# Symmetry and Structure within the Log-Brunn-Minkowski Conjecture

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# Brunn-Minkowski inequality

$K, C$ convex bodies in $\mathbb{R}^n$, $\alpha, \beta > 0$

$$\begin{aligned} \alpha K + \beta C &= \{\alpha x + \beta y : x \in K,\ y \in C\} \\ &= \{x \in \mathbb{R}^n : \langle u, x\rangle \leq \alpha h_K(u) + \beta h_C(u)\ \forall u \in S^{n-1}\} \end{aligned}$$

Brunn-Minkowski inequality $\alpha, \beta > 0$

$$V(\alpha\, K + \beta\, C)^{\frac{1}{n}} \geq \alpha\, V(K)^{\frac{1}{n}} + \beta\, V(C)^{\frac{1}{n}}$$

with equality iff $K$ and $C$ are homothetic ($K = \gamma C + x$, $\gamma > 0$).

Equivalent form $\lambda \in (0, 1)$

$$V((1-\lambda)\, K + \lambda\, C) \geq V(K)^{1-\lambda} V(C)^{\lambda}.$$

# Surface area measure, Minkowski's first inequality

$S_K$ - surface area measure on $S^{n-1}$ of a convex body $K$ in $\mathbb{R}^n$
$\partial K$ is $C^2_+ \Longrightarrow dS_K = \kappa^{-1}\, d\mathcal{H}^{n-1}$
$\kappa(u) =$Gaussian curvature at $x \in \partial K$ where $u$ is normal.

Minkowski problem Monge-Ampere equation on $S^{n-1}$:

$$\det(\nabla^2 h + h\, I_{n-1}) = \kappa^{-1}$$

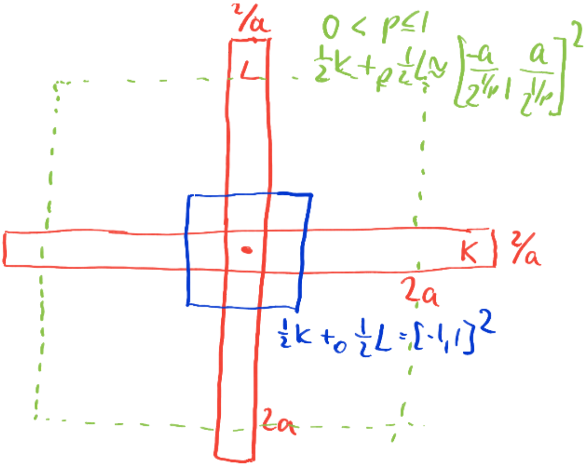
where $h(u) = h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ support function.

Minkowski's first inequality If $V(K) = V(C)$, then

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K.$$

Equality $\Longleftrightarrow$ $K$ and $C$ are translates.

# Lp sum of coordinate boxes




$$\tfrac{1}{2}K +_p \tfrac{1}{2}L = \left\{x \in \mathbb{R}^2 : \langle x,u\rangle \le \left(\tfrac{1}{2}h_K(u)^p + \tfrac{1}{2}h_L(u)^p\right)^{\frac{1}{p}} \ \forall u \in S^1\right\} \quad p > 0$$

$$\tfrac{1}{2}K +_0 \tfrac{1}{2}L = \left\{x \in \mathbb{R}^2 : \langle x,u\rangle \le \sqrt{h_K(u)\, h_L(u)} \ \forall u \in S^1\right\}$$

# Logarithmic sum - approximate estimate

Lemma (Pavlos Kalantzopoulos, K.B.)

