

REMARKS ON THE BRASCAMP–LIEB INEQUALITY

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ABSTRACT. We prove several variants of the Brascamp–Lieb inequality which are adapted to additional symmetries, weights, and non-Euclidean radial structures. First, an improved Brascamp–Lieb inequality for even functions yields a strengthened Prékopa–Leindler theorem for even strongly log-concave functions. We also record a Legendre-transform proof of a generalized log-Sobolev inequality for log-concave functions. Further results include a Brascamp–Lieb inequality with a concave weight, an unconditional second-order Poincaré inequality, a rotation-trick improvement for arbitrary even log-concave measures, and a Poincaré-type inequality associated with smooth norms.

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1. INTRODUCTION

The Brascamp–Lieb inequality [5] is a fundamental functional inequality with deep connections to convex geometry. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 function such that $\nabla^2 V \succ 0$ at every point. Let μ be the probability measure on $\mathbb{R}^n$ with density $Z^{-1}e^{-V}$, where $Z = \int e^{-V}$ (unless explicitly stated otherwise, all integrations are over $\mathbb{R}^n$). For a C^1 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\int |f|d\mu < \infty$, define

$$\mathrm{Var}_\mu f = \int f^2 d\mu - \left(\int f d\mu \right)^2.$$

The Brascamp–Lieb inequality then states that for such a function f we have

$$(1) \quad \text{Var}_\mu f \leq \int \left\langle (\nabla^2 V)^{-1} \nabla f, \nabla f \right\rangle d\mu.$$

In particular, if $\nabla^2 V \succeq \alpha \text{Id}$ for some $\alpha > 0$, then

$$\text{Var}_\mu f \leq \frac{1}{\alpha} \int |\nabla f|^2 d\mu.$$

For example, the standard Gaussian measure γ , with density $\frac{d\gamma}{dx} = Z^{-1}e^{-|x|^2/2}$, satisfies a Poincaré inequality with constant 1; this was known as a folklore result decades before the work of Brascamp and Lieb. In this note we prove new variants of several functional inequalities related to the Brascamp–Lieb inequality.

We begin in Section 3, which concerns the Prékopa–Leindler inequality [18, 19, 17]. This inequality states that for all $0 \leq t \leq 1$ and for all measurable functions $f, g, h : \mathbb{R}^n \rightarrow (-\infty, \infty]$ satisfying

$$h((1-t)x + ty) \leq (1-t)f(x) + tg(y)$$

for all $x, y \in \mathbb{R}^n$, one has

$$(2) \quad \int e^{-h} \geq \left(\int e^{-f} \right)^{1-t} \left(\int e^{-g} \right)^t.$$

The Prékopa–Leindler inequality is a functional version of the Brunn–Minkowski inequality: For Borel sets $A, B \subseteq \mathbb{R}^n$, take $e^{-f} = \mathbf{1}_A$, $e^{-g} = \mathbf{1}_B$, and $e^{-h} = \mathbf{1}_{(1-t)A+tB}$ to obtain

$$\text{Vol}((1-t)A + tB) \geq \text{Vol}(A)^{1-t} \text{Vol}(B)^t.$$

The original paper [5] of Brascamp and Lieb studies both extensions of the Prékopa–Leindler inequality and the Brascamp–Lieb inequality, but these two directions are treated rather independently. Later, Bobkov and Ledoux [4] proved the Brascamp–Lieb inequality from the Prékopa–Leindler inequality. To sketch the connection, fix convex functions $f_0, f_1 : \mathbb{R}^n \rightarrow (-\infty, \infty]$ and consider

$$f_t(z) = \inf_{z=(1-t)x+ty} ((1-t)f_0(x) + tf_1(y)).$$

The Prékopa–Leindler inequality implies that

$$(3) \quad \alpha(t) = \int e^{-f_t}$$

is log-concave, i.e. $\log \alpha$ is a concave function. In particular this implies that $(\log \alpha)''(t) \leq 0$, which yields the Brascamp–Lieb inequality.

In Section 3 we reverse this line of reasoning. We first prove a strengthened Brascamp–Lieb inequality for even functions, and then deduce the following strengthened Prékopa–Leindler inequality.

Theorem 1.1. *Let $f, g : \mathbb{R}^n \rightarrow (-\infty, \infty]$ be even functions such that $f - c|x|^2$ and $g - c|x|^2$ are convex for some $c > 0$. Fix $0 \leq t \leq 1$, and assume that a measurable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies*

$$h((1-t)x + ty) \leq (1-t)f(x) + tg(y)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int e^{-h} e^{\frac{c}{2}|x|^2} \geq \left(\int e^{-f} e^{\frac{c}{2}|x|^2} \right)^{1-t} \left(\int e^{-g} e^{\frac{c}{2}|x|^2} \right)^t.$$

Here $|\cdot|$ denotes the Euclidean norm. The theorem does not follow from the usual Prékopa–Leindler inequality after subtracting $c|x|^2/2$, since the shifted functions need not satisfy the required pointwise hypothesis.

In Section 4 we turn our attention to log-Sobolev inequalities. The classical log-Sobolev inequality, due to Gross [12], may be written as

$$(4) \quad \text{Ent}(\phi) + n \log \left(\sqrt{2\pi e} \right) \int \phi \leq \frac{1}{2} \int \frac{|\nabla \phi|^2}{\phi},$$

where

$$\text{Ent}(\phi) = \int \phi \log \phi - \int \phi \cdot \log \int \phi$$

is the Euclidean entropy of a function $\phi : \mathbb{R}^n \rightarrow [0, \infty)$. There are several equivalent formulations of this inequality, including the standard one where the entropy is taken with respect to the Gaussian measure; see e.g. [1, Theorem 1.3.8].

Bobkov and Ledoux [4] showed that the Prékopa–Leindler inequality also implies Gross’s log-Sobolev inequality. For log-concave functions, one obtains a more general inequality using the same function $\alpha(t)$ from (3). Since $\log \alpha(t)$ is concave by Prékopa–Leindler, we have

$$(\log \alpha)'(0) \geq \log \alpha(1) - \log \alpha(0).$$

This gives the following theorem.

Theorem 1.2. *Let $G : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a proper lower semicontinuous convex function such that*

$$\int_{\mathbb{R}^n} e^{-G} dx = 1.$$

Then, for every positive integrable log-concave function $\phi : \mathbb{R}^n \rightarrow (0, \infty)$,

$$\int_{\mathbb{R}^n} G^* \left(-\frac{\nabla \phi}{\phi} \right) \phi dx \geq n \int_{\mathbb{R}^n} \phi dx + \text{Ent}(\phi),$$

where

$$G^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - G(x) \}$$

and

$$\text{Ent}(\phi) = \int_{\mathbb{R}^n} \phi \log \phi \, dx - \left(\int_{\mathbb{R}^n} \phi \, dx \right) \log \left(\int_{\mathbb{R}^n} \phi \, dx \right).$$

The gradient is understood almost everywhere, and the integral on the left is understood in the extended sense.

Taking

$$G(x) = \frac{|x|^2}{2} + \frac{n}{2} \log(2\pi),$$

we have

$$G^*(y) = \frac{|y|^2}{2} - \frac{n}{2} \log(2\pi),$$

and Theorem 1.2 implies the classical log-Sobolev inequality (4), restricted here to the log-concave case. Taking

$$G(x) = \begin{cases} \log \text{Vol}(B_2^n), & |x| \leq 1, \\ +\infty, & |x| > 1, \end{cases}$$

we have $G^*(y) = |y| - \log \text{Vol}(B_2^n)$, and hence obtain

$$\text{Ent}(\phi) + (n + \log \text{Vol}(B_2^n)) \int \phi \leq \int |\nabla \phi|.$$

Bobkov and Ledoux organized the computation in [4] somewhat differently, obtaining less general inequalities but without the restriction to log-concave functions.

In the remaining sections of this paper, we present several extensions and variants of the Brascamp–Lieb inequality. Our interest comes from the connection of these inequalities to convex geometry, and in particular to the B-conjecture. In [8], Cordero-Erausquin, Fradelizi and Maurey proved that if γ is the standard Gaussian measure on $\mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is an origin-symmetric convex body, then

$$(5) \quad t \mapsto \gamma(e^t K)$$

is log-concave. This answered a question which Latała attributes to Banaszczyk; see [16]. The authors of [8] conjectured that (5) remains true with the Gaussian measure replaced by any even log-concave measure on $\mathbb{R}^n$. This conjecture became known as the B-conjecture.

The B-conjecture is known in dimension $n = 2$ due to the works of Böröczky–Lutwak–Yang–Zhang [3] and Saroglou [20], and in all dimensions when the measure is a certain Gaussian mixture, due to Eskenazis–Nayar–Tkocz [10]. More relevant to us is the work [9], where the conjecture was proved for rotation-invariant log-concave measures. The proof relied crucially on an improved Brascamp–Lieb inequality for even functions. Let $V(x) = v(|x|)$ be a C^2 convex function on $\mathbb{R}^n$ which depends only on $|x|$. Let μ be the probability measure with density $Z^{-1}e^{-V}$, and let ν be

any even probability measure which is more log-concave than μ , meaning that $\frac{d\nu}{d\mu}$ is an even log-concave function. Then, for every even C^1 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(6) \quad \text{Var}_\nu f \leq \int \left\langle \left(\nabla^2 V + \frac{v'(|x|)}{|x|} \text{Id} \right)^{-1} \nabla f, \nabla f \right\rangle d\nu.$$

Since $v'(|x|)/|x| \geq 0$, this is an improvement over the classical Brascamp–Lieb inequality in the case $\nu = \mu$ and f is even. For the Gaussian measure, $v(t) = t^2/2$, and (6) reduces to a Poincaré inequality with constant $1/2$, which was precisely the original result of [8].

Using standard techniques, (6) was derived in [9] from the following Poincaré-type inequality: for every C^2 even function $u : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(7) \quad \int \frac{v'(|x|)}{|x|} |\nabla u|^2 d\nu \leq \int \|\nabla^2 u\|_{HS}^2 d\nu,$$

where $\|\nabla^2 u\|_{HS}^2 = \sum_{i,j} u_{x_i x_j}^2$ is the Hilbert–Schmidt norm. While [9] treats the rotation-invariant case, the same argument gives the following abstract criterion.

Proposition 1.3. *Let μ be an even log-concave probability measure with C^2 density $Z^{-1}e^{-V}$. Assume that there exists a matrix-valued function $T : \mathbb{R}^n \rightarrow M_{n \times n}(\mathbb{R})$ with the following properties:*

- (1) $T(x)$ is symmetric and $\nabla^2 V(x) + T(x)$ is positive definite for all $x \in \mathbb{R}^n$.
- (2) $T(x)x = \nabla V(x)$ for all $x \in \mathbb{R}^n$.
- (3) For every probability measure ν which is even and more log-concave than μ , and for every C^2 even function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, one has

$$(8) \quad \int \langle T \nabla u, \nabla u \rangle d\nu \leq \int \|\nabla^2 u\|_{HS}^2 d\nu.$$

Then, for all such ν and all C^1 even functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(9) \quad \text{Var}_\nu f \leq \int \left\langle (\nabla^2 V + T)^{-1} \nabla f, \nabla f \right\rangle d\nu,$$

and the measure μ satisfies the B-conjecture.

We prove Proposition 1.3 in Section 6. This proposition illustrates how second-order improvements of the Brascamp–Lieb inequality can have geometric consequences. The results below are motivated by this connection, although the particular extensions proved here are mostly of independent analytic interest.

In Section 5, we prove a Brascamp–Lieb inequality with a concave weight.

