A variant of Kuperberg's proof of the Bourgain-Milman theorem

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Some points in the slides require additional explanation, either because what is written is not literally true as it stands, or is not obvious.

These points are indicated by a '(!)' and will be explained verbally during the lecture.

Let K be a convex body in $\mathbb{R}^n$.

$$K^\circ := \{\xi; \xi \cdot x \leq 1; x \in K\}.$$

The Mahler volume of K is

$$M(K) = |K||K^\circ|.$$

Kuperberg's theorem:

Theorem

$$M(K) \geq \pi^n/n!$$

if K is symmetric.

Scheme of Kuperberg's proof: Define a quantity $Q(K)$ ('energy' or 'Gauss linking integral') such that

$$M(K) \geq Q(K)$$

and

$$Q(K) \geq \pi^n/n!.$$

Strangely, $Q(K)$ is minimized when K is a ball.

Variant: Let ϕ be a convex function on $\mathbb{R}^n$. The Legendre transform is defined as

$$\phi^*(\xi) = \sup_x \xi \cdot x - \phi(x).$$

Theorem

$$\int e^{-\phi} \int e^{-\phi^*} \geq \pi^n,$$

if ϕ is symmetric.

Scheme of proof: Define a quantity $Q(\phi)$ such that

$$M(\phi) := \int e^{-\phi} \int e^{-\phi^*} \geq Q(\phi)$$

and

$$Q(\phi) \geq \pi^n.$$

$Q(\phi)$ is minimized for $\phi = \phi_0 = x^2/2$.

What is $Q(\phi)$?

Let

$$\lambda = \{(x, \xi); \xi = \partial\phi(x)/\partial x\} \subset R^{2n}.$$

Let

$$\Lambda = \Lambda_\phi = \lambda \times \lambda = \{(x, \xi, y, \eta); \xi = \partial\phi(x)/\partial x, \eta = \partial\phi(y)/\partial y\}.$$

Now write $z = x + iy$, $\zeta = \xi + i\eta$ and let

$$\Omega = ((i/2) \sum dz_j \wedge d\bar{\zeta}_j)^n / n! = \omega^n / n!.$$

ω can be seen as a holomorphic symplectic form on $\mathbb{C}^{2n}$ (!)

We first put

$$I(\phi) = \int_\Lambda e^{-(1/2)z\cdot\bar{\zeta}}\Omega.$$

Then we let

$$Q(\phi) = 2^{-n}\int_\Lambda |e^{-(1/2)z\cdot\bar{\zeta}}\Omega| = 2^{-n}\int_\Lambda e^{-(1/2)(\xi\cdot x+\eta\cdot y)}|\Omega|$$

$$|I(\phi)| \leq \int_{\Lambda} e^{-1/2(\xi \cdot x + \eta \cdot y)} |\Omega| = 2^n Q(\phi).$$

The point is that $I(\phi)$ is independent of ϕ since it is the integrand of a closed form. (!)Take $\phi = \phi_0 = x^2/2$. Then $\Lambda = \{z = \zeta\}$, and

$$\Omega = (i/2 \sum dz_j \wedge d\bar{z}_j)^n / n! = dm,$$

volume form on $\mathbb{C}^n_z$. Hence

$$I(\phi) = I(\phi_0) = \int_{\mathbb{C}^n} e^{-(1/2)|z|^2} dm = (2\pi)^n.$$

Hence $O(\phi) \geq \pi^n$.

It remains to prove the estimate from above of Q,

$$Q(\phi) \leq M(\phi).$$

Recall

$$\phi^*(\xi) = \sup_x \xi \cdot x - \phi(x), \quad \text{eq. for } \xi = \partial\phi(x).$$

Hence

$$\phi(x) + \phi^*(\xi) = \xi \cdot x, \quad \text{on } \Lambda, \quad \text{and} \quad \phi(y) + \phi^*(\eta) = \eta \cdot y.$$

Let

$$\pi : \Lambda \to \mathbb{R}^{2n}_{ts}, t = (x+y)/2, s = (\xi - \eta)/2.$$

Lemma

π is injective, and surjective if ϕ grows faster than any linear function.

To prove the lemma, let for t fixed.

$$A_t = \{x + y = 2t\}.$$

Put

$$\Phi(x) = \phi(x) + \phi(2t - x).$$

Then

$$\partial\Phi/\partial x = \partial\phi(x)/\partial x - \partial\phi(y)/\partial y = \xi - \eta.$$

Hence injective if ϕ is strictly convex and surjective if ϕ grows faster than linearily. $\square$

Pulling back the Mahler integral to Λ we get

$$M(\phi) = \int_{\mathbb{R}^{2n}} e^{-\phi(t)+\phi^*(s)} dtds = \int_\Lambda e^{-(\phi+\phi^*)\circ\pi} \pi^*(dtds).$$