*If $\lambda \in (0,1)$ and the centroid of the convex bodies $K$ and $L$ in $\mathbb{R}^n$ is the origin, then*

$$\gamma_1 V(K)^{1-\lambda} V(L)^{\lambda} \leq V\big((1-\lambda)K +_0 \lambda L\big) \leq \gamma_2 V(K)^{1-\lambda} V(L)^{\lambda}$$

*where $\gamma_2 > \gamma_1 > 0$ depend on $n$.*

- Logarithmic Brunn-Minkowski conjecture: $\gamma_1 = 1$
- No reasonable estimate is known for $\gamma_1$ and $\gamma_2$

# Logarithmic Minkowski problem - Cone volume measure

$dV_K = \frac{1}{n} h_K dS_K$ - cone volume measure on $S^{n-1}$ if $o \in K$ (Gromov, Milman, 1986) - also called $L_0$ surface area measure

- $K$ polytope, $F_1, \ldots, F_k$ facets, $u_i$ exterior unit normal at $F_i$

$$V_K(\{u_i\}) = \frac{h_K(u_i)\mathcal{H}^{n-1}(F_i)}{n} = V(\mathrm{conv}\{o, F_i\}).$$

- $V_K(S^{n-1}) = V(K)$.

Monge-Ampere type differential equation on $S^{n-1}$ for $h = h_K$ if $\mu$ has a density function $f$:

$$h \det(\nabla^2 h + h\, I) = f$$

- B., Lutwak, Yang, Zhang solved in the even case
- Partial result by Chen, Li, Zhu in the general case

## Logarithmic ($L_0$) Brunn-Minkowski conjecture

$\lambda \in [0,1]$, $o \in K, C$

$$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle u, x\rangle \le h_K(u)^{1-\lambda} h_C(u)^{\lambda}\ \forall u \in S^{n-1}\}$$

$$\lambda K +_0 (1-\lambda)C \ \subset\ \lambda K + (1-\lambda)C$$

Conjecture (Logarithmic Brunn-Minkowski conjecture)

*$\lambda \in (0,1)$, $K$, $C$ are o-symmetric*

$$V((1-\lambda)K +_0 \lambda C) \ge V(K)^{1-\lambda} V(C)^{\lambda}$$

*with equality iff $K$ and $C$ have dilated direct summands.*

Conjecture (Logarithmic Minkowski conjecture)

*For o-symmetric $K$, $C$, if $V(K) = V(C)$, then*

$$\int_{S^{n-1}} \log h_C \, dV_K \ge \int_{S^{n-1}} \log h_K \, dV_K,$$

*with equality iff $K$ and $C$ have dilated direct summands.*

# $L_p$ surface area measures

$L_p$ surface area measures (Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p}\, dS_K = n h_K^{-p}\, dV_K$$

Examples $S_{K,1} = S_K$ and $S_{K,0} = nV_K$

Monge-Ampere differential equation on $S^{n-1}$ for $h = h_K$

$$h^{1-p} \det(\nabla^2 h + h\, I) = f$$

Remark

- Minimize $\int_{S^{n-1}} h_C^p \, d\mu$ under the condition $V(C) = 1$ (Chou&Wang, Chen&Li&Zhu, B&Bianchi&Colesanti)
- Conjectured to have unique even solution if $0 < p < 1$

# $L_p$ Brunn-Minkowski inequality/conjecture

$p > 0$, $\lambda \in (0,1)$, $o \in \mathrm{int} K, \mathrm{int} L$

$\lambda K +_p (1-\lambda)L = \{x \in \mathbb{R}^n : \langle u, x\rangle^p \le \lambda h_K(u)^p + (1-\lambda) h_L(u)^p \ \forall u\}$

$p \ge 1$ $\quad h_{\lambda K +_p (1-\lambda)L} = \left(\lambda h_K^p + (1-\lambda) h_L^p\right)^{1/p}$

$L_p$ BM inequality($p \ge 1$)/conjecture($0 < p < 1$,$o$-symm)

$$V((1-\lambda)K +_p \lambda L)^{\frac{p}{n}} \ge (1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}$$

with equality iff $K$ and $L$ are dilated. Equivalent

$$V(\lambda K +_p (1-\lambda)L) \ge V(K)^{\lambda} V(L)^{1-\lambda}$$

$L_p$ Minkowski inequality/conjecture for $p > 0$

$$\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p dV_K \ge V(K)\left(\frac{V(L)}{V(K)}\right)^{\frac{p}{n}}$$

with equality in the $o$-symmetric case iff $K$ and $L$ are dilates.