Theorem 1.4. *Let $K \subset \mathbb{R}^n$ be a convex domain, and let $V \in C^2(\mathbb{R}^n)$ satisfy*

$$0 < Z := \int_{\mathbb{R}^n} e^{-V} dx < \infty, \quad \nabla^2 V(x) \geq 0 \quad \text{for every } x \in K.$$

Let

$$d\mu = Z^{-1}e^{-V} dx,$$

and let $g : K \rightarrow [0, \infty)$ be a finite concave function. Then every $f \in C^1(K)$ satisfying

$$\int_K |f| d\mu < \infty, \quad \int_K f d\mu = 0,$$

satisfies

$$(10) \quad \int_K g f^2 d\mu \leq \int_K g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu,$$

where the inequality is understood in the extended sense if the right-hand side is infinite.

The restriction to a convex set K is standard. Indeed, one may approximate $\mathbf{1}_K$ by functions of the form e^{-W} , where $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth and convex, apply Brascamp–Lieb to $V + W$, and use

$$\nabla^2(V + W) = \nabla^2 V + \nabla^2 W \succeq \nabla^2 V.$$

The new point in Theorem 1.4 is the presence of the concave weight g . Since every non-negative concave function is log-concave, one could apply Brascamp–Lieb to $V - \log g$, but this gives the result under the centering condition $\int_K g f d\mu = 0$. The theorem shows that the same conclusion holds under the different condition $\int_K f d\mu = 0$.

In Section 6, we turn to the unconditional case. A function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is unconditional if

$$u(x_1, \dots, x_n) = u(\epsilon_1 x_1, \dots, \epsilon_n x_n)$$

for all $x \in \mathbb{R}^n$ and all $(\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$. Unconditional measures and sets are defined similarly. The inequality (8) holds for unconditional functions in a particularly strong form.

Theorem 1.5. *Let μ be an even log-concave probability measure on $\mathbb{R}^n$ with C^2 density $Z^{-1}e^{-V}$. Fix any matrix-valued function $T : \mathbb{R}^n \rightarrow M_{n \times n}(\mathbb{R})$ such that $T(x)$ is symmetric, $T_{ij}(x)x_i x_j \geq 0$ for all $1 \leq i, j \leq n$, and $T(x)x = \nabla V(x)$. Let ν be any measure such that $\frac{d\nu}{d\mu}$ is unconditional and log-concave, and let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 unconditional function. Then*

$$\int \langle T \nabla u, \nabla u \rangle d\nu \leq \int \|\nabla^2 u\|_{HS}^2 d\nu.$$

If μ itself is also unconditional, the argument of Proposition 1.3 applies on the subspace of unconditional functions and gives the improved Brascamp–Lieb inequality

$$\text{Var}_\nu f \leq \int \left\langle (\nabla^2 V + T)^{-1} \nabla f, \nabla f \right\rangle d\nu$$

for every admissible T and every unconditional function f . In particular, the map $t \mapsto \mu(e^t K)$ is log-concave when both μ and K are unconditional. We do not dwell on this point, since the B-conjecture in the unconditional setting has simpler proofs, as was observed in [2].

In Section 7, we prove a general improvement of the Brascamp–Lieb inequality for even functions.

Theorem 1.6. *Let μ be an even probability measure on $\mathbb{R}^n$ with density $Z^{-1}e^{-V}$, where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 and convex. Define*

$$H_V(x) = \int (\nabla^2 V(x) + \nabla^2 V(y))^{-1} d\mu(y).$$

Then every C^1 even function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$\mathrm{Var}_\mu f \leq \int \langle H_V \nabla f, \nabla f \rangle d\mu.$$

If V is not strongly convex, The quadratic form defined by H_V is still well-defined by

$$\langle H_V(x)z, z \rangle = \lim_{\epsilon \rightarrow 0^+} \int \left\langle (\nabla^2 V(x) + \nabla^2 V(y) + \epsilon \mathrm{Id})^{-1} z, z \right\rangle d\mu(y) \in [0, \infty].$$

Since $H_V \preceq (\nabla^2 V)^{-1}$, Theorem 1.6 is a strengthening of the Brascamp–Lieb inequality. If $\nabla^2 V \succeq \mathrm{Id}$, then $H_V \preceq \frac{1}{2}\mathrm{Id}$, and we recover the improved Poincaré inequality from [8],

$$\mathrm{Var}_\mu f \leq \frac{1}{2} \int |\nabla f|^2 d\mu.$$

Surprisingly, this inequality and its connection to the B-conjecture were known already to Brascamp and Lieb. It appeared in [6], but was not noticed by the community until very recently (see, e.g. the discussion in [7]). The proof of [6] is very different than the known one from [8], and relies on a “rotation trick”. A variant of the same trick proves Theorem 1.6.

Finally, in Section 8, we extend (7) to non-Euclidean norms.

Theorem 1.7. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ whose unit ball*

$$K = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

is smooth and has strictly positive Gaussian curvature at every boundary point. Let $v : [0, \infty) \rightarrow \mathbb{R}$ be C^1 and increasing, and let μ be the measure on $\mathbb{R}^n$ with density $e^{-v(\|x\|)}$. Let ν be any even measure which is more log-concave than μ . Then for every odd C^1 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(11) \quad \int \frac{v'(\|x\|)}{\|x\|} f^2 d\nu \leq \int \left\langle (\nabla^2 v)^{-1} \nabla f, \nabla f \right\rangle d\nu,$$

where $\alpha(x) = \frac{1}{2} \|x\|^2$.

When $v(t) = t^2/2$, the result follows immediately from the classical Brascamp–Lieb inequality. When K is the Euclidean ball, applying (11) to $f = u_{x_i}$ and summing over i recovers (7). For a general norm, the same argument gives, for even C^2 functions $u : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(12) \quad \int \frac{v'(\|x\|)}{\|x\|} |\nabla u|^2 d\nu \leq \int \sum_{i=1}^n \left\langle (\nabla^2 \alpha)^{-1} \nabla u_{x_i}, \nabla u_{x_i} \right\rangle d\nu.$$

The right-hand side of (12) does not seem to have a simple invariant form, and we therefore do not know how to deduce an analogue of (6) from it.

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AI use statement. The mathematical content of this paper and the initial draft were generated solely by the authors over the course of several years, with the exception of Remarks 5.2 and 5.3. Recently, ChatGPT-5.6 pro (Sol Ultra) was used during the editorial process to identify mathematical and typographical errors, improve the writing, make some arguments more detailed, and to generate the Remarks 5.2 and 5.3. For these remarks, the authors prompted chatGPT to provide potential applications of the results in Section 5, and amid many ideas that it suggested, the content of these two remarks was selected by the authors for inclusion in the paper.

2. PRELIMINARIES

We now collect the analytic and convex-geometric facts used throughout the paper. All integrations by parts below are first justified for smooth compactly supported functions and smooth uniformly convex potentials. The stated forms then follow by standard approximation arguments (see e.g. [9, Theorem 4]). When a Hessian is only non-negative, all expressions such as $\langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle$ are understood as limits after replacing V by $V + \varepsilon |x|^2/2$ and then letting $\varepsilon \rightarrow 0$.

If μ has density $Z^{-1}e^{-V}$, write

$$L_\mu u = \Delta u - \langle \nabla V, \nabla u \rangle,$$

so we have the integration by parts formula

$$(13) \quad \int g L_\mu u d\mu = - \int \langle \nabla g, \nabla u \rangle d\mu.$$

We will also use the Bochner identity

$$(14) \quad \int (L_\mu u)^2 d\mu = \int \left(\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \right) d\mu.$$

For completeness, we recall a standard proof of the Brascamp–Lieb inequality:

Theorem 2.1 (Brascamp–Lieb inequality). *Let μ be a probability measure with density $Z^{-1}e^{-V}$, where $V \in C^2$ and $\nabla^2 V \succ 0$. Then for every smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we have*

$$\text{Var}_\mu f \leq \int \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu.$$

Proof. As already stated, we may assume by standard approximation arguments that f is compactly supported. We may also assume without loss of generality that $\int f d\mu = 0$. By the standard theory of elliptic PDEs (see e.g. [11]) we may then find $u : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth such that $L_\mu u = f$.

Using (13) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \int f^2 d\mu &= \int f L_\mu u d\mu = - \int \langle \nabla f, \nabla u \rangle d\mu \\ &\leq \left(\int \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu \right)^{1/2} \left(\int \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu \right)^{1/2} \\ &\leq \left(\int \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu \right)^{1/2} \left(\int f^2 d\mu \right)^{1/2}, \end{aligned}$$

where in the last inequality we also used Bochner’s identity and the fact that $f = L_\mu u$. The claim now follows immediately. $\square$

Legendre transform. Given a convex function $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$, its Legendre transform is

$$f^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - f(y)).$$

If f is smooth and strictly convex, then

$$(15) \quad f(x) + f^*(\nabla f(x)) = \langle \nabla f(x), x \rangle,$$

and $\nabla f^*(\nabla f(x)) = x$.

Given convex functions $f_0, f_1 : \mathbb{R}^n \rightarrow (-\infty, \infty]$ and $0 \leq t \leq 1$, consider the inf-convolution f_t defined by

$$f_t(z) = ((1-t)f_0 \star t f_1)(z) = \inf_{(1-t)x + ty = z} ((1-t)f_0(x) + t f_1(y)).$$

Then

$$(16) \quad f_t^* = (1-t)f_0^* + t f_1^*.$$

In other words, the Legendre transform linearizes the inf-convolution.

Of key importance for us will be the first and second variation formulas for the Legendre transform. These are well-known, but we provide a proof sketch for convenience:

Lemma 2.2. *Let $I \subset \mathbb{R}$ be an open interval, and let*

$$(x, t) \mapsto f_t(x)$$

be a C^2 function on $\mathbb{R}^n \times I$. Assume that there exists $m > 0$ such that

$$\nabla^2 f_t(x) \succeq m \text{Id}$$

for every $x \in \mathbb{R}^n$ and $t \in I$. Then f_t^* is finite and C^2 on $\mathbb{R}^n$. Moreover, for every $y \in \mathbb{R}^n$,

$$\frac{d}{dt} f_t^*(y) = -\dot{f}_t(\nabla f_t^*(y)),$$

and

$$\begin{aligned} \frac{d^2}{dt^2} f_t^*(y) &= -\ddot{f}_t(\nabla f_t^*(y)) \\ &\quad + \left\langle (\nabla^2 f_t(\nabla f_t^*(y)))^{-1} \nabla \dot{f}_t(\nabla f_t^*(y)), \nabla \dot{f}_t(\nabla f_t^*(y)) \right\rangle. \end{aligned}$$

Proof. The lower Hessian bound implies that f_t is superlinear and strictly convex. Hence, for every $y \in \mathbb{R}^n$, the supremum defining $f_t^*(y)$ is attained at a unique point

$$x_t = \nabla f_t^*(y),$$

characterized by

$$\nabla f_t(x_t) = y.$$

Thus

$$f_t^*(y) = \langle y, x_t \rangle - f_t(x_t).$$

Differentiating with respect to t , and using $\nabla f_t(x_t) = y$, gives

$$\frac{d}{dt} f_t^*(y) = -\dot{f}_t(x_t).$$

Differentiating the identity $\nabla f_t(x_t) = y$, we obtain

$$\nabla \dot{f}_t(x_t) + \nabla^2 f_t(x_t) \dot{x}_t = 0,$$

and therefore

$$\dot{x}_t = -(\nabla^2 f_t(x_t))^{-1} \nabla \dot{f}_t(x_t).$$

Differentiating the first-variation formula once more now gives

$$\begin{aligned} \frac{d^2}{dt^2} f_t^*(y) &= -\ddot{f}_t(x_t) - \left\langle \nabla \dot{f}_t(x_t), \dot{x}_t \right\rangle \\ &= -\ddot{f}_t(x_t) + \left\langle (\nabla^2 f_t(x_t))^{-1} \nabla \dot{f}_t(x_t), \nabla \dot{f}_t(x_t) \right\rangle. \end{aligned}$$

Substituting $x_t = \nabla f_t^*(y)$ proves the result. $\square$

3. SECOND VARIATION OF PRÉKOPA–LEINDLER

In this section we prove Theorem 1.1. We begin with an improved Brascamp–Lieb inequality for even functions.

