On optimal approximation of functions by log-polynomials

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(joint with D. Alonso-Gutiérrez and R. Villa)

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John-Löwner's ellipsoid (Löwner s.XX, John 1948)

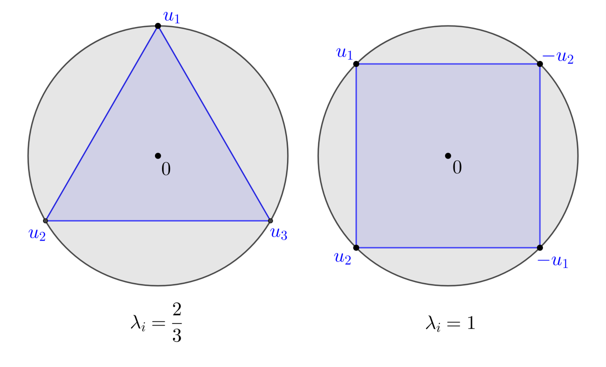
$K \in \mathcal{K}^n$, exists a unique ellipsoid $\mathcal{E}$ with $K \subset \mathcal{E}$ and $|\mathcal{E}|$ is minimized among all ellipsoids containing K.

Touching conditions (John 1948, Ball 1992)

$K \in \mathcal{K}^n$, if $K \subset B_n$, are equivalent:

- B_n is the Löwner ellipsoid of K.
- There exist $u_i \in \partial K \cap \partial B_n$ and $\lambda_i > 0$ such that

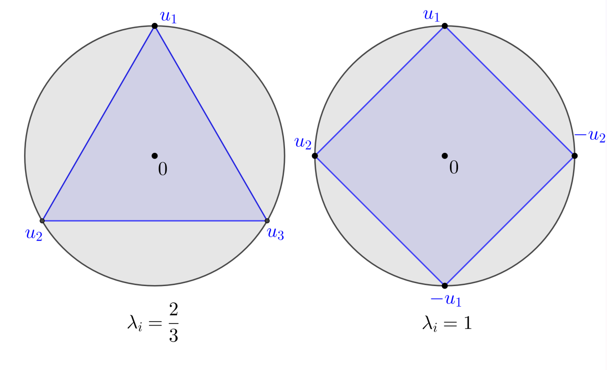
$$\sum_{i=1}^{m} \lambda_i u_i u_i^T = \mathrm{I}_n \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i u_i = 0.$$



Volume ratio (Ball 1992, Barthe 1998)

$K \in \mathcal{K}^n$ (resp. $K \in \mathcal{K}_0^n$) and $\mathcal{E}_K$ its Löwner ellipsoid. Then

$$\frac{|\mathcal{E}_K|}{|K|} \le \frac{|\mathcal{E}_{S_n}|}{|S_n|} \quad \left(\text{resp. } \frac{|\mathcal{E}_K|}{|K|} \le \frac{|\mathcal{E}_{C_n^*}|}{|C_n^*|}\right)$$



Authors refining aspects of John-Löwner ellipsoid:

David Alonso-Gutiérrez, Keith Ball, Alexander Barbinok, Karoly Böröczky, Niufa Fang, Apostolos Giannopoulos, Martin Grötschel, Peter Gruber, Martin Henk, Han Huang, Grigory Ivanov, Hugo Jiménez, Fritz John, Boaz Klartag, Yehoram Gordon, Gangsong Leng, Ben Li, Alexander Litvak, László Lovász, Karel Löwner, Erwin Lutwak, Songjun Lv, Mathieu Meyer, Vitali Milman, Marton Naszódi, Alain Pajor, Michael Schmuckenschläger, Alexander Schrijver, Franz Schuster, Carsten Schütt, Igor Tsiutsiurupa, Rafael Villa, Elisabeth M. Werner, Donghua Wu, Ge Xiong, Deane Yang, Wuyang Yu, Gaoyong Zhang, Jiazu Zhou, Du Zou . . .

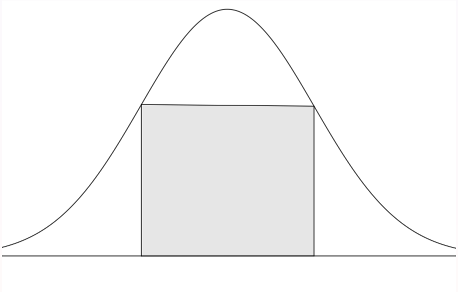
First extension

Ellipsoids of log-conc. functions (Alonso, G.M., Jiménez, Villa '18)

$f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = 1$, $\exists$ unique $t \in (0,1]$ and $\mathcal{E}$ s.t.
$t\chi_{\mathcal{E}}(x) \leq f(x)$ and

$$\int_{\mathbb{R}^n} t\chi_{\mathcal{E}}(x)dx = t|\mathcal{E}|$$

is the maximum among all ellipsoids below f.



