# Stability of the Prekopa-Leindler inequality and the unconditional Logarithmic Brunn-Minkowski Inequality

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Asymptotic Geometry Seminar, April 2, 2021

# Sketch of the talk

Part 1 Stability of the Prekopa-Leindler inequality

- for log-concave functions - detailed discussion
- general case - announcement

Part 2 Stability of the Bollobas-Leader inequality for coordinatewise product of unconditional bodies

Part 3 Stability of the Log-Minkowski inequality

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \geq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)}$$

with hyperplane symmetries where $dV_K = \frac{1}{n} h_K \, dS_K$ is the cone-volume measure (including stability of the Logarithmic Brunn-Minkowski inequality with hyperplane symmetries)

# Prekopa-Leindler inequality

$f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ log-concave
$\iff f((1-\alpha)x + \alpha y) \geq f(x)^{1-\alpha} f(y)^{\alpha} \; \forall x, y \in \mathbb{R}^n, \forall \alpha \in (0,1)$
$\iff f = e^{-\varphi}$ for convex $\varphi : \mathbb{R} \to (-\infty, \infty]$

## Theorem (Prekopa, Leindler, Dubuc)

*If $\lambda \in (0,1)$ and $h, f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfy $o < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$, $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} h \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \cdot \left( \int_{\mathbb{R}^n} g \right)^{\lambda},$$

*equality implying that for $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$, there exist $w \in \mathbb{R}^n$ and log-concave function $\tilde{h}$ such that $h = \tilde{h}$, $f(x) = a^{-\lambda} \tilde{h}(x - \lambda w)$, $g(y) = a^{1-\lambda} \tilde{h}(y + (1-\lambda)w)$ almost everywhere.*

Remark $\sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ may not be measurable, but it is log-concave, if $f, g$ log-concave

# Brunn-Minkowski inequality and its stability

$K, C$ convex bodies in $\mathbb{R}^n$

Brunn-Minkowski inequality $\lambda \in (0,1)$

$$V((1-\lambda)\,K + \lambda\,C) \geq V(K)^{1-\lambda} V(C)^{\lambda}.$$

$$\alpha = V(K)^{\frac{-1}{n}},\ \beta = V(C)^{\frac{-1}{n}}, \quad \sigma = \max\left\{\frac{V(C)}{V(K)}, \frac{V(K)}{V(C)}\right\}$$

$$A(K,C) = \min_{x\in\mathbb{R}^n} V\Big((\alpha K)\Delta(x+\beta C)\Big)$$

Theorem [Figalli, Maggi, Pratelli $\sim$ 2010]

$$V\Big(\tfrac{1}{2}\,K + \tfrac{1}{2}\,C\Big) \geq \sqrt{V(K)\cdot V(C)}\left[1 + \frac{(\sigma-1)^2}{32n\sigma^2} + \frac{\gamma(n)}{\sigma^{\frac{1}{n}}} \cdot A(K,C)^2\right].$$

Remark First stability version by Minkowski

# Stability of Prekopa-Leindler for log-concave functions

## Theorem (B., De)

*If $\tau \in (0, \frac{1}{2}]$, $\lambda \in [\tau, 1-\tau]$ and $f, g$ are log-concave on $\mathbb{R}^n$ satisfying*

$$\int_{\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}\, dz \leq (1+\varepsilon) \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^{\lambda}$$

*for $\varepsilon > 0$, then for $a = \int_{\mathbb{R}^n} f \,/\, \int_{\mathbb{R}^n} g$, there exists $w \in \mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} |f(x) - a \cdot g(x+w)|\, dx \leq c^n n^n \left( \frac{\varepsilon}{\tau} \right)^{\frac{1}{19}} \int_{\mathbb{R}^n} f.$$

# Stability of Prekopa-Leindler - annoucement

## Theorem (B., Figalli, Ramos)

*For $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, if $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are measurable functions such that $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ holds for $x, y \in \mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} h \leq (1+\varepsilon) \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g \right)^{\lambda},$$