Proposition 3.1. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be an even C^2 function which satisfies $\nabla^2 V \succeq c\text{Id}$ for some $c > 0$. Let μ be the probability measure with density $\frac{1}{Z}e^{-V}$. Then for every smooth even function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we have*

$$\text{Var}_\mu f \leq \int \left\langle (\nabla^2 V + c\text{Id})^{-1} \nabla f, \nabla f \right\rangle d\mu.$$

Note that one can obtain Proposition 3.1 as a corollary of Theorem 1.6. Still, we will give a more standard L^2 -type proof.

Proof. We may assume f is compactly supported and $\int f d\mu = 0$. Choose $u : \mathbb{R}^n \rightarrow \mathbb{R}$ which solves $L_\mu u = f$. Since V and f are even, the solution u is also even. Hence each derivative u_i is odd and has mean zero. Using the fact that $\nabla^2 V \succeq c\text{Id}$ and the usual Brascamp–Lieb inequality for μ we obtain

$$c \int |\nabla u|^2 d\mu = c \sum_{i=1}^n \int u_i^2 d\mu \leq \sum_{i=1}^n \int |\nabla u_i|^2 d\mu = \int \|\nabla^2 u\|_{HS}^2 d\mu.$$

Using this and the Bochner identity (14) we deduce that

$$\int f^2 d\mu \geq \int \left\langle (\nabla^2 V + c\text{Id}) \nabla u, \nabla u \right\rangle d\mu.$$

The rest of the proof is identical to the proof given in Theorem 2.1. Using integration by parts and the Cauchy–Schwarz inequality we have

$$\begin{aligned} \int f^2 d\mu &= \int f L_\mu u d\mu = - \int \langle \nabla f, \nabla u \rangle d\mu \\ &\leq \left(\int \left\langle (\nabla^2 V + c\text{Id})^{-1} \nabla f, \nabla f \right\rangle d\mu \right)^{1/2} \\ &\quad \times \left(\int \left\langle (\nabla^2 V + c\text{Id}) \nabla u, \nabla u \right\rangle d\mu \right)^{1/2} \\ &\leq \left(\int \left\langle (\nabla^2 V + c\text{Id})^{-1} \nabla f, \nabla f \right\rangle d\mu \right)^{1/2} \left(\int f^2 d\mu \right)^{1/2}. \end{aligned}$$

This proves the result. $\square$

Next, we explain the well-known connection between the Prékopa–Leindler inequality and the Brascamp–Lieb inequality.

Proposition 3.2. *Let $f_0, f_1 \in C^2(\mathbb{R}^n)$, and assume that there exist constants $0 < m \leq M < \infty$ such that*

$$m\text{Id} \preceq \nabla^2 f_0(x), \nabla^2 f_1(x) \preceq M\text{Id} \quad \text{for every } x \in \mathbb{R}^n.$$

For $0 \leq t \leq 1$, define

$$f_t^* = (1-t)f_0^* + tf_1^*$$

and

$$\psi_t(x) = (f_1^* - f_0^*)(\nabla f_t(x)).$$

Let ν be a nonzero Borel measure on $\mathbb{R}^n$. Assume that, for some a satisfying $0 < a < m/2$,

$$\int_{\mathbb{R}^n} e^{-a|x|^2} d\nu(x) < \infty.$$

Set

$$\alpha(t) = \int_{\mathbb{R}^n} e^{-f_t} d\nu$$

and let μ_t be the probability measure defined by

$$\frac{d\mu_t}{d\nu} = \frac{e^{-f_t}}{\alpha(t)}.$$

Then α is positive and twice differentiable on $(0, 1)$, and

$$(\log \alpha)''(t) \leq 0$$

is equivalent to

$$\text{Var}_{\mu_t}(\psi_t) \leq \int_{\mathbb{R}^n} \langle (\nabla^2 f_t)^{-1} \nabla \psi_t, \nabla \psi_t \rangle d\mu_t.$$

Consequently, α is log-concave on $[0, 1]$ if and only if the preceding inequality holds for every $0 < t < 1$.

Proof. The Hessian bounds on f_0 and f_1 imply

$$\frac{1}{M}\text{Id} \preceq \nabla^2 f_0^*, \nabla^2 f_1^* \preceq \frac{1}{m}\text{Id}.$$

Therefore

$$\frac{1}{M}\text{Id} \preceq \nabla^2((1-t)f_0^* + tf_1^*) \preceq \frac{1}{m}\text{Id},$$

and hence

$$m\text{Id} \preceq \nabla^2 f_t \preceq M\text{Id}$$

for every $0 \leq t \leq 1$.

Applying Lemma 2.2 to the family

$$(1-t)f_0^* + tf_1^*,$$

whose Legendre transform is f_t , gives

$$\dot{f}_t = -\psi_t.$$

The second-variation formula gives

$$\ddot{f}_t = \langle (\nabla^2 f_t)^{-1} \nabla \psi_t, \nabla \psi_t \rangle.$$

The uniform Hessian bounds also imply, uniformly in $t \in [0, 1]$,

$$f_t(x) \geq \frac{m}{2}|x|^2 - C(1 + |x|),$$

as well as

$$|\psi_t(x)| \leq C(1 + |x|^2), \quad |\nabla \psi_t(x)| \leq C(1 + |x|).$$

Together with the assumption on ν , these estimates justify differentiation under the integral sign. We therefore have

$$\alpha'(t) = - \int_{\mathbb{R}^n} \dot{f}_t e^{-f_t} d\nu = \int_{\mathbb{R}^n} \psi_t e^{-f_t} d\nu$$

and

$$\begin{aligned} \alpha''(t) &= \int_{\mathbb{R}^n} (\dot{f}_t^2 - \ddot{f}_t) e^{-f_t} d\nu \\ &= \int_{\mathbb{R}^n} (\psi_t^2 - \langle (\nabla^2 f_t)^{-1} \nabla \psi_t, \nabla \psi_t \rangle) e^{-f_t} d\nu. \end{aligned}$$

Dividing by $\alpha(t)$, we obtain

$$(\log \alpha)''(t) = \text{Var}_{\mu_t}(\psi_t) - \int_{\mathbb{R}^n} \langle (\nabla^2 f_t)^{-1} \nabla \psi_t, \nabla \psi_t \rangle d\mu_t.$$

This proves the equivalence. $\square$

Proof of Theorem 1.1. Write $f_0 = f$ and $f_1 = g$. It is enough to prove the result when

$$h = f_t = ((1-t)f_0^* + tf_1^*)^*$$

because the assumption of the theorem is exactly that $h \leq f_t$, and hence $e^{-h} \geq e^{-f_t}$.

By approximation, we may assume f_0 and f_1 satisfy the regularity assumptions of Proposition 3.2 (Since $m \geq 2c$, the integrability hypothesis in Proposition 3.2 is satisfied by choosing $c/2 < a < m/2$). The assumptions of the theorem then imply that $\nabla^2 f_0, \nabla^2 f_1 \succeq 2c \cdot \text{Id}$. Using the relation

$$f_t^* = (1-t)f_0^* + tf_1^*,$$

and the formula $\nabla^2 f_t^*(x) = (\nabla^2 f_t(\nabla f_t^*(x)))^{-1}$, it follows that $\nabla^2 f_t \succeq 2c \cdot \text{Id}$ for all $t \in [0, 1]$.

Next we define

$$\alpha(t) = \int e^{-f_t(x)} e^{\frac{c}{2}|x|^2} dx = \int e^{-f_t} d\nu,$$

where ν has density $e^{\frac{c}{2}|x|^2}$. By Proposition 3.2 we know that the inequality $(\log \alpha)''(t) \leq 0$ is equivalent to

$$(17) \quad \text{Var}_{\mu_t}(\psi_t) \leq \int \langle (\nabla^2 f_t)^{-1} \nabla \psi_t, \nabla \psi_t \rangle d\mu_t,$$

where $\psi_t = (f_1^* - f_0^*) \circ \nabla f_t$ and $\frac{d\mu_t}{d\nu} = \frac{1}{Z_t} e^{-f_t}$. In other words, μ_t has density proportional to e^{-V_t} , where $V_t = f_t - \frac{c}{2}|x|^2$. But then $\nabla^2 V_t \succeq (2c - c) \cdot \text{Id} = c \text{Id}$, and $\nabla^2 f_t = \nabla^2 V_t + c \text{Id}$, so the validity of (17) follows immediately from Proposition 3.1 (note that ψ_t is even since f_0^* , f_1^* and f_t are all even).

It follows that $(\log \alpha)''(t) \leq 0$ for all $t \in [0, 1]$, so α is log-concave. Therefore

$$\alpha(t) \geq \alpha(0)^{1-t} \alpha(1)^t,$$

which is the desired inequality. $\square$

4. A GENERALIZED LOG-SOBOLEV INEQUALITY

We start by restating Theorem 1.2. Throughout this section, all unspecified integrals are over $\mathbb{R}^n$.

Theorem 4.1 (Minkowski's first inequality for functions, or generalized log-Sobolev inequality). *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex, and let $G : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Assume that*

$$0 < Z_F := \int_{\mathbb{R}^n} e^{-F} dx < \infty, \quad 0 < Z_G := \int_{\mathbb{R}^n} e^{-G} dx < \infty.$$

Then

$$(18) \quad \int_{\mathbb{R}^n} G^*(\nabla F) e^{-F} dx \geq \int_{\mathbb{R}^n} F^*(\nabla F) e^{-F} dx + Z_F \log \frac{Z_G}{Z_F}.$$

Equivalently,

$$(19) \quad \int_{\mathbb{R}^n} G^*(\nabla F) e^{-F} dx \geq n Z_F - \int_{\mathbb{R}^n} F e^{-F} dx + Z_F \log \frac{Z_G}{Z_F}.$$

Here ∇F is understood almost everywhere, and the integral on the left-hand side is understood as an extended integral in $(-\infty, +\infty]$.

In particular, if $Z_F = Z_G$, then

$$\int_{\mathbb{R}^n} G^*(\nabla F) e^{-F} dx \geq \int_{\mathbb{R}^n} F^*(\nabla F) e^{-F} dx.$$

Thus, among the admissible convex functions G satisfying $Z_G = Z_F$, the left-hand side is minimized at $G = F$.

Proof. We divide the proof into the regular case and an approximation argument.