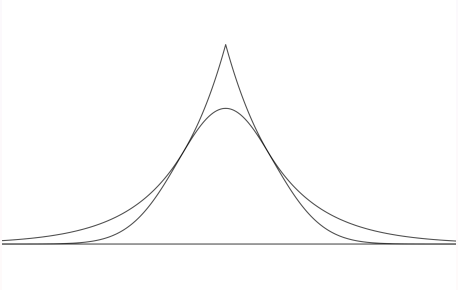
Polarity: $f = e^{-u}$ then $f^\circ = e^{-u^*}$, u^* Legendre transform.

Ellipsoids of log-conc. functions II (Li, Shütt, Werner 2019)

$f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = 1$, $\exists$ unique $s \in [1, +\infty)$ and $\mathcal{E}$ s.t. $f(x) \leq s e^{-\|x\|_\mathcal{E}}$ and

$$\int_{\mathbb{R}^n} s e^{-\|x\|_\mathcal{E}} dx = s n! |\mathcal{E}|$$

is the minimum among all ellipsoids above f.



Authors extending geometric results to functional settings:

David Alonso-Gutiérrez, Leticia Alves, Shiri Artstein-Avidan, Julio Bernués, Herm Brascamp, Umut Caglar, Andrea Colesanti, Niufa Fang, Dan Florentin, Matthieu Fradelizi, Ilaria Fragalá, Julián Haddad, María Hernández Cifre, Grigory Ivanov, Hugo Jiménez, Boaz Klartag, Elliott Lieb, Joram Lindenstrauss, Erwin Lutwak, Mathieu Meyer, Vitali Milman, Marcos Montenegro, Liran Rotem, Michael Roysdon, Carsten Schütt, Alexander Segal, Igor Tsiutsiurupa, Rafael Villa, Ellisabeth Werner, Deane Yang, Jesús Yepes Nicolás, Gaoyong Zhang, Jiazu Zhou, Artem Zvavitch...

Second extension

- d-homogeneous polynomials $\mathbb{H}_d(\mathbb{R}^n)$:

$$g(x) = \sum_{\alpha \in \mathbb{N}^n_d} g_\alpha x^\alpha = \sum_{\alpha \in \mathbb{N}^n_d} g_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \sum_{i=1}^n \alpha_i = d.$$

d-Löwner-Lasserre polynomial (Lasserre 2015)

$K \in \mathcal{K}^n$, $\exists$ unique $g_d \in \mathbb{H}_d(\mathbb{R}^n)$ s.t.

$$K \subset G_1(g_d) = \{x \in \mathbb{R}^n : g_d(x) \leq 1\}$$

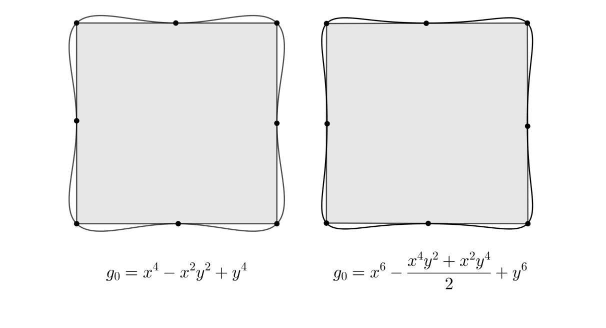
which minimizes $|G_1(g)|$ among all such g.

Touching conditions (Lasserre 2015)

$K \in \mathcal{K}^n$, if $g_d \in \mathbb{H}_d(\mathbb{R}^n)$ with $K \subset G_1(g_d)$, are equivalent:

- g_d is the d-Löwner-Lasserre polynomial of K.
- There exist $y_i \in K \cap \partial G_1(g_d)$ and $\lambda_i > 0$ such that

$$\int_{\mathbb{R}^n} x^\alpha e^{-g_d(x)} dx = \sum_{i=1}^{s} \lambda_i y_i^\alpha, \quad \text{for every } \alpha \in \mathbb{N}_d^n.$$

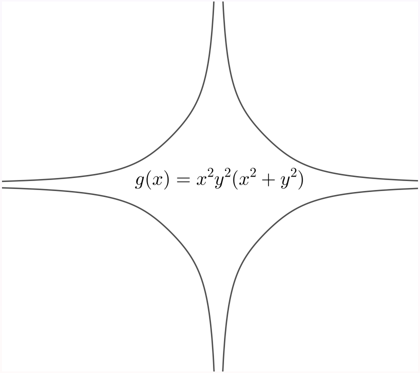


Level sets of polynomials?

$$|G_1(g)| < +\infty!$$

(1) d odd implies $|G_1(g)| = +\infty$. Thus d only even!

(2) $G_1(g)$ can be non-convex and unbounded!