*then setting $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$, $\exists$ log-concave $\tilde{h}$ and $w \in \mathbb{R}^n$ with*

$$\int_{\mathbb{R}^n} |a^{\lambda} f - \tilde{h}(\cdot + \lambda w)| < M_n \frac{\varepsilon^{Q_n(\tau)}}{\tau^{N_n}} \int_{\mathbb{R}^n} h$$

$$\int_{\mathbb{R}^n} |a^{\lambda-1} g - \tilde{h}(\cdot + (\lambda - 1) w)| < M_n \frac{\varepsilon^{Q_n(\tau)}}{\tau^{N_n}} \int_{\mathbb{R}^n} h$$

$$\int_{\mathbb{R}^n} |h - \tilde{h}| < M_n \frac{\varepsilon^{Q_n(\tau)}}{\tau^{N_n}} \int_{\mathbb{R}^n} h$$

# Very little history

- Stability of Brunn-Minkowski w.r.t. Hausdorff distance: Diskant, Groemer (Isoper. Inequality: Schneider, Fuglede)
- Optimal stability of Isoperimetric Inequality w.r.t. symmetric difference metric: Fusco, Maggi, Pratelli $\sim$ 2008
- Optimal stability of Brunn-Minkowski for convex bodies w.r.t. symmetric difference metric: Figalli, Maggi, Pratelli, constant improved by Kolesnikov, Milman
- Optimal stability of Brunn-Minkowski for general sets for $n = 1$ w.r.t. symmetric difference metric: Freiman
- Optimal stability of Brunn-Minkowski for general sets, $n = 2$ w.r.t. symm. diff. metric: van Hintum, Spink, Tiba, 2020
- Stability of Brunn-Minkowski for general sets w.r.t. symmetric difference metric: Figalli, Jerisson $\sim$ 2017
- Weak Stability of Prekopa-Leindler for log-concave functions: Bucur, Fragala, 2014

# Tools for stability of P-L for log-concave functions

- We may assume that $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$ and $\sup f = 1$
- We may assume that $\lambda = \tau = \frac{1}{2}$
- $n = 1$ (Keith Ball, B. $\sim$ 2010)
  $f, g$ are log-concave probability densities on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \sup_{z=\frac{1}{2}x+\frac{1}{2}y} \sqrt{f(x)g(y)}\, dz \leq 1 + \varepsilon$$

for $\varepsilon \in (0,1)$, then there exists $w \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} |f(x) - g(x+w)|\, dx \leq c\varepsilon^{\frac{1}{3}} |\log \varepsilon|^{\frac{4}{3}}.$$

Remark Error might be of order $\varepsilon^{\frac{1}{2}}$ (exponent $\frac{1}{3}$ above can't be larger than $\frac{1}{2}$)

# Level sets of a log-concave function

## Lemma (Lovasz-Vempala)

*If $\psi$ is a log-concave probability density on $\mathbb{R}^n$ and $t \in (0,1)$, then*

$$V\left(\{\psi > (1-t)\|\psi\|_\infty\}\right) \geq \frac{1}{n!+1} \cdot \frac{t^n}{\|\psi\|_\infty}.$$

## Lemma (Lovasz-Vempala)

*If $s \in (0, e^{-4(n-1)})$ and $\psi$ is a log-concave probability density on $\mathbb{R}^n$, then*

$$\begin{aligned} V\left(\{\psi > s\|\psi\|_\infty\}\right) &< \frac{2}{n!} \cdot \frac{|\ln s|^n}{\|\psi\|_\infty}, \\ \int_{\{\psi < s\|\psi\|_\infty\}} \psi &< \frac{e^{n-1}}{(n-1)^{n-1}} \cdot s \cdot |\ln s|^{n-1}. \end{aligned}$$