Step 1: the regular case. Assume that $F, G \in C^2(\mathbb{R}^n)$ and that there exist constants $0 < m \leq M < \infty$ such that

$$mI \leq \nabla^2 F(x), \nabla^2 G(x) \leq MI \quad \text{for every } x \in \mathbb{R}^n.$$

For $0 \leq t \leq 1$, define

$$H_t = (1-t)F^* + tG^*, \quad F_t = H_t^*, \quad \alpha(t) = \int_{\mathbb{R}^n} e^{-F_t} dx.$$

The Prékopa–Leindler inequality implies that $t \mapsto \log \alpha(t)$ is concave. Since $F_0 = F$ and $F_1 = G$, we obtain

$$(20) \quad \frac{\alpha'(0)}{\alpha(0)} \geq \log \alpha(1) - \log \alpha(0) = \log \frac{Z_G}{Z_F}.$$

For every x and t , the supremum defining $F_t(x)$ has a unique maximizer $y_t(x)$ under our assumption. Thus

$$F_t(x) = \langle x, y_t(x) \rangle - (1-t)F^*(y_t(x)) - tG^*(y_t(x)).$$

Differentiating this identity in t , and using Lemma 2.2, gives

$$\frac{\partial}{\partial t} F_t(x) = F^*(y_t(x)) - G^*(y_t(x)).$$

At $t = 0$, the maximizing point is

$$y_0(x) = \nabla F(x).$$

Consequently,

$$\left. \frac{\partial}{\partial t} F_t(x) \right|_{t=0+} = F^*(\nabla F(x)) - G^*(\nabla F(x)).$$

The two-sided Hessian bounds imply uniform quadratic estimates for F_t , for t in a neighborhood of zero. More precisely, there are constants $c, C > 0$ such that

$$F_t(x) \geq c|x|^2 - C$$

and

$$\left| \frac{\partial}{\partial t} F_t(x) \right| \leq C(1 + |x|^2).$$

Therefore differentiation under the integral sign is justified, and

$$\alpha'(0) = \int_{\mathbb{R}^n} (G^*(\nabla F) - F^*(\nabla F)) e^{-F} dx.$$

Substitution into (20) gives

$$\int_{\mathbb{R}^n} G^*(\nabla F) e^{-F} dx \geq \int_{\mathbb{R}^n} F^*(\nabla F) e^{-F} dx + Z_F \log \frac{Z_G}{Z_F}.$$

In view of our regularity assumption, Fenchel's identity and integration by parts give

$$F^*(\nabla F(x)) = \langle x, \nabla F(x) \rangle - F(x)$$

and

$$\int_{\mathbb{R}^n} \langle x, \nabla F(x) \rangle e^{-F(x)} dx = nZ_F.$$

Hence

$$\int_{\mathbb{R}^n} F^*(\nabla F) e^{-F} dx = nZ_F - \int_{\mathbb{R}^n} F e^{-F} dx.$$

This proves both formulations in the regular case.

Step 2: approximation.

We first record some standard integrability facts. Put

$$f = e^{-F}.$$

Since F is finite and convex and f is integrable, there exist constants $c, C > 0$ such that

$$f(x) \leq C e^{-c|x|}.$$

Moreover,

$$(21) \quad f \in W^{1,1}(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} (1 + |x|) |\nabla f(x)| dx < \infty.$$

For completeness, these assertions follow from the coarea formula. Indeed, the superlevel sets of f are bounded convex sets contained in balls whose radii are bounded by $C(1 + |\log t|)$. Monotonicity of the surface area of convex bodies under inclusion therefore gives

$$\int_0^{\|f\|_\infty} (1 + |\log t|)^n dt < \infty$$

as an upper bound for the relevant coarea integrals.

In particular,

$$(22) \quad \int_{\mathbb{R}^n} \nabla F e^{-F} dx = - \int_{\mathbb{R}^n} \nabla f dx = 0.$$

Also, $F e^{-F} = -f \log f$ is integrable.

If $x_0 \in \text{dom } G$, then

$$G^*(y) \geq \langle x_0, y \rangle - G(x_0).$$

Together with (21), this shows that the negative part of $G^*(\nabla F) e^{-F}$ is integrable. Thus the left-hand side of the theorem is well-defined as an extended integral.

Part 2a: regular F and G with bounded effective domain. Assume that F satisfies the regularity assumptions of Step 1 and that

$$D := \overline{\text{dom } G}$$

is compact. Set

$$H = G^*.$$

Then H is finite and globally Lipschitz. Let L be a Lipschitz constant for H . Choose a smooth nonnegative symmetric mollifier ρ , supported in the Euclidean unit ball and satisfying $\int \rho = 1$, and put

$$\rho_\delta(y) = \delta^{-n} \rho(y/\delta), \quad K_\delta = H * \rho_\delta, \quad H_\delta(y) = K_\delta(y) + \frac{\delta}{2} |y|^2.$$

Since ρ_δ is centered, convexity and Lipschitz continuity of H give

$$(23) \quad H(y) \leq K_\delta(y) \leq H(y) + c_\rho L \delta,$$

where c_ρ depends only on ρ . Moreover,

$$\delta I \leq \nabla^2 H_\delta \leq \left(\delta + \frac{C_\rho L}{\delta} \right) I.$$

Define

$$G_\delta = H_\delta^*.$$

The preceding two-sided Hessian bounds imply corresponding two-sided positive Hessian bounds for G_δ . Thus G_δ satisfies the regularity assumptions of Step 1.

We claim that

$$(24) \quad \int_{\mathbb{R}^n} e^{-G_\delta} dx \longrightarrow Z_G.$$

Let

$$G_\delta^{\text{MY}}(x) = \inf_{z \in \mathbb{R}^n} \left\{ G(z) + \frac{|x - z|^2}{2\delta} \right\}$$

be the Moreau envelope of G . From (23) and the order-reversing property of the Legendre transform, we obtain

$$G - c_\rho L \delta \leq K_\delta^* \leq G.$$

Since

$$G_\delta = K_\delta^* \square \frac{|\cdot|^2}{2\delta},$$

it follows that

$$(25) \quad G_\delta^{\text{MY}} - c_\rho L \delta \leq G_\delta \leq G.$$

Because $G_\delta^{\text{MY}} \rightarrow G$ pointwise in the extended sense, we have $G_\delta \rightarrow G$ pointwise in the extended sense.

Since D is compact and G is bounded below on its effective domain, there is $b \in \mathbb{R}$ such that

$$G_\delta^{\text{MY}}(x) \geq b + \frac{\text{dist}(x, D)^2}{2\delta}.$$

Consequently, for $0 < \delta \leq 1$, the functions e^{-G_δ} are dominated by a fixed integrable function proportional to

$$\exp\left(-\frac{\text{dist}(x, D)^2}{2}\right).$$

Dominated convergence proves (24).

Applying Step 1 to F and G_δ , whose conjugate is H_δ , gives

$$\int_{\mathbb{R}^n} H_\delta(\nabla F) e^{-F} dx \geq nZ_F - \int_{\mathbb{R}^n} F e^{-F} dx + Z_F \log \frac{\int e^{-G_\delta}}{Z_F}.$$

On the other hand, (23) gives

$$H_\delta(y) \leq H(y) + c_\rho L\delta + \frac{\delta}{2}|y|^2.$$

Since the Hessian bounds on F imply

$$\int_{\mathbb{R}^n} |\nabla F|^2 e^{-F} dx < \infty,$$

we may let $\delta \downarrow 0$ and obtain

$$(26) \quad \int_{\mathbb{R}^n} G^*(\nabla F) e^{-F} dx \geq nZ_F - \int_{\mathbb{R}^n} F e^{-F} dx + Z_F \log \frac{Z_G}{Z_F}.$$

Part 2b: arbitrary F and G with bounded effective domain. Continue to assume that $\overline{\text{dom } G}$ is compact, and write again

$$H = G^*.$$

Thus H is finite and globally Lipschitz.

Let γ_s denote the centered Gaussian density with covariance sI , and define

$$f_s = f * \gamma_s, \quad F_s = -\log f_s.$$

By Prékopa's theorem, F_s is convex. It is also smooth, and the Gaussian differentiation formula gives

$$0 \leq \nabla^2 F_s \leq \frac{1}{s}I.$$

For $\varepsilon > 0$, define

$$F_{s,\varepsilon}(x) = F_s(x) + \frac{\varepsilon}{2}|x|^2.$$

Then

$$\varepsilon I \leq \nabla^2 F_{s,\varepsilon} \leq \left(\frac{1}{s} + \varepsilon\right) I.$$

Therefore Part 2a applies to $F_{s,\varepsilon}$ and G .

Letting $\varepsilon \downarrow 0$, by monotone convergence theorem, we have

$$\int_{\mathbb{R}^n} e^{-F_{s,\varepsilon}} dx \longrightarrow \int_{\mathbb{R}^n} f_s dx = Z_F$$

and, by dominated convergence,

$$\int_{\mathbb{R}^n} F_{s,\varepsilon} e^{-F_{s,\varepsilon}} dx \longrightarrow \int_{\mathbb{R}^n} F_s f_s dx.$$

Since H is Lipschitz, ∇F_s has at most linear growth, and f_s has exponential tails, we also have

$$\int_{\mathbb{R}^n} H(\nabla F_{s,\varepsilon}) e^{-F_{s,\varepsilon}} dx \longrightarrow \int_{\mathbb{R}^n} H(\nabla F_s) f_s dx.$$

It follows from (26) that

$$(27) \quad \int_{\mathbb{R}^n} H(\nabla F_s) f_s dx \geq n Z_F - \int_{\mathbb{R}^n} F_s f_s dx + Z_F \log \frac{Z_G}{Z_F}.$$

We now compare the regularized left-hand side with the original one. Since

$$\nabla f = -f \nabla F,$$

differentiation of the convolution gives

$$\nabla F_s(x) f_s(x) = \int_{\mathbb{R}^n} \nabla F(y) f(y) \gamma_s(x-y) dy.$$

Thus $\nabla F_s(x)$ is the barycenter of $\nabla F(y)$ with respect to the probability measure

$$\frac{f(y) \gamma_s(x-y)}{f_s(x)} dy.$$

Convexity of H therefore yields

$$H(\nabla F_s(x)) f_s(x) \leq \int_{\mathbb{R}^n} H(\nabla F(y)) f(y) \gamma_s(x-y) dy.$$

Integrating in x gives the crucial estimate

$$(28) \quad \int_{\mathbb{R}^n} H(\nabla F_s) f_s dx \leq \int_{\mathbb{R}^n} H(\nabla F) f dx.$$

Finally, $f_s \rightarrow f$ pointwise and locally uniformly. The exponential tail estimate for f implies, uniformly for $0 < s \leq 1$, that

$$f_s(x) \leq C' e^{-c'|x|}.$$

It follows by dominated convergence that

$$\int_{\mathbb{R}^n} f_s \log f_s dx \longrightarrow \int_{\mathbb{R}^n} f \log f dx,$$

or equivalently,

$$(29) \quad \int_{\mathbb{R}^n} F_s f_s dx \longrightarrow \int_{\mathbb{R}^n} F e^{-F} dx.$$

Combining (27), (28), and (29), and then letting $s \downarrow 0$, proves (19) whenever $\overline{\text{dom } G}$ is compact.

Part 2c: arbitrary G . Choose $x_0 \in \text{dom } G$. For all sufficiently large R , let

$$G_R = G + \iota_{\overline{B_R}},$$

where $\iota_{\overline{B_R}}$ is zero on $\overline{B_R}$ and $+\infty$ outside $\overline{B_R}$. Then

$$e^{-G_R} = e^{-G} \mathbf{1}_{\overline{B_R}}, \quad Z_{G_R} \uparrow Z_G.$$

Moreover,

$$G_R^*(y) = \sup_{x \in \overline{B_R}} \{ \langle x, y \rangle - G(x) \} \uparrow G^*(y).$$

For every $R \geq |x_0|$,

$$G_R^*(y) \geq \langle x_0, y \rangle - G(x_0).$$

Consequently,

$$G_R^*(\nabla F) - \langle x_0, \nabla F \rangle + G(x_0)$$

is a nonnegative increasing sequence. By monotone convergence and (21),

$$\int_{\mathbb{R}^n} G_R^*(\nabla F) e^{-F} dx \uparrow \int_{\mathbb{R}^n} G^*(\nabla F) e^{-F} dx.$$

Applying the already proved compact-domain case to G_R and letting $R \rightarrow \infty$ proves (19) for arbitrary G .

