

$n \geqslant 2$, $\mathbf{B} \in \mathcal{S}^n$, $x \in \partial\mathbf{B}$. Fix some positive $\delta < 1/n$. Then, if $d \in G^n(\delta)$ deep illuminates x, we will have

$$x + \varepsilon d \in \mathrm{int}(\mathbf{B})$$

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Also, a useful strengthening of this is the following:

Lemma 2 (Sun-V.)

$n \geqslant 2$, $\mathbf{B} \in \mathcal{S}^n$, $x \in \partial\mathbf{B}$. Write $M_x := \{k \in [n] : |x_k| = \|x\|_\infty\}$. Fix some positive $\delta < 1/n$. If $d \in G^n(\delta)$ deep illuminates the projection $\mathrm{P}_{M_x}(x)$ of x (proj. onto the coord. subspace $[e_k : k \in M_x]$), then

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for some $\varepsilon > 0$.

How to use these?

The above say that, if $\mathbf{B} \in \mathcal{S}^n$, and if S is any subset of $G^n(\delta)$ (with $\delta < 1/n$) which deep illuminates every non-zero vector of $\mathbb{R}^n$, then S is an illuminating set for $\mathbf{B}$.

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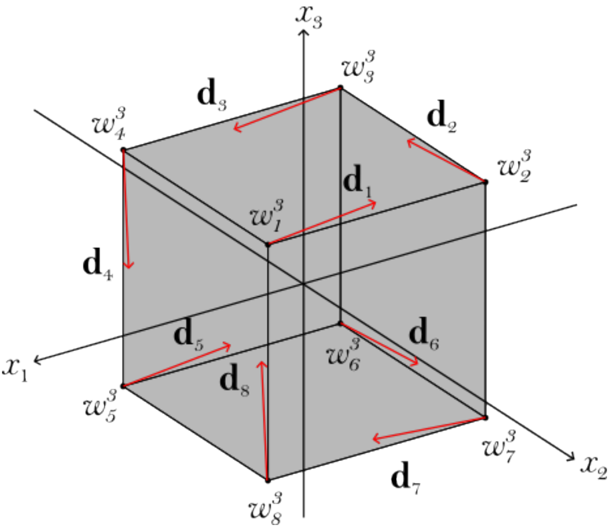
Combine this with the following existence result:

Theorem A (Sun-V.)

For all $n \geqslant 2$, there exists a subset $\mathcal{I}^n(\delta)$ of $G^n(\delta)$ with $|\mathcal{I}^n(\delta)| = 2^n$ which deep illuminates all non-zero vectors of $\mathbb{R}^n$.

Thus $\mathcal{I}^n(\delta)$ illuminates all $\mathbf{B} \in \mathcal{S}^n$ (as long as $\delta < \frac{1}{n}$).

$\rightsquigarrow$ We get a **common illuminating set of the 'right' size**, but, attention, we haven't treated equality cases yet.


x_3
w_3^3
$\mathbf{d}_3$
$\mathbf{d}_2$
w_4^3
w_2^3
$\mathbf{d}_1$
w_1^3
$\mathbf{d}_4$
$\mathbf{d}_5$
$\mathbf{d}_6$
w_6^3
x_1
$\mathbf{d}_8$
w_5^3
w_7^3
x_2
$\mathbf{d}_7$
w_8^3

– Set $$\begin{aligned}\mathcal{I}^2(\delta) &= \{(+1,+\delta),(-1,-\delta),(+\delta,-1),(-\delta,+1)\}\\ &= \{\pm(+1,+\delta),\ \pm(+\delta,-1)\}.\end{aligned}$$

Combinatorial construction

– Set
$$\mathcal{I}^2(\delta) = \{(+1, +\delta), (-1, -\delta), (+\delta, -1), (-\delta, +1)\}$$
$$= \{\pm(+1, +\delta),\ \pm(+\delta, -1)\}.$$

– To construct $\mathcal{I}^3(\delta)$:

$$((+1, +\delta), +\delta),\ ((-1, -\delta), -\delta),\ ((+\delta, -1), -\delta),\ ((-\delta, +1), +\delta)$$

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Equality cases for 1-symmetric convex bodies

Lemma 3 (Sun-V.)

Let $n \geqslant 3$, $\delta \in (0,1)$, and consider the combinatorially constructed $\mathcal{I}^n(\delta)$. Set

$$\mathcal{I}^n_{-2}(\delta) := \mathcal{I}^n(\delta) \setminus \left\{\pm(+\delta, +\delta, \ldots, +\delta, +\delta, -\delta, +1)\right\}.$$

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Theorem B (Sun-V.)

$n \geqslant 3$, $\mathbf{B} \in \mathcal{S}^n$ and suppose that $\mathrm{dist}(\mathbf{B}, [-1,+1]^n) > 1$ (equivalently, $m_{\mathbf{B}} < n$). Then we can find some sufficiently small $\eta = \eta_{\mathbf{B}} > 0$ so that

$$\left[\mathcal{I}^n_{-2}(\tfrac{1}{n+1}) \setminus \left\{\pm(+1, +\tfrac{1}{n+1}, +\tfrac{1}{n+1}, \ldots, +\tfrac{1}{n+1}, +\tfrac{1}{n+1}, +\tfrac{1}{n+1})\right\}\right]$$
$$\bigcup \left\{\pm(+1, +\tfrac{1}{n+1}, +\tfrac{1}{n+1}, \ldots, +\tfrac{1}{n+1}, \eta, +\tfrac{1}{n+1})\right\}$$

illuminates $\mathbf{B}$.
$\rightsquigarrow \mathfrak{I}(\mathbf{B}) \leqslant 2^n - 2$

For ‘thicker’ 1-symmetric convex bodies, or for ‘thick’ 1-unconditional convex bodies we can do a bit better

Theorem C (Sun-V.)

$n \geqslant 3$, and let $\mathbf{B} \in \mathcal{U}^n$, $\mathbf{B} \neq [-1,1]^n$, such that

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Then we can find sufficiently small $\delta = \delta_{\mathbf{B}} > 0$ such that

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illuminates $\mathbf{B}$ (and thus $\mathfrak{I}(\mathbf{B}) \leqslant 2^n - 2$).

This is used to prove...

Theorem 2

$n \geqslant 3$, $K \subset \mathbb{R}^n$ 1-unconditional, **not** an affine image of the cube, having Property (†): that is,

$$\text{if } x \in \mathrm{ext}(K), \text{ then } x_i \neq 0 \text{ for all } i \in [n]. \tag{†}$$

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Cases where the inductive hypothesis cannot "kick in": can show that there is a 'maximal' unit subcube *(can even reduce to the case where we have all 'maximal' unit subcubes, that is, to the assumptions in Thm C)*.

Thank you for your attention!