# Comparing level sets of $f, g, h$

$$h(z) = \sup_{z=\frac{1}{2}x+\frac{1}{2}y} \sqrt{f(x)g(y)}$$

$$\frac{1}{2}\{f > t\} + \frac{1}{2}\{g > s\} \subset \left\{h > \sqrt{ts}\right\} \quad \text{for } t, s > 0$$

$$F(t) = V(\{f > t\}) \quad G(s) = V(\{g > s\}) \quad H(r) = V(\{h > r\})$$

$$\Longrightarrow H(\sqrt{ts}) \geq \sqrt{F(t) \cdot G(s)}$$

$$\Longrightarrow \sqrt{\int_0^\infty F \cdot \int_0^\infty G} \leq \int_0^\infty H \leq (1+\varepsilon)\sqrt{\int_0^\infty F \cdot \int_0^\infty G}$$

$\Longrightarrow_{\textit{PL } n=1 \text{ \& } \textit{BM stab}}$ $\{f > t\}$ and $\{g > t\}$ are "almost translates"

# log-concave function $\mapsto$ convex body

$\eta \approx \varepsilon^{1/18}$

$K = \{ (x, \ln t) \in \mathbb{R}^{n+1} : x \in \{f > \eta\} \ \& \ \eta \le t \le f(x) \}$

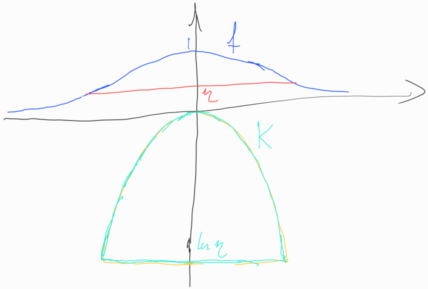
$C = \{ (x, \ln t) \in \mathbb{R}^{n+1} : x \in \{g > \eta\} \ \& \ \eta \le t \le g(x) \}$

$L = \{ (x, \ln t) \in \mathbb{R}^{n+1} : x \in \{h > \eta\} \ \& \ \eta \le t \le h(x) \}$

$h(\tfrac{1}{2}x + \tfrac{1}{2}y) \ge f(x)^{\frac{1}{2}} g(y)^{\frac{1}{2}} \Rightarrow \tfrac{1}{2}K + \tfrac{1}{2}C \subset L$

$|V(K) - V(L)| = \text{small} \qquad |V(C) - V(L)| = \text{small}$

BM stab $\Longrightarrow$ $K, C, L$ are essentially translates



# Part 2
# Coordinatewise product of unconditional convex bodies

$K$, $C$ unconditional ($(x_1, \ldots, x_n) \in K \Longrightarrow (\pm x_1, \ldots, \pm x_n) \in K$)

$$K^{1-\lambda} \cdot C^{\lambda} = \Big\{ \Big( \pm|x_1|^{1-\lambda}|y_1|^{\lambda}, \ldots, \pm|x_n|^{1-\lambda}|y_n|^{\lambda} \Big) \\ (x_1, \ldots, x_n) \in K \;\&\; (y_1, \ldots, y_n) \in C \Big\}$$

**Theorem (Bollobas&Leader, Saroglou)**

*If $K$ and $C$ are unconditional convex bodies and $\lambda \in (0,1)$, then*

$$V(K^{1-\lambda} \cdot C^{\lambda}) \geq V(K)^{1-\lambda} V(C)^{\lambda},$$

*with equality* $\Longleftrightarrow \exists\, \Phi$ *positive definit diagonal matrix s.t.* $K = \Phi C$

# Stability of Bollobas-Leader

Theorem (B., De)

*If* $\tau \in (0, \frac{1}{2}]$, $\lambda \in [\tau, 1-\tau]$ *and unconditional convex bodies* $K$ *and* $C$ *in* $\mathbb{R}^n$ *satisfy*

$$V(K^{1-\lambda} \cdot C^{\lambda}) \leq (1+\varepsilon) V(K)^{1-\lambda} V(C)^{\lambda},$$