It remains to identify the first two terms on the right-hand side. The exponential tail estimate implies that F is coercive, so it has a minimizer $a \in \mathbb{R}^n$. By convexity,

$$\langle x - a, \nabla F(x) \rangle \geq 0$$

at every differentiability point of F .

Let χ_R be radial cutoff functions centered at a , increasing pointwise to one, such that

$$|(x - a) \cdot \nabla \chi_R(x)| \leq C$$

and $\nabla \chi_R$ is supported in $\{R \leq |x - a| \leq 2R\}$. Integration by parts gives

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_R \langle x - a, \nabla F \rangle e^{-F} dx &= - \int_{\mathbb{R}^n} \chi_R \langle x - a, \nabla f \rangle dx \\ &= n \int_{\mathbb{R}^n} \chi_R f dx + \int_{\mathbb{R}^n} \langle x - a, \nabla \chi_R \rangle f dx. \end{aligned}$$

The last term tends to zero as $R \rightarrow \infty$. Since the integrand on the left is nonnegative, monotone convergence yields

$$\int_{\mathbb{R}^n} \langle x - a, \nabla F \rangle e^{-F} dx = n Z_F.$$

Together with (22), this gives

$$\int_{\mathbb{R}^n} \langle x, \nabla F \rangle e^{-F} dx = nZ_F.$$

Finally, Fenchel's identity gives almost everywhere

$$F^*(\nabla F(x)) = \langle x, \nabla F(x) \rangle - F(x).$$

Therefore

$$\int_{\mathbb{R}^n} F^*(\nabla F) e^{-F} dx = nZ_F - \int_{\mathbb{R}^n} F e^{-F} dx.$$

Substituting this identity into (19) proves (18). $\square$

Remark 4.2. Let $K, L \subset \mathbb{R}^n$ be convex bodies. We first assume that $|K| = |L|$. For $m > 0$, set

$$F_m = md_K, \quad G_m = md_L,$$

and

$$A_m = \int_{\mathbb{R}^n} e^{-md_K} dx, \quad B_m = \int_{\mathbb{R}^n} e^{-md_L} dx.$$

The conjugates of these functions are

$$F_m^* = h_K + \iota_{mB_2^n}, \quad G_m^* = h_L + \iota_{mB_2^n}.$$

Since $\nabla F_m = m\nabla d_K \in mB_2^n$ almost everywhere, Theorem 4.1 gives

$$(30) \quad \int_{\mathbb{R}^n} (h_L(m\nabla d_K) - h_K(m\nabla d_K)) e^{-md_K} dx \geq A_m \log \frac{B_m}{A_m}.$$

For any convex body M , the coarea formula gives

$$\begin{aligned} \int_{\mathbb{R}^n} h_M(m\nabla d_K) e^{-md_K} dx &= m \int_0^\infty e^{-mr} \int_{\partial(K+rB_2^n)} h_M(n_{K+rB_2^n}) d\mathcal{H}^{n-1} dr \\ &= n \int_0^\infty e^{-s} V_1\left(K + \frac{s}{m} B_2^n, M\right) ds. \end{aligned}$$

Here we used

$$\int_{\partial C} h_M(n_C) d\mathcal{H}^{n-1} = nV_1(C, M).$$

Therefore, by continuity of mixed volumes and dominated convergence,

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} h_M(m\nabla d_K) e^{-md_K} dx = nV_1(K, M).$$

Indeed, for $m \geq 1$ the integrand is dominated by

$$ne^{-s} V_1(K + sB_2^n, M),$$

which is integrable because $V_1(K + sB_2^n, M)$ is a polynomial in s .

Moreover,

$$A_m \longrightarrow |K|, \quad B_m \longrightarrow |L|,$$

by dominated convergence. Since $|K| = |L|$, the right-hand side of (30) tends to zero. Passing to the limit gives

$$V_1(K, L) \geq V_1(K, K) = |K|.$$

For arbitrary K and L , apply the equal-volume case to K and λL , where

$$\lambda = \left(\frac{|K|}{|L|} \right)^{1/n}.$$

Using $V_1(K, \lambda L) = \lambda V_1(K, L)$, we obtain

$$V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}},$$

which is Minkowski's first inequality.

Recall that, for a nonnegative integrable function ϕ and a probability measure μ , its entropy relative to μ is

$$\text{Ent}_\mu(\phi) := \int \phi \log \phi \, d\mu - \left(\int \phi \, d\mu \right) \log \left(\int \phi \, d\mu \right),$$

where $0 \log 0 = 0$. Jensen's inequality gives $\text{Ent}_\mu(\phi) \geq 0$. For Lebesgue measure we write $\text{Ent}(\phi)$. Notice that entropy relative to Lebesgue measure need not be nonnegative.

If $Z_G = 1$ and $\phi = e^{-F}$, then

$$\text{Ent}(\phi) = - \int_{\mathbb{R}^n} F e^{-F} \, dx - \left(\int_{\mathbb{R}^n} e^{-F} \, dx \right) \log \left(\int_{\mathbb{R}^n} e^{-F} \, dx \right).$$

Thus Theorem 4.1 implies the following.

Corollary 4.3. *Let $\phi = e^{-F}$ be a nonzero integrable log-concave function, and let $G : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that*

$$\int_{\mathbb{R}^n} e^{-G} \, dx = 1.$$

Under the hypotheses of Theorem 4.1,

$$\int_{\mathbb{R}^n} G^* \left(-\frac{\nabla \phi}{\phi} \right) \phi \, dx \geq n \int_{\mathbb{R}^n} \phi \, dx + \text{Ent}(\phi).$$

The quotient $-\nabla \phi / \phi$ is taken on $\{\phi > 0\}$; the nonsmooth cases below follow by convex approximation.

We now record several special choices of G .

Let $\omega_n = |B_2^n|$ and choose

$$G^*(y) = |y| - \log \omega_n.$$

Then

$$G(x) = \iota_{B_2^n}(x) + \log \omega_n, \quad \int_{\mathbb{R}^n} e^{-G} dx = 1.$$

The preceding corollary therefore gives the following result; see also Bobkov and Ledoux [4].

Corollary 4.4 (L^1 log-Sobolev inequality). *Let ϕ be a nonzero nonnegative integrable log-concave function in $BV(\mathbb{R}^n)$. Then*

$$\text{Ent}(\phi) + (n + \log \omega_n) \int_{\mathbb{R}^n} \phi dx \leq |D\phi|(\mathbb{R}^n).$$

In particular, if $\phi \in W^{1,1}(\mathbb{R}^n)$, then

$$\text{Ent}(\phi) + (n + \log \omega_n) \int_{\mathbb{R}^n} \phi dx \leq \int_{\mathbb{R}^n} |\nabla \phi| dx.$$

Alternatively, choose

$$G(x) = \frac{|x|^2}{2} + \frac{n}{2} \log(2\pi).$$

Then

$$G^*(y) = \frac{|y|^2}{2} - \frac{n}{2} \log(2\pi), \quad \int_{\mathbb{R}^n} e^{-G} dx = 1.$$

For a positive log-concave function ϕ , the entropy form of Theorem 4.1 gives

$$(31) \quad \text{Ent}(\phi) + n \log(\sqrt{2\pi} e) \int_{\mathbb{R}^n} \phi dx \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla \phi|^2}{\phi} dx.$$

Writing $\phi = f^2$ gives

$$(32) \quad \text{Ent}(f^2) + n \log(\sqrt{2\pi} e) \int_{\mathbb{R}^n} f^2 dx \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

for nonnegative log-concave f . In fact, (32) is valid for every $f \in W^{1,2}(\mathbb{R}^n)$ and is equivalent to the classical Gaussian log-Sobolev inequality (in the partial case of log-concave functions):

Corollary 4.5 (Classical Gaussian log-Sobolev inequality). *Let γ be the standard Gaussian probability measure on $\mathbb{R}^n$. Then, for every log-concave $g \in W^{1,2}(\gamma)$,*

$$\text{Ent}_\gamma(g^2) \leq 2 \int_{\mathbb{R}^n} |\nabla g|^2 d\gamma.$$

Remark 4.6. *In fact the above result is classical and holds for all $W^{1,2}(\gamma)$, not just log-concave ones.*

Indeed, let

$$\rho(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$$

be the Gaussian density and set $f = g\sqrt{\rho}$. A direct integration by parts gives the identity

$$\begin{aligned} 2 \int_{\mathbb{R}^n} |\nabla f|^2 dx - \text{Ent}(f^2) - n \log(\sqrt{2\pi} e) \int_{\mathbb{R}^n} f^2 dx \\ = 2 \int_{\mathbb{R}^n} |\nabla g|^2 d\gamma - \text{Ent}_\gamma(g^2), \end{aligned}$$

which proves the equivalence.

More generally, let $p > 1$, let $q = p/(p-1)$, and set

$$Z_{n,q} = \int_{\mathbb{R}^n} e^{-|x|^q/q} dx = |S^{n-1}| \int_0^\infty r^{n-1} e^{-r^q/q} dr.$$

Choose

$$G^*(y) = \frac{|y|^p}{p} - \log Z_{n,q}.$$

Since

$$G(x) = \frac{|x|^q}{q} + \log Z_{n,q},$$

we have $\int e^{-G} = 1$.

Corollary 4.7 (L^p log-Sobolev inequality). *Let $p > 1$, let $q = p/(p-1)$, and let f be a nonzero nonnegative log-concave function in $W^{1,p}(\mathbb{R}^n)$. Then*

$$\text{Ent}(f^p) + C_{n,p} \int_{\mathbb{R}^n} f^p dx \leq p^{p-1} \int_{\mathbb{R}^n} |\nabla f|^p dx,$$

where

$$C_{n,p} = n + \log Z_{n,q}.$$

Equivalently,

$$C_{n,p} = n + \log \left(|S^{n-1}| \int_0^\infty r^{n-1} e^{-r^q/q} dr \right).$$

Proof. Apply Corollary 4.3 to $\phi = f^p$. On $\{f > 0\}$,

$$-\frac{\nabla \phi}{\phi} = -p \frac{\nabla f}{f},$$

and hence

$$\begin{aligned} \int_{\mathbb{R}^n} G^* \left(-\frac{\nabla \phi}{\phi} \right) \phi \, dx &= p^{p-1} \int_{\mathbb{R}^n} |\nabla f|^p \, dx \\ &\quad - (\log Z_{n,q}) \int_{\mathbb{R}^n} f^p \, dx. \end{aligned}$$

The conclusion follows. $\square$

The same L^p inequality is valid without the log-concavity assumption, by the sharp Euclidean L^p log-Sobolev inequality. The argument above only proves it directly on the log-concave class.

We finish with a version relative to an arbitrary log-concave probability measure.

Theorem 4.8 (Weighted generalized log-Sobolev inequality). *Let*

$$d\mu(x) = Z^{-1} e^{-V(x)} \, dx,$$

where $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous and convex, and $0 < Z < \infty$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and continuously differentiable, and let $G : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex. Assume that

$$0 < \int e^{-f} \, d\mu < \infty, \quad 0 < \int e^{-G} \, d\mu < \infty,$$

and that, for some $\delta > 0$,

$$\int \exp \left(-f + \delta [G^*(\nabla f) - f^*(\nabla f)]_+ \right) \, d\mu < \infty,$$

while

$$\int [G^*(\nabla f) - f^*(\nabla f)]_- e^{-f} \, d\mu < \infty.$$

Then

$$\begin{aligned} \int G^*(\nabla f) e^{-f} \, d\mu &\geq \text{Ent}_\mu(e^{-f}) + \int \langle x, \nabla f(x) \rangle e^{-f} \, d\mu \\ (33) \quad &\quad + \left(\int e^{-f} \, d\mu \right) \log \left(\int e^{-G} \, d\mu \right). \end{aligned}$$

In particular, if

$$\int e^{-G} \, d\mu = 1$$

and f attains its minimum at the origin, then

$$\int G^*(\nabla f) e^{-f} \, d\mu \geq \text{Ent}_\mu(e^{-f}).$$

This applies, in particular, when f is even.