Topological structure of $\mathbb{F}_d(\mathbb{R}^n)$

- $\mathbb{F}_d(\mathbb{R}^n)$: d-homogeneous polynomial with $|G_1(g)| < +\infty$

Lemma 4

(i) $\mathbb{F}_d(\mathbb{R}^n)$ convex cone, not closed, non empty interior.

- $g_0(x) = \sum_{i=1}^n x_i^d \in \mathrm{int}\mathbb{F}_d(\mathbb{R}^n)$
- $tg_0 \in \mathbb{F}_d(\mathbb{R}^n)$, $t > 0$, but $0 \notin \mathbb{F}_d(\mathbb{R}^n)$

Topological structure of $\mathbb{F}_d(\mathbb{R}^n)$

Lemma 4

(ii) $g \in \mathbb{F}_2(\mathbb{R}^n) \iff G_1(g)$ bounded (ellipsoid). $\mathbb{F}_2(\mathbb{R}^n)$ is open.

- Sylvester Law Inertia: $g(x) = \sum_{i=1}^n \alpha_i x_i^2$.
  $g \in \mathbb{F}_2(\mathbb{R}^n)$ iff $\alpha_i > 0$.

Lemma 4

(iii) $g \in \mathbb{F}_4(\mathbb{R}^2) \iff G_1(g)$ bounded. $\mathbb{F}_4(\mathbb{R}^2)$ is open.

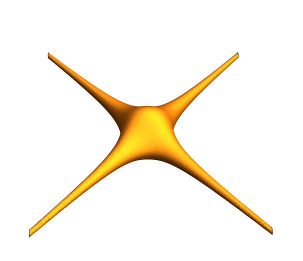
- Salmon 1859: $g(x, y) = ax^4 + 2bx^2y^2 + cy^4$.
  $g \in \mathbb{F}_4(\mathbb{R}^2)$ iff $a, c > 0$ and $b > -\sqrt{ac}$.

Topological structure of $\mathbb{F}_d(\mathbb{R}^n)$

Lemma 4

(iv) There exists $g \in \mathbb{F}_4(\mathbb{R}^n)$, $n \geq 3$, s.t. $G_1(g)$ is unbounded. $\mathbb{F}_4(\mathbb{R}^n)$ is not open.

- $g(x) = x^4 + y^4 + z^4 - 2\sqrt{2}x^2yz$ has $G_1(g)$ unbounded with $|G_1(g)| < +\infty$.
- $h(x) = g(x) + \sum_{i=4}^n x_i^4$ has $G_1(h)$ unbounded with $|G_1(h)| < +\infty$.
- $\sum_{i=1}^n x_i^4 - tx^2yz \notin \mathbb{F}_4(\mathbb{R}^n)$, $t > 2\sqrt{2}$.

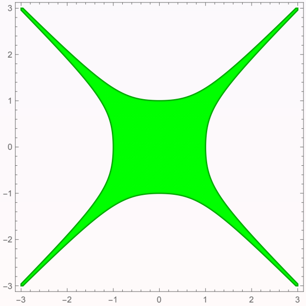


Topological structure of $\mathbb{F}_d(\mathbb{R}^n)$

Lemma 4

(v) $d \geq 6$, $n \geq 2$, there exists $g \in \mathbb{F}_d(\mathbb{R}^n)$ s.t. $G_1(g)$ is unbounded. $\mathbb{F}_d(\mathbb{R}^n)$ is not open.

- $(x^2 - y^2)^2(x^2 + y^2) \in \mathbb{F}_6(\mathbb{R}^2)$ and $G_1(g)$ is unbounded.
- Same with $g(x, y) = (x^2 - y^2)^2(x^{d-4} + y^{d-4})$.
- Same conclusion adding x_i^d, $i \geq 3$.
- $g - (1 - t)x_1^d \notin \mathbb{F}_d(\mathbb{R}^n)$, $t < 1$.



Problem 1

Given $f : \mathbb{R}^n \to \mathbb{R}_+$ with $\|f\|_\infty = f(0) = 1$ and d even,

$$f(x) \leq te^{-g(x)^{\frac{1}{d}}}$$

for $t \geq 1$ and $g \in \mathbb{H}_d(\mathbb{R}^n)$. Find (t_1, g_1) minimizing

$$\int_{\mathbb{R}^n} te^{-g(x)^{\frac{1}{d}}} dx = tn!|G_1(g)|.$$

The good news

Convexity of functional

$W : \mathbb{R}_+ \times \mathbb{F}_d(\mathbb{R}^n) \to \mathbb{R}_+$ given by $W(r,g) = e^r |G_1(g)|$ is log-convex and strictly convex.