Lemma

$\pi^*(dtds) = 2^{-n}|\Omega|.$

Accepting this we get

$$M(\phi) = 2^{-n} \int_\Lambda e^{-(\phi((x+y)/2)+\phi^*((\xi-\eta)/2))} |\Omega| \geq$$

$$2^{-n} \int_\Lambda e^{-(1/2)(\phi(x)+\phi(y)+\phi^*(\xi)+\phi^*(\eta))} |\Omega| =$$

$$2^{-n} \int_\Lambda e^{-(1/2)(x\cdot\xi+y\cdot\eta})|\Omega| = Q(\phi).$$

□

It remains to prove that

$$\pi^*(dtds) = 2^{-n}|\Omega|.$$

Recall that $\Omega = \omega^n/n!$,

$$\omega = (i/2)\sum dz_j \wedge d\bar{\zeta}_j =$$

$$(i/2)(\sum dx_j \wedge d\xi_j + dy_j \wedge d\eta_j + i\sum dy_j \wedge d\xi_j - dx_j \wedge d\eta_j).$$

Put $\tau = \sum dt_j \wedge ds_j$; $dtds = \pm\tau^n/n!$. Then

$$\pi^*(\tau) = (1/4)\sum(dx_j + dy_j) \wedge (d\xi_j - d\eta_j) =$$

$$(1/4)\sum dx_j \wedge d\xi_j - dy_j \wedge d\eta_j + dy_j \wedge d\xi_j - dx_j \wedge d\eta_j).$$

Compare and use $\sum dx_j \wedge d\xi_j = \sum dy_j \wedge d\eta_j = 0$. (!) □

Remarks

As we said ω is a holomorphic symplectic form on $\mathbb{C}^{2n}$. Its real and imaginary parts are both real symplectic forms. The real part vanishes on Λ.
The imaginary part is a symplectic form on Λ, i. e. nondegenerate there.
The point of the main lemma is that $x + y = 2t$ and $\xi - \eta = 2s$ are Darboux coordinates on Λ; they transform the imaginary part to the standard symplectic form on $\mathbb{R}^{2n}$.

Comments on Nazarov's proof

Nazarov considers Bergman spaces of the form

$$A^2_K = \{f \in H; \int_{x \in K} |f(x+iy|^2 dxdy < \infty\}.$$

The Bergman kernel for such a space is

$$B(z) = \sup_f |f(z)|^2 / \|f\|^2.$$

His main technical result is

Theorem

$$B(0) \geq c^n |K|^{-2}.$$

The main difficulty in an estimate of the Bergman kernel from below is that one needs to construct a function f which has a large value at a point compared to its norm. He uses Hörmander's L^2-estimates for $\bar{\partial}$.

The next step is to couple the thm with an estimate from above (which is more elementary)

$$B(0) \leq \pi^n |K^\circ|/|K|.$$

The result is

$$c^n |K|^{-2} \leq B(0) \leq \pi^n |K^\circ|/|K|,$$

which gives the BM-theorem.

More generally, we can consider Bergman spaces defined by a convex function ϕ

$$A^2_\phi = \{f \in H; \int |f(x+iy|^2 e^{-\phi(x)} dxdy < \infty\}.$$

The analog of the upper estimate is then

$$B(0) \leq \pi^n \frac{\int e^{-\phi^*}}{\int e^{-\phi}}.$$

$$A^2_{\phi+\psi} = \{f \in H; \int |f(x+iy|^2 e^{-(\phi(x)+\psi(x))} dxdy < \infty\}.$$

Let

$$B^2_{\phi,\psi} = \{f \in H; \int |f(x+iy|^2 e^{-(\phi(x)+\psi(y))} dxdy < \infty\}$$

and let B' be the Bergman kernel for the second space.

Theorem

$$B'(0) \leq C^n B(0).$$

The main interest of the theorem is that $B'(0)$ is very easy to estimate from below, since $f = 1$ lies in $B^2_{\phi,\psi}$ (but not in $A^2_{\phi+\psi}$).

$$B'(0) \geq (\int e^{-\phi} \int e^{-\psi})^{-1}.$$

Hence, with $\psi = \phi$,

$$(\int e^{-\phi} \int e^{-\phi})^{-1} \leq B'(0) \leq C^n B(0) \leq C_1^n \frac{\int e^{-\phi^*}}{\int e^{-\phi}},$$

which gives BM-theorem again.

One instance of the thm is easy to see directly: If ψ is a quadratic form, e.g. $\psi(x) = x^2$.

Since $x^2 - y^2 = \operatorname{Re} z^2$ one finds that $B(0) = B'(0)$. (!)

I do not know if one can take $C = 1$ in the theorem. If so, the above argument gives the same bound as Kuperberg's.

The proof of the theorem uses the family of spaces

$$A_s^2 = \{f \in H; \int |f(x + iy|^2 e^{-(\phi(z)+\psi(sz))} dxdy < \infty\}.$$

Here $\phi(z) = \phi(\mathrm{Re}\, z)$ and s is a complex parameter.

Since $\phi(z) + \phi(sz)$ is plurisubharmonic in (s, z) it follows from an earlier result of mine that the logarithm of the Bergman kernel $\log B_s(0)$ is subharmonic in s.

One can therefore estimate $B'(0) = B_{-i}$ by $B_s(0)$ for s real by the Poisson integral representation.