Proof. For $0 < t < 1$, let

$$f_t = ((1-t)f^* + tG^*)^*.$$

For every $x, y \in \mathbb{R}^n$, the definition of the Legendre transform gives

$$f_t((1-t)x + ty) \leq (1-t)f(x) + tG(y).$$

Combining this with the convexity of V and applying the Prékopa–Leindler inequality, we obtain

$$(34) \quad \int e^{-f_t} d\mu \geq \left(\int e^{-f} d\mu \right)^{1-t} \left(\int e^{-G} d\mu \right)^t.$$

On the other hand, taking $\nabla f(x)$ as a competitor in the supremum defining $f_t(x)$, we find

$$f_t(x) \geq f(x) - t(G^*(\nabla f(x)) - f^*(\nabla f(x))).$$

It follows that

$$\left(\int e^{-f} d\mu \right)^{1-t} \left(\int e^{-G} d\mu \right)^t \leq \int \exp(-f + t(G^*(\nabla f) - f^*(\nabla f))) d\mu.$$

The two sides agree at $t = 0$. The integrability assumptions allow us to differentiate this inequality from the right at $t = 0$, giving

$$\left(\int e^{-f} d\mu \right) \log \frac{\int e^{-G} d\mu}{\int e^{-f} d\mu} \leq \int (G^*(\nabla f) - f^*(\nabla f)) e^{-f} d\mu.$$

Finally,

$$f^*(\nabla f(x)) = \langle x, \nabla f(x) \rangle - f(x)$$

and

$$\text{Ent}_\mu(e^{-f}) = - \int f e^{-f} d\mu - \left(\int e^{-f} d\mu \right) \log \left(\int e^{-f} d\mu \right).$$

Substitution gives (33).

If f attains its minimum at the origin, convexity gives

$$\langle x, \nabla f(x) \rangle \geq f(x) - f(0) \geq 0.$$

Thus, when $\int e^{-G} d\mu = 1$, the additional terms on the right-hand side of (33) are nonnegative. Finally, every even convex function attains its minimum at the origin. $\square$

Related inequalities obtained by Prékopa–Leindler methods were studied by Bobkov and Ledoux [4].

5. A BRASCAMP–LIEB INEQUALITY WITH A CONCAVE WEIGHT

We begin with a weighted Bochner identity (see [15], [13]).

Lemma 5.1 (Weighted Bochner identity). *Let $d\mu = e^{-V}dx$, where $V \in C^2(\mathbb{R}^n)$, and let g, u be smooth. Assume that all the terms below are integrable and that the integrations by parts have no boundary contribution. Then*

$$\begin{aligned} \int_{\mathbb{R}^n} g(L_\mu u)^2 d\mu &= \int_{\mathbb{R}^n} g \left(\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \right) d\mu \\ &\quad - \int_{\mathbb{R}^n} \langle \nabla^2 g \nabla u, \nabla u \rangle d\mu - 2 \int_{\mathbb{R}^n} \langle \nabla g, \nabla u \rangle L_\mu u d\mu. \end{aligned}$$

Proof for completeness. Recall that

$$L_\mu = \Delta - \langle \nabla V, \nabla \rangle$$

and that

$$\frac{1}{2} L_\mu |\nabla u|^2 = \langle \nabla u, \nabla L_\mu u \rangle + \|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle.$$

Integration by parts gives

$$\begin{aligned} \int g(L_\mu u)^2 d\mu &= - \int L_\mu u \langle \nabla g, \nabla u \rangle d\mu - \int g \langle \nabla L_\mu u, \nabla u \rangle d\mu \\ &= - \int L_\mu u \langle \nabla g, \nabla u \rangle d\mu \\ &\quad + \int g \left(\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \right) d\mu - \frac{1}{2} \int g L_\mu |\nabla u|^2 d\mu. \end{aligned}$$

Moreover,

$$-\frac{1}{2} \int g L_\mu |\nabla u|^2 d\mu = \frac{1}{2} \int \langle \nabla g, \nabla |\nabla u|^2 \rangle d\mu = \int \langle \nabla^2 u \nabla g, \nabla u \rangle d\mu.$$

Finally,

$$\begin{aligned} 0 &= \int \operatorname{div}_\mu (\langle \nabla g, \nabla u \rangle \nabla u) d\mu \\ &= \int (\langle \nabla^2 g \nabla u, \nabla u \rangle + \langle \nabla^2 u \nabla g, \nabla u \rangle + \langle \nabla g, \nabla u \rangle L_\mu u) d\mu. \end{aligned}$$

Substitution gives the asserted identity. $\square$

Proof of Theorem 1.4. We first prove the assertion on a bounded convex domain D with smooth boundary. Assume that V and f are smooth on a neighborhood of $\overline{D}$, that $g \in C^2(\overline{D})$ is nonnegative and concave, and that

$$\int_D f d\mu = 0.$$

Let u be the solution, unique up to an additive constant, of the weighted Neumann problem

$$\begin{cases} L_\mu u = f & \text{in } D, \\ \partial_\nu u = 0 & \text{on } \partial D, \end{cases}$$

where ν is the outward unit normal. The displayed centering condition is precisely the compatibility condition for this problem.

We use the convention

$$\Pi(X, Y) = \langle D_X \nu, Y \rangle$$

for the second fundamental form of ∂D . Thus $\Pi \succeq 0$ because D is convex. The boundary version of Lemma 5.1 is

$$\begin{aligned} \int_D g(L_\mu u)^2 d\mu &= \int_D g \left(\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \right) d\mu \\ &\quad - \int_D \langle \nabla^2 g \nabla u, \nabla u \rangle d\mu - 2 \int_D \langle \nabla g, \nabla u \rangle L_\mu u d\mu \\ (35) \quad &\quad + \int_{\partial D} g \Pi(\nabla_{\partial D} u, \nabla_{\partial D} u) e^{-V} d\mathcal{H}^{n-1}. \end{aligned}$$

Indeed, the proof of the lemma applies verbatim except when $L_\mu |\nabla u|^2$ is integrated by parts. Tangentially differentiating the Neumann condition gives

$$\frac{1}{2} \partial_\nu |\nabla u|^2 = -\Pi(\nabla_{\partial D} u, \nabla_{\partial D} u).$$

This produces the last term in (35). The integration of

$$\operatorname{div}_\mu (\langle \nabla g, \nabla u \rangle \nabla u)$$

has no boundary contribution because its normal component contains the factor $\partial_\nu u$.

Using $L_\mu u = f$, integration by parts and (35), we obtain

$$\begin{aligned} \int_D g f^2 d\mu &= 2 \int_D g f L_\mu u d\mu - \int_D g (L_\mu u)^2 d\mu \\ &= -2 \int_D g \langle \nabla f, \nabla u \rangle d\mu \\ &\quad - \int_D g \left(\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \right) d\mu \\ &\quad + \int_D \langle \nabla^2 g \nabla u, \nabla u \rangle d\mu \\ &\quad - \int_{\partial D} g \Pi(\nabla_{\partial D} u, \nabla_{\partial D} u) e^{-V} d\mathcal{H}^{n-1}. \end{aligned}$$

Here the two terms containing $f \langle \nabla g, \nabla u \rangle$ cancel. Since $g \geq 0$, $\nabla^2 g \preceq 0$, and $\text{II} \succeq 0$, it follows that

$$\int_D g f^2 d\mu \leq \int_D g (-2 \langle \nabla f, \nabla u \rangle - \langle \nabla^2 V \nabla u, \nabla u \rangle) d\mu.$$

For a positive definite matrix A and vectors a, b ,

$$-2 \langle a, b \rangle - \langle Ab, b \rangle \leq \langle A^{-1}a, a \rangle.$$

Taking

$$A = \nabla^2 V, \quad a = \nabla f, \quad b = \nabla u,$$

we conclude that

$$\int_D g f^2 d\mu \leq \int_D g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu.$$

If $\nabla^2 V$ is only positive semidefinite, the same argument remains valid provided the inverse quadratic form is understood in the extended sense

$$(36) \quad \langle A^{-1}a, a \rangle := \sup_{z \in \mathbb{R}^n} \{2 \langle a, z \rangle - \langle Az, z \rangle\} \in [0, +\infty].$$

Indeed, the preceding pointwise estimate follows by choosing $z = -b$. When A is positive definite, (36) is the usual inverse quadratic form.

We now pass to an arbitrary convex set K with $0 < \mu(K) < \infty$. Choose an increasing sequence of bounded smooth convex domains D_j such that

$$\overline{D_j} \subset \text{int } K, \quad \bigcup_{j=1}^{\infty} D_j = \text{int } K.$$

Set

$$a_j = \frac{1}{\mu(D_j)} \int_{D_j} f d\mu.$$

Since $f \in L^1(K, \mu)$ and $\int_K f d\mu = 0$, we have $a_j \rightarrow 0$.

Fix j . A finite concave function is continuous in the interior of its domain. Since $\overline{D_j} \subset \text{int } K$, we may convolve g with a standard mollifier on a neighborhood of $\overline{D_j}$. In this way we obtain smooth nonnegative concave functions $g_{j,\varepsilon}$ such that

$$g_{j,\varepsilon} \longrightarrow g \quad \text{uniformly on } \overline{D_j}.$$

If g is not identically zero, its nonnegativity and concavity imply that $g > 0$ on $\text{int } K$; hence g has a positive minimum on $\overline{D_j}$. Thus, whenever the right-hand side of the desired inequality is finite, its unweighted quadratic-form integrand is integrable on D_j . If the right-hand side is infinite, the conclusion is automatic. Consequently, uniform convergence is sufficient in the following passage to the limit. (The case

$g \equiv 0$ is trivial.) Applying the smooth-domain inequality to $f - a_j$ and $g_{j,\varepsilon}$, and then letting $\varepsilon \downarrow 0$, gives

$$\int_{D_j} g(f - a_j)^2 d\mu \leq \int_{D_j} g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu.$$

Finally, Fatou's lemma on the left and monotone convergence on the right yield

$$\begin{aligned} \int_K g f^2 d\mu &\leq \liminf_{j \rightarrow \infty} \int_{D_j} g(f - a_j)^2 d\mu \\ &\leq \lim_{j \rightarrow \infty} \int_{D_j} g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu \\ &= \int_K g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu. \end{aligned}$$

The boundary of a full-dimensional convex set has Lebesgue measure zero, so replacing $\text{int } K$ by K does not change the integrals. This proves the theorem. $\square$

Remark 5.2. *The proof in fact yields a stronger form of Theorem 1.4. Assume first that g is positive and of class C^2 , and set*

$$A_g = \nabla^2 V - \frac{\nabla^2 g}{g}.$$

Indeed, in the weighted Bochner identity the two corresponding quadratic terms combine as

$$\int_K g \langle \nabla^2 V \nabla u, \nabla u \rangle d\mu - \int_K \langle \nabla^2 g \nabla u, \nabla u \rangle d\mu = \int_K g \langle A_g \nabla u, \nabla u \rangle d\mu.$$

Consequently, if A_g is positive definite, then for every f such that

$$\int_K f d\mu = 0$$

the same argument gives

$$\int_K g f^2 d\mu \leq \int_K g \langle A_g^{-1} \nabla f, \nabla f \rangle d\mu.$$

If g is concave, then

$$A_g \geq \nabla^2 V.$$

Thus, whenever the relevant matrices are positive definite,

$$A_g^{-1} \leq (\nabla^2 V)^{-1},$$

and the preceding inequality strengthens (10).