Proof: First, notice that

$$|G_1(g)| = \frac{1}{n!} \int e^{-g(x)^{\frac{1}{d}}} dx = \frac{1}{\Gamma(\frac{n}{d}+1)} \int e^{-g(x)} dx.$$

Let (r_i, g_i), $i = 1, 2$. By Hölder inequality

$$\int e^{-((1-\theta)g_1(x)+\theta g_2(x))} \leq \left(\int e^{-g_1(x)} \right)^{1-\theta} \left(\int e^{-g_2(x)} \right)^{\theta}$$

and thus

$$W((1-\theta)(r_1,g_1) + \theta(r_2,g_2)) \leq W(r_1,g_1)^{1-\theta} W(r_2,g_2)^{\theta},$$

i.e. W is log-convex.

By AG-mean inequality

$$W((1-\theta)(r_1, g_1) + \theta(r_2, g_2)) \leq (1-\theta)W(r_1, g_1) + \theta W(r_2, g_2),$$

thus W is convex.

Strict convexity follows from equality cases of Hölder and AG. $\square$

The other news

Example 1: Non-convexity of domain

Let

$$f(x) = \chi_{B_n}(x) \quad \text{and} \quad g_i(x) = r_i^d |x|^d,\ r_0, r_1 > 0.$$

Then

$$f(x) \le e^{r_i - g_i(x)^{\frac{1}{d}}},\ i = 0, 1,$$

but

$$f(x) \not\le e^{r_\theta - g_\theta(x)^{\frac{1}{d}}}$$

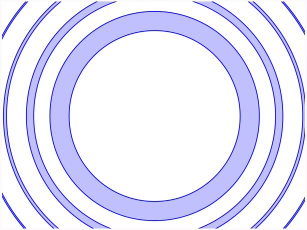
for every $\theta \in (0,1)$, $r_\theta = (1-\theta) r_1 + \theta r_2$ and $g_\theta = (1-\theta) g_1 + \theta g_2$.

The other news

Example 2: Non negative integrable is not enough

$$f = \chi_A \quad \text{where} \quad A = \bigcup_{k=1}^{+\infty} \{x \in \mathbb{R}^n : k \leq |x| \leq k + \frac{1}{2^k}\}.$$

Then $\int f < +\infty$ but $f(x) \leq t \exp(-g(x)^{\frac{1}{d}})$ is equivalent to $g(x) \leq (\log t)^d$, thus $g \equiv 0$ and $|G_1(g)| = +\infty$!



- Log-concave integrable functions $\mathcal{F}(\mathbb{R}^n)$

Theorem 1

$f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = f(0) = 1$, d even, there exists $(t_1, g_1) \in [1, +\infty) \times \mathbb{F}_d(\mathbb{R}^n)$ minimizing Problem 1.

Some lemmas

Lemma 1

$$f(x) \leq te^{-g(x)^{\frac{1}{d}}} \quad \Longleftrightarrow \quad H_t(f) \subset G_1(g),$$

where

$$H_t(f) = \bigcup_{\lambda \in (0,1)} \log(t/\lambda)^{-1} K_\lambda(f).$$

Lemma 2

$$\log t_0 H_{t_0}(f) \subset \log t_1 H_{t_1}(f) \text{ for every } 1 < t_0 < t_1.$$

Lemma 3

$H_t(f)$ bounded $t > 1$, $H_1(f)$ unbounded $\Longleftrightarrow |H_1(f)| = +\infty$.

Reformulation Problem 1

Given $f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = f(0) = 1$, find $t_1 \geq 1$ and $g_1 \in \mathbb{F}_d(\mathbb{R}^n)$ s.t.

$$H_{t_1}(f) \subset G_1(g_1)$$

and

$$t_1 |G_1(g_1)| = \inf_{t \geq 1} \left(t \inf_{H_t(f) \subset G_1(g)} |G_1(g)| \right) = \inf_{t \geq 1} t \cdot v(t) = \inf_{t \geq 1} \phi(t).$$

By Lemma 3 $H_t(f)$ bounded, by Lasserre's theorem infimum is minimum!

Proving Theorem 1

Lemma 4

$t_0, t_1, d \geq 1$, $\theta \in [0,1]$, $a \in (0,1]$. Then

$$(1-\theta)\left(\log \frac{t_0}{a}\right)^d + \theta\left(\log \frac{t_1}{a}\right)^d \leq \left(\log \frac{t_\theta}{a}\right)^d,$$

where

$$(\log t_\theta)^d = (1-\theta)(\log t_0)^d + \theta(\log t_1)^d.$$

Proving Theorem 1

Lemma 5

$f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = f(0) = 1$. Then

1. $v(t)$ is decreasing.
2. $(\log t)^n v(t)$ is increasing.
3. $t_0, t_1 > 1$, $\theta \in [0,1]$, then

$$v(t_\theta) \leq v(t_0)^{1-\theta} v(t_1)^\theta,$$

   where $(\log t_\theta)^d = (1-\theta)(\log t_0)^d + \theta(\log t_1)^d$.