# Related - Gardner-Zvavitch Conjecture

Livshyts, Marsiglietti, Nayar, Zvavitch
logarithmic B-M conjecture $\Longrightarrow$ Gardner-Zvavitch Conjecture

$$\gamma((1-\lambda)K+\lambda L)^{\frac{1}{n}} \geq (1-\lambda)\gamma(K)^{\frac{1}{n}}+\lambda\gamma(L)^{\frac{1}{n}}$$

for $o$-symmetric $K, L$ and the Gaussian measure $\gamma$ on $\mathbb{R}^n$.
($\gamma$ can be replaced by any even log-concave measure)

Theorem (Eskenazis, Moschidis)

*For o-symmetric convex bodies $K, C$, we have*

$$\gamma((1-\lambda)K+\lambda L)^{\frac{1}{n}} \geq (1-\lambda)\gamma(K)^{\frac{1}{n}}+\lambda\gamma(L)^{\frac{1}{n}}$$

- Argument based on Kolesnikov & Livshyts's approach, stemming from Kolesnikov & Emanuel Milman's work on the Hilbert operator

## The $L_p$ Minkowski conjecture for $p_0 < p < 1$

$p_0 = 1 - \frac{c}{n^{3/2}}$

Theorem (Chen, Huang, Li, Liu)

$p_0 < p < 1$, *$K, L$ o-symmetric*

$$V((1-\lambda)K +_p \lambda L) \geq V(K)^{1-\lambda} V(L)^{\lambda}$$

Idea $\partial K$, $\partial L$ are $C^2_+$ and $S_{K,p} = S_{L,p} \Longrightarrow K = L$

Step 1 (Kolesnikov, Emanuel Milman)
$\partial M$ is $C^2_+$, $\|h_K - h_M\|_{C^2} < \varepsilon_M$ and $\|h_L - h_M\|_{C^2} < \varepsilon_M$ for $\varepsilon_M > 0$
(spectral gap for Hilbert operator)

Step 2 From local to global Works for any $0 \leq p < 1$

- Schauder estimates in PDE (Chen, Huang, Li, Liu)
- Deforming strongly isomorphic polytopes (Eli Putterman)

# Kolesnikov, Emanuel Milman approach (extending Colesanti&Livshyts&Marsiglietti)

$$D^2 h = \nabla^2 h + h\, I_{n-1} \quad \text{for } h \in C^2(S^{n-1})$$

Mixed discriminant For $h_1, \ldots, h_{n-1} \in C^2(S^{n-1})$

$$S(h_1, \ldots, h_{n-1}) = D_{n-1}(D^2 h_1, \ldots, D^2 h_{n-1})$$

Hilbert-Brunn-Minkowski operator $\partial K$ $C^2_+$, $z \in C^2(S^{n-1})$

$$L_K z = \frac{S(z h_K, h_K, \ldots, h_K)}{S(h_K, \ldots, h_K)} - z$$

Theorem (Hilbert-Kolesnikov-Milman)

$L_K : C^2(S^{n-1}) \to C(S^{n-1})$ *elliptic with self-adjoint extension to* $L^2(dV_K)$

# Spectral properties of $-L_K$

Trivial eigenvalues of $-L_K$

- $\lambda_0(-L_K) = 0$ (corresponding to constant functions)
- linear functions (that are odd) have eigenvalue 1 with multiplicity $n$

Theorem (Hilbert)

$K \in \mathcal{K}^2_+ \Longrightarrow \lambda_1(-L_K) \geq 1$

**Remark:** Equivalent with Brunn-Minkowski inequality

Fact $\lambda_{1,e}(-L_K) = \lambda_{n+1}(-L_K)$ for $K \in \mathcal{K}^2_{+,e}$

$\lambda_{1,e} =$first positive eigenvalue when restricted to even functions

Theorem (Kolesnikov&Milman)

$p \in [0,1)$

*local $L_p$-Brunn-Minkowski conjecture* $\Longleftrightarrow$

$\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}$ *for* $\forall K \in \mathcal{K}^2_{+,e}$

# Eli Putterman's formulation

Equivalent to $L_p$ B-M conjecture

$p \in [0,1)$, $K$, $L$ $o$-symmetric

$$V(K)\left((n-1)V(L[2],K[n-2]) + \frac{1-p}{n}\int_{S^{n-1}} \frac{h_L^2}{h_K}\,dS_K\right) \le (n-p)V(L,K[n-1])^2.$$