More generally, concavity is not needed for the estimate involving A_g : it is enough to assume that

$$g \nabla^2 V - \nabla^2 g$$

is positive definite. A corresponding degenerate statement can be obtained by approximation when this matrix is only positive semidefinite.

Remark 5.3 (Bounded domains, balls, and polytopes). *The preceding strengthening is particularly useful on bounded domains. Let $K \subset \mathbb{R}^n$ be bounded and take $V = 0$, so that μ is normalized Lebesgue measure on K . Retaining the Hessian term in the proof and using*

$$\|\nabla^2 u\|_{\text{HS}}^2 \geq \frac{1}{n}(\Delta u)^2$$

gives the following dimensional estimate. If g is positive and concave and $-\nabla^2 g$ is positive definite, then

$$\int_K g f^2 dx \leq \frac{n}{n+1} \int_K g^2 \left\langle (-\nabla^2 g)^{-1} \nabla f, \nabla f \right\rangle dx, \quad \int_K f dx = 0.$$

The normalization of Lebesgue measure is immaterial, since the same constant occurs on both sides.

Equivalently, let

$$d\nu(x) = \frac{g(x) dx}{\int_K g dx}.$$

Applying the preceding inequality to

$$f - \frac{1}{|K|} \int_K f dx$$

and then minimizing the left-hand side over constants gives

$$\text{Var}_\nu(f) \leq \frac{n}{n+1} \int_K \left\langle \left(-\frac{\nabla^2 g}{g} \right)^{-1} \nabla f, \nabla f \right\rangle d\nu.$$

For example, on the Euclidean unit ball, take

$$g(x) = 1 - |x|^2.$$

Since

$$-\nabla^2 g = 2I,$$

we obtain

$$\int_{B_2^n} (1 - |x|^2) f^2 dx \leq \frac{n}{2(n+1)} \int_{B_2^n} (1 - |x|^2)^2 |\nabla f|^2 dx, \quad \int_{B_2^n} f dx = 0.$$

Equivalently, if ν is the probability measure on B_2^n with density proportional to $1 - |x|^2$, then

$$\text{Var}_\nu(f) \leq \frac{n}{2(n+1)} \int_{B_2^n} (1 - |x|^2) |\nabla f|^2 d\nu.$$

In dimension one this inequality is sharp, with equality for $f(x) = x$. The use of a weight vanishing on the boundary is justified by replacing g with $g + \varepsilon$ and letting $\varepsilon \downarrow 0$.

There is also a natural polytope example. Let

$$P = \{x \in \mathbb{R}^n : \ell_i(x) = b_i - \langle a_i, x \rangle > 0, \quad i = 1, \dots, m\}$$

be a bounded full-dimensional polytope, where the inequalities define the facets of P , each facet being listed once. Set

$$g(x) = \left(\prod_{i=1}^m \ell_i(x) \right)^{1/m}.$$

The function g is concave. Define

$$v_i(x) = \frac{a_i}{\ell_i(x)}, \quad \bar{v}(x) = \frac{1}{m} \sum_{i=1}^m v_i(x).$$

A direct calculation gives

$$-\frac{\nabla^2 g(x)}{g(x)} = \frac{1}{m} \sum_{i=1}^m v_i(x) \otimes v_i(x) - \bar{v}(x) \otimes \bar{v}(x).$$

Denote this matrix by $H_P(x)$. Thus $H_P(x)$ is the covariance matrix, with equal weights, of the vectors

$$\frac{a_i}{\ell_i(x)}.$$

It is a centered version of the logarithmic-barrier metric.

For every $x \in P$, the matrix $H_P(x)$ is positive definite. Indeed, if

$$\langle H_P(x)z, z \rangle = 0,$$

then the numbers $\langle v_i(x), z \rangle$ are all equal. If their common value is zero, then P contains the entire line $x + \mathbb{R}z$. If their common value is nonzero, all the supporting hyperplanes of P pass through a common point and P contains an unbounded ray. Both conclusions contradict the boundedness of P .

Consequently,

$$\int_P g f^2 dx \leq \frac{n}{n+1} \int_P g \langle H_P^{-1} \nabla f, \nabla f \rangle dx, \quad \int_P f dx = 0.$$

If

$$d\nu_P(x) = \frac{g(x) dx}{\int_P g dx},$$

then one also has the Poincaré-type inequality

$$\text{Var}_{\nu_P}(f) \leq \frac{n}{n+1} \int_P \langle H_P^{-1} \nabla f, \nabla f \rangle d\nu_P.$$

When P is a simplex, $m = n + 1$, and after an affine identification with the standard simplex the measure ν_P is the symmetric Dirichlet distribution whose parameters are

$$\alpha_1 = \cdots = \alpha_{n+1} = 1 + \frac{1}{n+1} = \frac{n+2}{n+1}.$$

The polytope estimate follows from the smooth case by approximation from the interior. All the preceding inequalities are affine invariant.

6. AN IMPROVED BRASCAMP–LIEB INEQUALITY IN THE UNCONDITIONAL CASE

In this section we give the short proof of Theorem 1.5.

Proof of Theorem 1.5. Write $d\nu = e^{-U}d\mu$, where U is convex and unconditional. Since u is unconditional, each partial derivative $\partial_i u$ is odd in the i -th variable, and we may write

$$\partial_i u = x_i g_i$$

with g_i smooth and even in the i -th coordinate. Using $T_{ij}x_i x_j \geq 0$ and $ab \leq (a^2 + b^2)/2$,

$$\begin{aligned} \int \langle T\nabla u, \nabla u \rangle d\nu &= \sum_{i,j=1}^n \int T_{ij}x_i x_j g_i g_j d\nu \leq \sum_{i,j=1}^n \int T_{ij}x_i x_j \frac{g_i^2 + g_j^2}{2} d\nu \\ &= \sum_{i,j=1}^n \int T_{ij}x_i x_j g_i^2 d\nu = \sum_{i=1}^n \int (Tx)_i x_i g_i^2 d\nu \\ &= \sum_{i=1}^n \int \partial_i V x_i g_i^2 d\nu. \end{aligned}$$

For each i , integration by parts with $d\nu = Z_\nu^{-1}e^{-V-U}dx$ gives

$$\int \partial_i V x_i g_i^2 d\nu = \int (g_i^2 + 2x_i g_i \partial_i g_i - x_i g_i^2 \partial_i U) d\nu.$$

Since U is convex and unconditional, $x_i \partial_i U \geq 0$. Hence

$$\int \partial_i V x_i g_i^2 d\nu \leq \int (g_i^2 + 2x_i g_i \partial_i g_i) d\nu \leq \int |\nabla(x_i g_i)|^2 d\nu.$$

Summing over i gives

$$\int \langle T\nabla u, \nabla u \rangle d\nu \leq \sum_{i=1}^n \int |\nabla \partial_i u|^2 d\nu = \int \|\nabla^2 u\|_{HS}^2 d\nu.$$

□

Corollary 6.1. *Let μ be an unconditional log-concave probability measure on $\mathbb{R}^n$ with C^2 density $Z^{-1}e^{-V}$. Let T satisfy the assumptions of Theorem 1.5, and assume in addition that $\nabla^2 V + T \succ 0$. Then for every probability measure ν with unconditional log-concave density with respect to μ , and every smooth unconditional $f : \mathbb{R}^n \rightarrow \mathbb{R}$,*

$$\mathrm{Var}_\nu f \leq \int \langle (\nabla^2 V + T)^{-1} \nabla f, \nabla f \rangle d\nu.$$

This corollary follows from Proposition 1.3, with the observation that the same proof continues to hold when the word “even” is replaced everywhere with the word “unconditional”. To avoid repetition we only prove Proposition 1.3 as stated:

Proof of Proposition 1.3. Let $d\nu = \frac{1}{Z} e^{-W} d\mu$, where W is convex and even. Let f be even and centered with respect to ν . We solve $L_\nu u = f$. Since ν and f are even, u may be chosen even. By the Bochner identity for the potential $V + W$,

$$\int f^2 d\nu = \int \left(\|\nabla^2 u\|_{HS}^2 + \langle (\nabla^2 V + \nabla^2 W) \nabla u, \nabla u \rangle \right) d\nu.$$

Using (8) and $\nabla^2 W \succeq 0$, we get

$$\int f^2 d\nu \geq \int \langle (\nabla^2 V + T) \nabla u, \nabla u \rangle d\nu.$$

On the other hand,

$$\int f^2 d\nu = - \int \langle \nabla f, \nabla u \rangle d\nu,$$

and (9) follows in the same way as in Theorem 2.1 or Proposition 3.1.

It remains to recall the standard implication to the B-conjecture. Let K be an origin-symmetric convex body, and put $F(t) = \mu(e^t K)$. After the change of variables $x = e^t y$, differentiation gives

$$\frac{F'(t)}{F(t)} = \int (n - \langle \nabla V(x), x \rangle) d\nu_t(x),$$

where ν_t is the normalized restriction of μ to $e^t K$. A second differentiation yields

$$(\log F)''(t) = \mathrm{Var}_{\nu_t} (\langle \nabla V, x \rangle) - \int (\langle \nabla^2 V x, x \rangle + \langle \nabla V, x \rangle) d\nu_t.$$

For $q(x) = \langle \nabla V(x), x \rangle$ we have

$$\nabla q = (\nabla^2 V)x + \nabla V = (\nabla^2 V + T)x.$$

Applying (9) to the even function q under ν_t gives

$$\mathrm{Var}_{\nu_t}(q) \leq \int \langle (\nabla^2 V + T)x, x \rangle d\nu_t = \int (\langle \nabla^2 V x, x \rangle + \langle \nabla V, x \rangle) d\nu_t.$$

Hence $(\log F)''(t) \leq 0$. □

7. THE BRASCAMP-LIEB ROTATION TRICK

In this section we prove Theorem 1.6:

Proof of Theorem 1.6. By adding a constant to V we may assume that $\int e^{-V} = 1$. Fix a C^1 even function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We use the identity

$$\mathrm{Var}_\mu f = \frac{1}{2} \iint (f(x) - f(y))^2 e^{-V(x)-V(y)} dx dy.$$

Next we perform the change of variables $x = \frac{u+v}{\sqrt{2}}$, $y = \frac{u-v}{\sqrt{2}}$, which has Jacobian $\equiv 1$, and obtain

$$\mathrm{Var}_\mu f = \frac{1}{2} \iint \left(f\left(\frac{u+v}{\sqrt{2}}\right) - f\left(\frac{u-v}{\sqrt{2}}\right) \right)^2 e^{-V\left(\frac{u+v}{\sqrt{2}}\right) - V\left(\frac{u-v}{\sqrt{2}}\right)} du dv.$$

For a fixed $u \in \mathbb{R}^n$ we define

$$\begin{aligned} f_u(v) &= f\left(\frac{u+v}{\sqrt{2}}\right) - f\left(\frac{u-v}{\sqrt{2}}\right) \\ V_u(v) &= V\left(\frac{u+v}{\sqrt{2}}\right) + V\left(\frac{u-v}{\sqrt{2}}\right). \end{aligned}$$

Note that f_u is odd and V_u is even, and so $\int f_u e^{-V_u} = 0$. Since V_u is convex it follows from the usual Brascamp-Lieb inequality that

$$\mathrm{Var}_\mu f = \frac{1}{2} \iint f_u(v)^2 e^{-V_u(v)} dv du \leq \frac{1}{2} \iint \langle (\nabla^2 V_u)^{-1} \cdot \nabla f_u, \nabla f_u \rangle e^{-V_u(v)} dv du.$$