Proving Theorem 1

Proof of Lemma 5:

1. $H_t(f)$ decreasing.
2. $(log\ t)H_t(f)$ increasing (Lemma 2).
3. $f(x) \le t_i e^{-g_{t_i}(x)^{\frac{1}{d}}}$ rewrites $g_{t_i}(x) \le \left(\log \frac{t_i}{f(x)}\right)^d$. Thus

$$(1-\theta)g_{t_0}(x) + \theta g_{t_1}(x) \le (1-\theta)\left(\log \frac{t_0}{f(x)}\right)^d + \theta \left(\log \frac{t_1}{f(x)}\right)^d$$
$$\le \left(\log \frac{t_\theta}{f(x)}\right)^d,$$

(cf. Lemma 4). By Lemma 1 $H_{t_\theta}(f) \subset G_1((1-\theta)g_{t_0} + \theta g_{t_1})$. Thus $|v(t_\theta)| \le |G_1((1-\theta)g_{t_0} + \theta g_{t_1})|$.

Proving Theorem 1

$$\begin{aligned}
|v(t_\theta)| &\leq |G_1((1-\theta)g_{t_0} + \theta g_{t_1})| \\
&= \frac{1}{\Gamma(\frac{n}{d}+1)} \int e^{-((1-\theta)g_{t_0}(x)+\theta g_{t_1}(x))} dx \\
&\leq \left(\frac{1}{\Gamma(\frac{n}{d}+1)} \int e^{-g_{t_0}(x)} dx \right)^{1-\theta} \left(\frac{1}{\Gamma(\frac{n}{d}+1)} \int e^{-g_{t_1}(x)} dx \right)^{\theta} \\
&= |v(g_0)|^{1-\theta} |v(g_1)|^{\theta}. \qquad \square
\end{aligned}$$

Proving Theorem 1

Proof of Theorem 1: Lemma 5, $s \to \log v(e^{s^{1/d}})$ is convex $t > 1$.

By Monotone Convergence Theorem $\lim_{t\to 1^+} |H_t(f)| = |H_1(f)|$.

(1) If $H_1(f)$ is bounded, using the characterization by touching points of Lasserre on each $H_t(f) \subset G_1(g_t)$, proving uniform boundedness of coefficients of g_t, we get a limit $g_0 = \lim_{t\to 1^+} g_t$. Thus $|G_1(g_0)| = \lim_{t\to 1^+} |G_1(g_t)|$.

(2) If $H_1(f)$ is unbounded, by MCT $\lim_{t\to 1^+} |H_t(f)| = +\infty$.

In both (1), (2), we get $\lim_{t\to 1^+} v(t) = v(1)$.

Proving Theorem 1

Lemma 5

$$\phi(t) = t \cdot v(t) = \frac{t}{(\log t)^n}(\log t)^n v(t)$$

product of two increasing functions in $[e^n, +\infty)$.

If $H_1(f)$ bounded, $\phi(t)$ attains its minimum by continuity in $[1, e^n]$.

If $H_1(f)$ unbounded, $\phi(t)$ attains it in $(1, e^n]$. $\square$

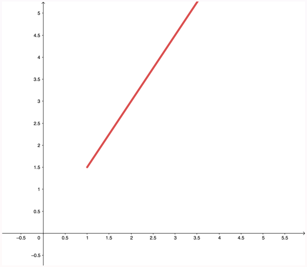
$t \cdot v(t)$?

Example 3

$$f(x) = e^{-\|x\|_K}$$

then $K_\lambda(f) = (\log(1/\lambda))K$, $H_t(f) = \mathrm{int} K$, $H_1(f) = K$, $g_t = g_1$, and thus

$$t \cdot v(t) = t|G_1(g_1)|, \quad \text{i.e.} \quad \min_{t \geq 1} \phi(t) = \phi(1).$$



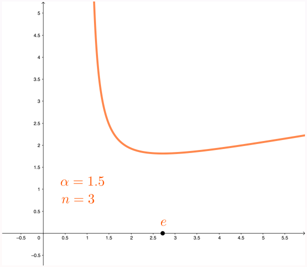
$t \cdot v(t)$?

Example 4

$$f(x) = e^{-\|x\|_K^\alpha}, \quad \alpha > 1$$

then $K_\lambda(f) = (\log(1/\lambda))^{\frac{1}{\alpha}} K$, $H_t(f) = \alpha^{-\frac{1}{\alpha}} (\alpha' \log t)^{-\frac{1}{\alpha'}} K$, $H_1(f) = \mathbb{R}^n$, $g_t = \alpha^{\frac{d}{\alpha}} (\alpha' \log t)^{\frac{d}{\alpha'}} g$, thus

$$t \cdot v(t) = t\alpha^{-\frac{n}{\alpha}} (\alpha' \log t)^{-\frac{n}{\alpha'}} |G_1(g)| \quad \text{i.e.} \quad \min_{t \geq 1} \phi(t) = \phi(e^{\frac{n}{\alpha'}}).$$



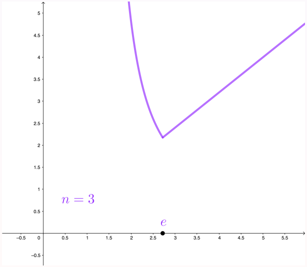
$t \cdot v(t)$?