- If $p = 1$, then we have Minkowski's second inequality
- For $p \in [0,1)$, the conjecture is stronger than Minkowski's second inequality because

$$V(K)\cdot\frac{1}{n}\int_{S^{n-1}} \frac{h_L^2}{h_K}\,dS_K \ge V(L,K[n-1])^2$$

by Hölder's inequality

# Known cases of the logarithmic B-M conjecture 1

- Interesting for any log-concave measure (like Gaussian) instead of volume
  log B-M conjecture for volume $\Longrightarrow$log B-M conjecture for any log-concave measure (Saroglou)
- $n = 2$ for volume (Stancu + BLYZ)
- $K$ and $L$ are unconditional for any log-concave measure - follows directly from Prékopa-Leindler (Bollobás&Leader + Cordero-Erausquin&Fradelizi&Maurey + Saroglou on coordinatewise product)
- $K$ and $L$ are dilates for the Gaussian measure (Cordero-Erausquin&Fradelizi&Maurey on $B$-conjecture)
- Holds for the volume in $\mathbb{R}^{2n} = \mathbb{C}^n$ if $K$ and $L$ are complex convex bodies (Rotem)

# Logarithmic B-M conjecture for almost ellipsoids

Chen, Huang, Li, Liu verified logarithmic Brunn-Minkowski conjecture if $K$ is close to be an ellipsoid
(based on the local result by Colesanti, Livshyts, Marsiglietti, reproved by Kolesnikov, Milman):
$\exists \varepsilon_n > 0$ such that if $K, L$ $o$-symmetric with $V(K) = V(L)$ and $E \subset K \subset (1+\varepsilon_n)E$ for an ellipsoid $E$, then

$$\int_{S^{n-1}} \log h_L \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K.$$

# Variants of $L_p$-BM by John Hosle, Kolesnikov, Livshyts

Conjecture (John Hosle, Kolesnikov, Livshyts)

*For $K, L$ o-symmetric and log-concave measure $d\mu(x) = e^{-V(x)}dx$ on $\mathbb{R}^n$, $V$ convex, if $0 < q \leq p \leq 1$, then*

$$\mu((1-\lambda)K +_p \lambda L)^{\frac{q}{n}} \geq (1-\lambda)\mu(K)^{\frac{q}{n}} + \lambda\mu(L)^{\frac{q}{n}},$$

*and $\mu((1-\lambda)K +_p \lambda L) \geq \mu(K)^{1-\lambda}\mu(L)^{\lambda}$ if $0 = q \leq p \leq 1$.*

- $q = 0$ is the $L_p$ Brunn-Minkowski conjecture for $\mu$
- $p = q = 1$ for $\mu = \gamma$ is Gardner-Zvavitch
- $(p, q)$ implies $(p', q')$ for $p' \geq p$ and $q' \leq q$.

Local form $K \in \mathcal{K}^2_+$, $II$ =second fund. form, $H_x = \mathrm{tr} II - \langle \nabla V, n_x \rangle$

$$\int_{\partial K} H_x f^2 - \langle II^{-1}\nabla_K f, \nabla_K f\rangle + \frac{(1-p)f^2}{\langle x, n_x\rangle}\, d\mu \leq \frac{n-q}{n\mu(K)}\left(\int_{\partial K} f\, d\mu\right)^2$$

# Some equivalent formulations of the $L_p$-Brunn-Minkowski conjecture for $o$-symmetric bodies, $0 \leq p < p_0 < 1$

- Monge-Ampere: $h^{1-p}\det(\nabla^2 h + h\,I) = f$ on $S^{n-1}$
  has unique even solution if $f$ is even positive and $C^\infty$

- Kolesnikov, Emanuel Milman: $\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}$
  for $K \in \mathcal{K}^2_{+,e}$

- Eli Putterman: $(n-p)V(L, K[n-1])^2 \geq$
  $V(K)\left((n-1)V(L[2], K[n-2]) + \frac{1-p}{n}\int_{S^{n-1}} \frac{h_L^2}{h_K}\,dS_K\right)$