*then there exists positive definite diagonal matrix* $\Phi$*,* $\det \Phi = V(K)/V(C)$ *such that*

$$V(K \Delta (\Phi C)) < c_0^n n^n \left(\frac{\varepsilon}{\tau}\right)^{\frac{1}{19}} V(K).$$

# Proof of Bollobas-Leader via Prekopa-Leindler ala Fradelizi, Cordero-Erausquin, Maurey

**Theorem (Prekopa, Leindler, Dubuc)**

*If $\lambda \in (0,1)$ and $h, f, g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ log-concave satisfy $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} h \geq \left( \int_{\mathbb{R}^n} f \right)^{1-\lambda} \cdot \left( \int_{\mathbb{R}^n} g \right)^{\lambda},$$

*equality implying that for $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, there exist $w \in \mathbb{R}^n$ such that $f(x) = a^{-\lambda} h(x - \lambda w)$, $g(y) = a^{1-\lambda} h(y + (1-\lambda) w)$.*

To prove Bollobas-Leader, apply Prekopa-Leindler to

$$\begin{aligned} f(x_1, \ldots, x_n) &= \mathbf{1}_K(e^{x_1}, \ldots, e^{x_n}) e^{x_1 + \ldots + x_n} \\ g(x_1, \ldots, x_n) &= \mathbf{1}_C(e^{x_1}, \ldots, e^{x_n}) e^{x_1 + \ldots + x_n} \\ h(x_1, \ldots, x_n) &= \mathbf{1}_{K^{1-\lambda} \cdot C^{\lambda}}(e^{x_1}, \ldots, e^{x_n}) e^{x_1 + \ldots + x_n} \end{aligned}$$

For stability Bollobas-Leader, use stability of Prekopa-Leindler for log-concave functions

# Part 3 - Stability of the Log-Minkowski inequality with hyperplane symmetry

support function $h_K(u) = \max_{x\in K}\langle x, u\rangle$

$S_K$ - surface area measure on $S^{n-1}$ of a convex body $K$ in $\mathbb{R}^n$

$\partial K$ is $C^2_+ \Longrightarrow dS_K = \kappa^{-1}\, d\mathcal{H}^{n-1}$ ($\kappa(u)$ =Gaussian curvature)

Minkowski's first inequality If $V(K) = V(C)$, then

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K.$$

Equality $\Longleftrightarrow$ $K$ and $C$ are translates.

Part 3$\subset$"$L_p$ Minkowski Inequalities", intiated by Firey, Lutwak, developed by Colesanti, Milman, Kolesnikov (local theory)
Livshyts
Chen&Huang&Li and Putterman ("continuity method")

# Logarithmic Minkowski conjecture

$dV_K = \frac{1}{n} h_K dS_K$ - cone volume measure on $S^{n-1}$ if $o \in K$
(Firey, 1974, Gromov, Milman, 1986) $= L_0$ surface area measure

## Conjecture (B, Lutwak, Yang, Zhang)

*If $K$ and $C$ are convex bodies whose centroid is the origin and $V(K) = V(C)$, then*

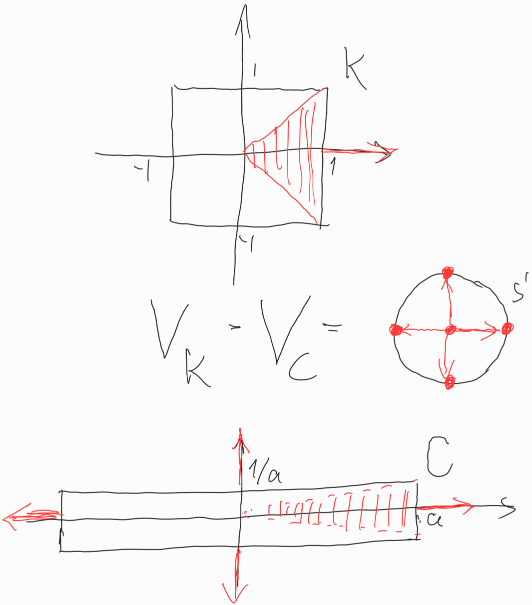
$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K. \tag{1}$$