Example 5

$$f(x) = \begin{cases} 1 & x \in K \\ e^{1-\|x\|_K} & \text{otherwise} \end{cases}$$

$$K_\lambda(f) = (1 + \log(1/\lambda))K, \quad H_t(f) = \begin{cases} \frac{1}{\log t} K & 1 \le t \le e \\ \operatorname{int} K & \text{otherwise,} \end{cases}$$

$$\phi(t) = \begin{cases} \frac{t}{(\log t)^n} |G_1(g)| & 1 \le t \le e \\ t|G_1(g)| & \text{otherwise,} \end{cases} \quad \text{i.e.} \quad \min_{t \ge 1} \phi(t) = \phi(e).$$



The other (late) news

Example 6: Quasi-concave but not log-concave?

Let $K \in \mathcal{K}^n$, $0 \in K$, $\alpha > n$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in K, \\ \|x\|_K^{-\alpha} & \text{otherwise.} \end{cases}$$

Then

$$\int f = \frac{\alpha}{\alpha - n}|K| < +\infty, \quad K_\lambda(f) = \lambda^{-\frac{1}{\alpha}} K,$$

and

$$H_t(f) = \bigcup_{\lambda \in (0,1)} \frac{1}{\lambda^{\frac{1}{\alpha}} \log(t/\lambda)} K = \mathbb{R}^n.$$

Problem 2

Given $f : \mathbb{R}^n \to \mathbb{R}_+$ with $\|f\|_\infty = f(0) = 1$ and d even,

$$f(x) \leq te^{-g(x)}$$

for $t \geq 1$ and $g \in \mathbb{H}_d(\mathbb{R}^n)$. Find (t_2, g_2) minimizing

$$\int_{\mathbb{R}^n} te^{-g(x)} dx = t\Gamma(\frac{n}{d} + 1)|G_1(g)|.$$

The good news II

The domain, i.e. $(r, g) \in \mathbb{R}_+ \times \mathbb{H}_d(\mathbb{R}^n)$ s.t.

$$f(x) \leq e^r e^{-g(x)}$$

is convex. In fact, if $f(x) \leq e^{r_i} e^{-g_i(x)}$, $i = 0, 1$, then

$$\begin{aligned} f(x) &\leq \left(e^{r_0} e^{-g_0(x)}\right)^{1-\theta} \left(e^{r_1} e^{-g_1(x)}\right)^{\theta} \\ &= e^{(1-\theta)r_0+\theta r_1} e^{-((1-\theta)g_0(x)+\theta g_1(x))}. \end{aligned}$$

The other news II

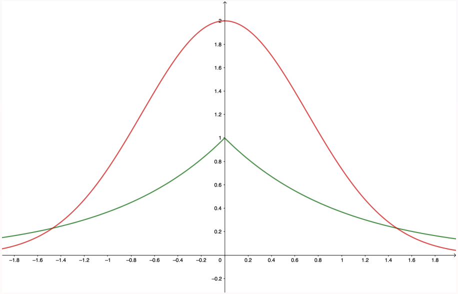
Example 7

$$f(x) = e^{-\|x\|_1},$$

$\forall g \in \mathbb{H}_d(\mathbb{R}^n), t \geq 1$

$$f(x) = e^{-(|x_1|+\cdots+|x_n|)} \not\leq t \cdot e^{-g(x)}$$

if $|x| \gg 0$!



Theorem 2

$f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = f(0) = 1$, d even, with

$$f(x) \leq te^{-g(x)}$$

and

$$\widehat{H}_1(f) = \bigcup_{\lambda \in (0,1)} \log(1/\lambda)^{-1/d} K_\lambda(f)$$

is bounded, there exists a unique $(t_2, g_2) \in [1, +\infty) \times \mathbb{F}_d(\mathbb{R}^n)$ minimizing Problem 2.

Some lemmas II

Lemma 6

$$f(x) \le te^{-g(x)} \quad \Longleftrightarrow \quad \widehat{H}_t(f) \subset G_1(g),$$

where

$$\widehat{H}_t(f) = \bigcup_{\lambda \in (0,1)} \log(t/\lambda)^{-1/d} K_\lambda(f).$$

Lemma 7

$$(\log t_0)^{1/d} \widehat{H}_{t_0}(f) \subset (\log t_1)^{1/d} \widehat{H}_{t_1}(f) \text{ for every } 1 < t_0 < t_1.$$

Lemma 8

Minimize $\widehat{\phi}(t) = t\widehat{v}(t) = t|G_1(\widehat{g}_t)|$.