- Kolesnikov: Formulation in terms of Optimal Transportation on $S^{n-1}$

# Logarithmic Brunn-Minkowski conjecture for general convex bodies

Conjecture (General log-Brunn-Minkowski Conjecture)

*For any $\lambda \in (0,1)$ and convex bodies $K$ and $L$ in $\mathbb{R}^n$, there exist $z_K \in K$ and $z_L \in L$ depending on both $K$ and $L$ such that*

$$V((1-\lambda)\cdot(K - z_K) +_0 \lambda \cdot (L - z_L)) \geq V(K)^{1-\lambda} V(L)^{\lambda}$$

*where $z_K = z_L = o$ if $K$ and $L$ are o-symmetric.*
*Equality $\iff$ $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ where $K_1, \ldots, K_m, L_1, \ldots, L_m$ are invariant under $A_1, \ldots, A_n$, $\sum_{i=1}^m \dim K_i = n$ and $K_i$ and $L_i$ are homothetic, $i = 1, \ldots, m$.*

Known in the plane by Xi and Leng ($z_K$ and $z_L$ are NOT the centroid in their argument)

# Logarithmic Brunn-Minkowski for bodies with many hyperplane symmetries

$A \in \mathrm{GL}(n, \mathbb{R})$ linear reflection if

- $A$ acts identically on an $(n-1)$-dimensional linear subspace $H$,
- $\exists u \in \mathbb{R}^n \setminus H$ with $A(u) = -u$

$A$ is an "orthogonal reflection" if $H = u^\perp$.

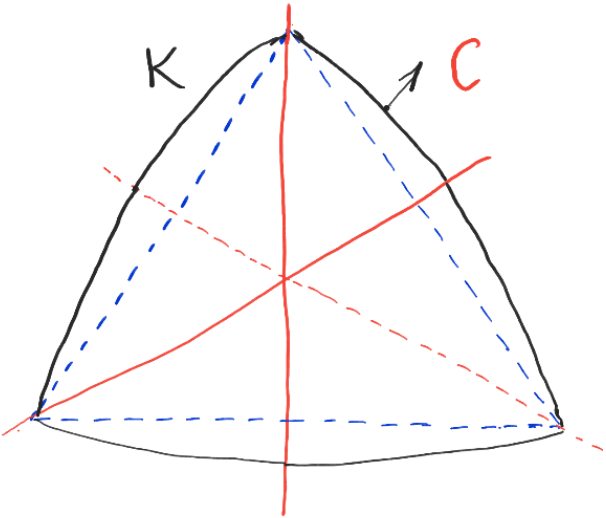
Theorem (Pavlos Kalantzopoulos, K.B.)

*If $\lambda \in (0,1)$ and the convex bodies $K$ and $L$ are invariant under linear reflections $A_1, \ldots, A_n$ are such that $H_1 \cap \ldots \cap H_n = \{o\}$ holds for the associated hyperplanes, then*

$$V((1-\lambda) \cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^\lambda.$$

*Equality $\Longleftrightarrow$ $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ where $K_1, \ldots, K_m, L_1, \ldots, L_m$ are invariant under $A_1, \ldots, A_n$, $\sum_{i=1}^m \dim K_i = n$ and $K_i$ and $L_i$ are homothetic, $i = 1, \ldots, m$.*

Convex body K with symmetries of a regular simplex



# Idea and some consequences of the argument

- Idea comes from Barthe & Fradelizi's work on Mahler's conjecture
- For a common subgroup $H$ of the symmetry groups of $K$ and $L$, $H$ has simplicial cone $C$ as fundamental domain, and reflections through the walls of $C$ are in $H$
- $C$ is mapped into a "coordinate corner" by a linear transform, and the known Log-Brunn-Minkowski inequality for unconditional bodies is used.

Some consequences for $K, L$ convex bodies with many hyperplane symmetries

- Gardner-Zvavitch
- $V_K = V_L \iff V(K) = V(L)$, $K = K_1 + \ldots + K_m$ and $L = L_1 + \ldots + L_m$ where $\sum_{i=1}^{m} \dim K_i = n$ and $K_i$ and $L_i$ are homothetic, $i = 1, \ldots, m$.