*Assuming $K$ is smooth, equality holds $\iff$ $K = C$.*

## Known results

- $K$ is close to some ellipsoid (Colesanti&Livshyts&Marsiglietti, Kolesnikov&Milman, Chen&Huang&Li&Liu)
- $K, C$ have complex symmetry (Rotem)
- $K, C$ - hyperplane symmetry (Saroglou, B&Kalantzopoulos)

# Coinciding cone volumes



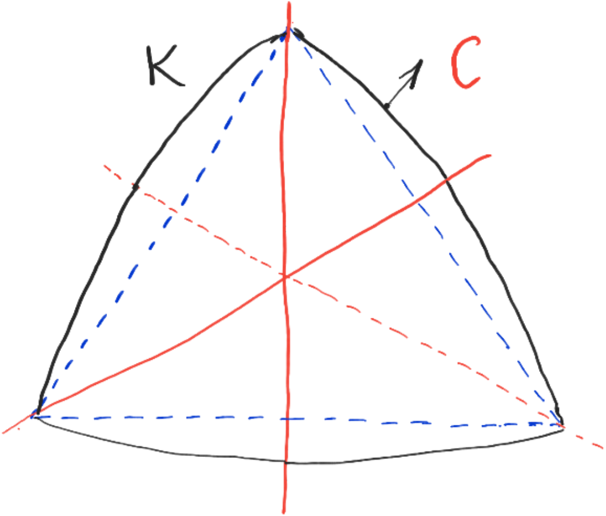
# Convex bodies with "hyperplane symmetries"

Linear reflection $A \in \mathrm{GL}(n)$, $A \neq \mathrm{Id}$, $A^2 = A$ and there exists linear hyperplane $H$ s.t. $Ax = x$ for $x \in H$.

a convex body $K$ has "hyperplane symmetries" $\Longleftrightarrow$
$K$ is invariant under some linear reflections $A_1, \ldots, A_n$ through hyperplanes $H_1, \ldots, H_n$ with $H_1 \cap \ldots \cap H_n = \{o\}$ $\Longleftrightarrow$
$K$ invariant under a Coxeter group $G \subset \mathrm{GL}(n)$ of rank $n$

- Idea comes from
  - Barthe & Fradelizi's work on Mahler's conjecture
  - Barthe & Cordero-Erausquin's work on the Slicing conjecture
- $G$ has a simplicial cone $C$ as fundamental domain, and reflections through the walls of $C$ generate $G$
- $C$ is mapped into a "coordinate corner" by a linear transform, and results about unconditional bodies are used.

Convex body K with symmetries of a regular simplex



# Stability of Log-Minkowski with hyperplane symmetries

## Theorem (B., De)

*If the convex bodies $K$ and $C$ in $\mathbb{R}^n$ are invariant under the Coxeter group $G \subset \mathrm{GL}(n)$ generated by $n$ independent linear reflections, and*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \frac{dV_K}{V(K)} \leq \frac{1}{n} \cdot \log \frac{V(C)}{V(K)} + \varepsilon$$

*for $\varepsilon > 0$, then for some $m \geq 1$, there exist compact convex sets $K_1, C_1, \ldots, K_m, C_m$ of dimension at least one and invariant under $G$ where $K_i$ and $C_i$ are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$ such that*

$$K_1 + \ldots + K_m \subset \quad K \quad \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(K_1 + \ldots + K_m)$$

$$C_1 + \ldots + C_m \subset \quad C \quad \subset \left(1 + c^n \varepsilon^{\frac{1}{95n}}\right)(C_1 + \ldots + C_m)$$

*where $c > 1$ is an absolute constant.*