- $\widehat{v}(t)$ decreasing and $(\log t)^{\frac{n}{d}} \widehat{v}(t)$ increasing.
- $r \to \widehat{\phi}(e^r)$ is log-convex and $\min_{t \ge 1} \widehat{\phi}(t) = \widehat{\phi}(t_0) = \widehat{\phi}(t_1)$ then $t_0 = t_1$.

Theorem 3

$f : \mathbb{R}^n \to \mathbb{R}_+$ bounded, $\|f\|_\infty = f(0) = 1$. Let $(t_2, g_2) \in (1, +\infty) \times \mathrm{int}(\mathbb{F}_d(\mathbb{R}^n))$ with $f(x) \leq t_2 e^{-g_2(x)}$.
They are equivalent:

1. (t_2, g_2) is the only solution to Problem 2.
2. $\exists\, x_1, \ldots, x_m \in \mathbb{R}^n$, $m \leq \binom{n+d-1}{d} + 1$, $f(x_i) = t_2 e^{-g_2(x_i)}$, and $\lambda_i > 0$, $i = 1, \ldots, m$, s.t.

$$t_2 \int_{\mathbb{R}^n} e^{-g_2(x)} dx = \sum_{i=1}^m \lambda_i, \text{ and}$$

$$t_2 \int_{\mathbb{R}^n} x^\alpha e^{-g_2(x)} dx = \sum_{i=1}^m \lambda_i x_i^\alpha, \text{ for all } \alpha \in \mathbb{N}_d^n.$$

$d = 2$ similar results by Ivanov and Tsiutsiurupa

Proving Theorem 3

Proof of Theorem 3: Problem 2 rewrites as

$$\min_{(r,g)\in C} e^r \int_{\mathbb{R}^n} e^{-g(x)} dx$$

s.t.

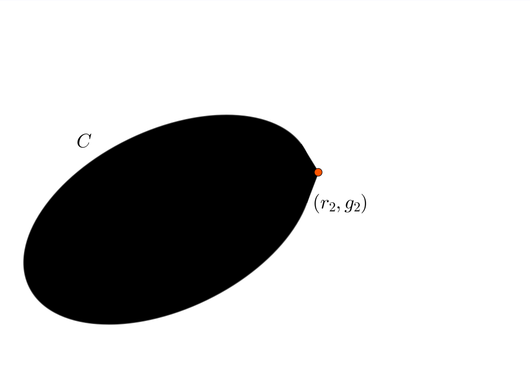
$$\begin{aligned} C &= \{(r,g) : f(x) \le e^r e^{-g(x)} \ \forall x \in S_f\} \\ &= \{(r,g) : r - g(x) \ge \log f(x) \ \forall x \in S_f\}, \end{aligned}$$

with $S_f = \{x \in \mathbb{R}^n : f(x) \neq 0\}$.

$$\begin{aligned} &= \{(r,g) : r - \sum_{\alpha\in\mathbb{N}^n_d} g_\alpha x^\alpha \ge \log f(x) \ \forall x \in S_f\} \\ &= \{(r,(g_\alpha)_\alpha) : \langle (r,(g_\alpha)_\alpha), (1, -(x^\alpha)_\alpha)\rangle \ge \log f(x) \ \forall x \in S_f\}. \end{aligned}$$

Proving Theorem 3

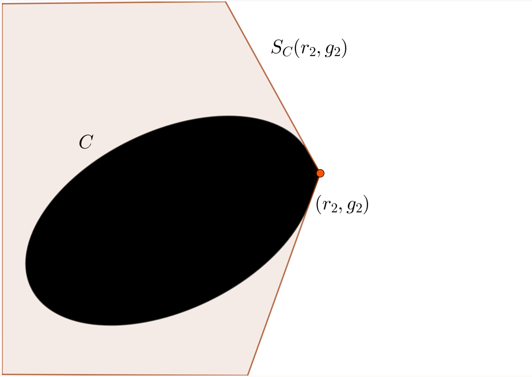
Notice that $(r_2, g_2) \in \partial C$ (attains minimum over convex func.).



Proving Theorem 3

$$S_C(r_2, g_2) = \{(r, (g_\alpha)_\alpha) : \langle (r, (g_\alpha)_\alpha), (1, -(x^\alpha)_\alpha)\rangle \geq \log f(x)\ \forall x \in S_f^*\},$$

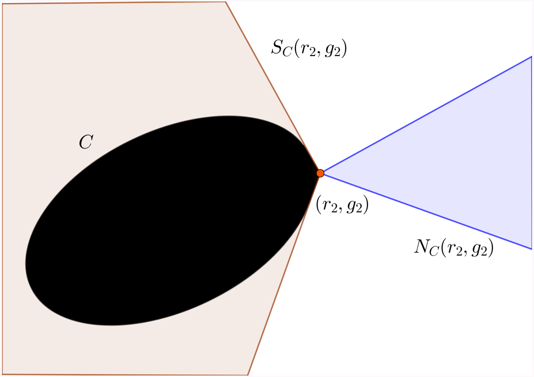
where $S_f^* = \{x \in S_f : r_2 - g_2(x) = \log f(x)\}$.



Proving Theorem 3

Thus

$$N_C(r_2, g_2) = \mathrm{pos}\{(-1, (x^\alpha)_\alpha) : x \in S_f^*\}.$$



Proving Theorem 3

$W(r,g) = e^r \int e^{-g(x)}dx$ differentiable, strictly convex, C is convex, by Karush-Kuhn-Tucker on minimum (r_2, g_2)

$$-\nabla W(r_2, g_2) \in N_C(r_2, g_2) = \mathrm{pos}\{(-1, (x^\alpha)_\alpha) : x \in S_f^*\}.$$

$$\text{Since } \nabla W(r,g) = \left(W(r,g), \left(-e^r \int x^\alpha e^{-g(x)} dx \right)_\alpha \right)$$

by Carathéodory Theorem applied to convex cone $N_C(r_2, g_2) \subset \mathbb{R}^{h_d(n)}$ tells $\exists\ x_1, \ldots, x_m$, $m \leq \binom{n+d-1}{d} + 1 = h_d(n)$, with $r_2 - g_2(x_i) = \log f(x_i)$ and $\lambda_i > 0$, $i = 1, \ldots, m$, s.t.

$$-\left(W(r_2, g_2), \left(-e^{r_2} \int x^\alpha e^{-g_2(x)} dx \right)_\alpha \right) = \sum_{i=1}^{m} \lambda_i (-1, (x_i^\alpha)_\alpha) \ \Box$$

Definition 1

$K \in \mathcal{K}^n$, let

$$\mathrm{o.v.r.}_d(K) := \left(\frac{|G_1(g_d)|}{|K|}\right)^{\frac{1}{n}}$$

be *d-outer volume ratio* of K.

Remark

$g \in \mathbb{H}_d(\mathbb{R}^n)$, then $G_1(g)$ is

- 0-symmetric and
- star-shaped to 0.

Benko & Kroó (2009)

$K \in \mathcal{K}_0^n$ is C^2 boundary, for every $\tau \in (0,1)$, $\exists\ g_d \in \mathbb{H}_d(\mathbb{R}^n)$ and $c > 0$ s.t.

$$|g_d(x) - 1| \leq c \cdot d^{-\tau} \quad \forall\, x \in \partial K.$$

Theorem 4

$K \in \mathcal{K}_0^n$. Then $\lim_{d\to+\infty} \mathrm{o.v.r.}_d(K) = 1$.

Proof: Benko and Kroó theorem and an approximation argument give the result. □

Theorem 5

$K \in \mathcal{K}^n$. Then $\limsup_{d\to+\infty} \mathrm{o.v.r.}_d(K) \leq 2$.

Proof: Rogers-Shephard inequality and Theorem 4 give the result. □

Definition 2

$f \in \mathcal{F}(\mathbb{R}^n)$, $\|f\|_\infty = f(0) = 1$, its *d-outer integral ratio* is

$$\mathrm{o.i.r.}_d(f) := \left(\frac{t_d \int e^{-g_d^{1/d}}}{\int f} \right)^{\frac{1}{n}},$$

where (r_d, g_d) minimizes Problem 1.

Theorem/Example 6

If $f(x) = e^{-\|x\|_K}$, $K \in \mathcal{K}_0^n$, then $\lim_{d \to +\infty} \mathrm{o.i.r.}_d(f) = 1$.

Bibliography

- D. Alonso-Gutiérrez, B. González Merino, R. Villa, Best approximation of functions by log-polynomials, arXiv:2007.07952.
- K. Ball, Volume ratios and a reverse isoperimetric inequality, J. London Math. Soc., 44 (1991), no. 2, 351-—359.
- F. Barthe, On a reverse form of the Brascamp-Lieb inequality, Invent. Math., 134 (1998), no. 2, 335-—361.
- P. M. Gruber, F. E. Schuster, An arithmetic proof of John's ellipsoid theorem, Arch. Math., 85 (2005), no. 1, 82-–88.
- B. Klartag, V. D. Milman, Geometry of Log-concave Functions and Measures, Geom. Dedicata, 112 (2005), no. 1, 169–182.
- J. B. Lasserre, A generalization of Löwner-John's ellipsoid theorem, Math. Program., 152 (2015), no. 1-2, 559-–591.

Thank you for your attention!!