Computing these derivatives (in v) we have

$$\nabla f_u(v) = \frac{1}{\sqrt{2}} \left(\nabla f\left(\frac{u+v}{\sqrt{2}}\right) + \nabla f\left(\frac{u-v}{\sqrt{2}}\right) \right)$$

as well as

$$\nabla^2 V_u(v) = \frac{1}{2} \left(\nabla^2 V\left(\frac{u+v}{\sqrt{2}}\right) + \nabla^2 V\left(\frac{u-v}{\sqrt{2}}\right) \right).$$

Therefore, performing a change of variables back to x and y , we obtain

$$\begin{aligned} \mathrm{Var}_\mu f &\leq \frac{1}{2} \iint \left\langle \left(\frac{\nabla^2 V(x) + \nabla^2 V(y)}{2} \right)^{-1} \left(\frac{\nabla f(x) + \nabla f(y)}{\sqrt{2}} \right), \left(\frac{\nabla f(x) + \nabla f(y)}{\sqrt{2}} \right) \right\rangle d\mu(x) d\mu(y) \\ &= \frac{1}{2} \iint \left\langle (\nabla^2 V(x) + \nabla^2 V(y))^{-1} (\nabla f(x) + \nabla f(y)), \nabla f(x) + \nabla f(y) \right\rangle d\mu(x) d\mu(y). \end{aligned}$$

We now expand the inner product, and note that the “cross term”

$$\left\langle (\nabla^2 V(x) + \nabla^2 V(y))^{-1} \nabla f(x), \nabla f(y) \right\rangle$$

is odd in y for every fixed x , so its integral vanishes. Using this and the symmetry between x and y we obtain

$$\begin{aligned} \text{Var}_\mu f &\leq \iint \left\langle (\nabla^2 V(x) + \nabla^2 V(y))^{-1} \nabla f(x), \nabla f(x) \right\rangle d\mu(x) d\mu(y) \\ &= \int \left\langle \left(\int (\nabla^2 V(x) + \nabla^2 V(y))^{-1} d\mu(y) \right) \cdot \nabla f(x), \nabla f(x) \right\rangle d\mu(x) \\ &= \int \langle H_V \nabla f, \nabla f \rangle d\mu, \end{aligned}$$

as claimed. $\square$

8. A POINCARÉ INEQUALITY FOR SMOOTH NORMS

In this section we prove Theorem 1.7. We begin by collecting the necessary ingredients. The first ingredient gives us a more geometric representation for the inverse Hessian $(\nabla^2 \alpha)^{-1}$.

Lemma 8.1. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with unit ball K , where K is smooth and has strictly positive Gaussian curvature. Define $\alpha(x) = \frac{1}{2} \|x\|^2$. Then for every $x \in \partial K$ and every $u, v \in \mathbb{R}^n$ we have*

$$\left\langle (\nabla^2 \alpha(x))^{-1} u, v \right\rangle = \langle x, u \rangle \langle x, v \rangle + \langle x, n_K \rangle \Pi^{-1}(P_{T_x \partial K} u, P_{T_x \partial K} v).$$

Here n_K denotes the outer unit normal to ∂K at x , $P_{T_x \partial K}$ denotes the orthogonal projection onto the tangent plane $T_x \partial K$, and Π^{-1} denotes the inverse of the second fundamental form of the hypersurface ∂K at the point x .

Proof. Let us simplify our notation, writing P for $P_{T_x \partial K}$. We will also use S for the shape operator $S : T_x \partial K \rightarrow T_x \partial K$, with the sign convention that $Su = \partial_u(n_K)$. Finally, we will suppress the dependence on x whenever appropriate, writing for example $(\nabla^2 \alpha)^{-1}$ instead of $(\nabla^2 \alpha(x))^{-1}$.

Since $\alpha \equiv \frac{1}{2}$ on ∂K we know that $\nabla \alpha = \lambda n_K$ for some $\lambda \in \mathbb{R}$. To compute λ we note that α is 2-homogeneous and therefore

$$1 = 2\alpha = \langle \nabla \alpha, x \rangle = \lambda \langle n_K, x \rangle,$$

and so $\nabla \alpha = \frac{n_K}{\langle n_K, x \rangle}$. Again, since α is 2-homogeneous, we have that

$$(37) \quad \nabla^2 \alpha \cdot x = \nabla \alpha = \frac{n_K}{\langle n_K, x \rangle}.$$

On the other hand, for all $\xi \in T_x \partial K$ we may use the fact that $\partial_\xi(n_K) = S\xi$. We have

$$\partial_\xi(\langle n_K, x \rangle) = \langle S\xi, x \rangle + \langle n_K, \xi \rangle = \langle S\xi, x \rangle,$$

and therefore

$$(38) \quad \nabla^2 \alpha \cdot \xi = \partial_\xi (\nabla \alpha) = \partial_\xi \left(\frac{n_K}{\langle n_K, x \rangle} \right) = \frac{S\xi}{\langle n_K, x \rangle} - \frac{\langle S\xi, x \rangle}{\langle n_K, x \rangle^2} n_K.$$

It follows from (37) and (38) that if we define temporarily

$$Au = \langle x, u \rangle x + \langle n_K, x \rangle S^{-1} Pu$$

then for all $u \in \mathbb{R}^n$ we have

$$\begin{aligned} (\nabla^2 \alpha)(Au) &= \langle x, u \rangle \frac{n_K}{\langle n_K, x \rangle} + \langle n_K, x \rangle \left(\frac{Pu}{\langle n_K, x \rangle} - \frac{\langle Pu, x \rangle}{\langle n_K, x \rangle^2} n_K \right) \\ &= Pu + \langle u - Pu, x \rangle \frac{n_K}{\langle n_K, x \rangle} \\ &= Pu + \langle \langle u, n_K \rangle n_K, x \rangle \frac{n_K}{\langle n_K, x \rangle} = Pu + \langle u, n_K \rangle n_K = u, \end{aligned}$$

which precisely means that $A = (\nabla^2 \alpha)^{-1}$. Therefore

$$\begin{aligned} \langle (\nabla^2 \alpha)^{-1} u, v \rangle &= \langle Au, v \rangle = \langle x, u \rangle \langle x, v \rangle + \langle n_K, x \rangle \langle S^{-1} Pu, v \rangle \\ &= \langle x, u \rangle \langle x, v \rangle + \langle n_K, x \rangle \Pi^{-1}(Pu, Pv) \end{aligned}$$

as claimed. $\square$

Our second ingredient is a Poincaré-type inequality on ∂K :

Lemma 8.2. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth, convex function. Let $\mu_{\partial K}$ be the measure on ∂K with density $e^{-V} \langle x, n_K \rangle$ with respect to $\mathcal{H}^{n-1}|_{\partial K}$. Then for every C^1 function $f : \partial K \rightarrow \mathbb{R}$ such that $\int_{\partial K} f d\mu_{\partial K} = 0$ we have*

$$\int_{\partial K} (n - 1 - \langle \nabla V, x \rangle) f^2 d\mu_{\partial K} \leq \int_{\partial K} \Pi^{-1}(\nabla_{\partial K} f, \nabla_{\partial K} f) \langle x, n_K \rangle d\mu_{\partial K}.$$

Lemma 8.2 is nothing more than a reformulation of a result of Kolesnikov–Milman from [15]. Specifically, restricting Theorem 1.1 of [15] to the manifold with boundary K and to the case $N = \infty$, one obtains the following:

Proposition 8.3. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth, convex function. Then for every C^1 function $g : \partial K \rightarrow \mathbb{R}$ such that $\int_{\partial K} g e^{-V} = 0$ we have*

$$\int_{\partial K} (H - \langle \nabla V, n_K \rangle) g^2 e^{-V} \leq \int_{\partial K} \Pi^{-1}(\nabla_{\partial K} g, \nabla_{\partial K} g) e^{-V},$$

where H denotes the mean curvature.

Here we use the convention that an integral on ∂K is always with respect to the Hausdorff measure $\mathcal{H}^{n-1}|_{\partial K}$ unless explicitly stated otherwise. Lemma 8.2 is obtained from Proposition 8.3 using a careful integration by parts:

Proof of Lemma 8.2. Given $f : \partial K \rightarrow \mathbb{R}$ we define $g : \partial K \rightarrow \mathbb{R}$ by $g = f \langle x, n_K \rangle$. Since

$$\int_{\partial K} g e^{-V} = \int_{\partial K} f \langle x, n_K \rangle e^{-V} = \int_{\partial K} f d\mu_{\partial K} = 0,$$

we deduce from Proposition 8.3 that

$$(39) \quad \int_{\partial K} (H - \langle \nabla V, n_K \rangle) g^2 e^{-V} \leq \int_{\partial K} \Pi^{-1} (\nabla_{\partial K} g, \nabla_{\partial K} g) e^{-V}.$$

Similarly to Lemma 8.1, let us write P for the orthogonal projection to $T_x \partial K$ and $S : T_x \partial K \rightarrow T_x \partial K$ for the shape operator. Note that for $h = \langle x, n_K \rangle$ and $\xi \in T_x \partial K$ we have

$$\partial_\xi h = \langle \xi, n_K \rangle + \langle x, S\xi \rangle = \langle x, S\xi \rangle = \langle Px, S\xi \rangle = \langle SPx, \xi \rangle,$$

so $\nabla_{\partial K} h = SPx$. Therefore

$$\begin{aligned} \Pi^{-1} (\nabla_{\partial K} g, \nabla_{\partial K} g) &= \langle S^{-1} \nabla_{\partial K} (fh), \nabla_{\partial K} (fh) \rangle \\ &= h^2 \langle S^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + 2hf \langle S^{-1} \nabla_{\partial K} h, \nabla_{\partial K} f \rangle + f^2 \langle S^{-1} \nabla_{\partial K} h, \nabla_{\partial K} h \rangle \\ &= h^2 \Pi^{-1} (\nabla_{\partial K} f, \nabla_{\partial K} f) + \underbrace{h \langle Px, \nabla_{\partial K} (f^2) \rangle}_{(*)} + \underbrace{f^2 \langle Px, \nabla_{\partial K} h \rangle}_{(**)}. \end{aligned}$$

We now integrate by parts the term $(*)$. Note that

$$\begin{aligned} \operatorname{div}_{\partial K} (Px) &= \operatorname{div}_{\partial K} x - \operatorname{div}_{\partial K} (hn_K) = n - 1 - \langle \nabla_{\partial K} h, n_K \rangle - h \operatorname{div}_{\partial K} (n_K) \\ &= n - 1 - hH, \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\partial K} h \langle Px, \nabla_{\partial K} (f^2) \rangle e^{-V} &= - \int_{\partial K} \operatorname{div}_{\partial K} (h e^{-V} \cdot Px) f^2 \\ &= - \int_{\partial K} (\langle (\nabla_{\partial K} h - h \nabla_{\partial K} V), Px \rangle + h(n - 1 - hH)) f^2 e^{-V} \\ &= - \int_{\partial K} f^2 \langle Px, \nabla_{\partial K} h \rangle e^{-V} + \int_{\partial K} f^2 h \langle \nabla_{\partial K} V, Px \rangle e^{-V} \\ &\quad - \int_{\partial K} (n - 1 - hH) h f^2 e^{-V}. \end{aligned}$$

The first term in the expression above precisely cancels the term $(**)$, so the right-hand side of (39) can be written as

$$\int_{\partial K} \Pi^{-1} (\nabla_{\partial K} f, \nabla_{\partial K} f) h d\mu_{\partial K} + \int_{\partial K} f^2 \langle \nabla_{\partial K} V, Px \rangle d\mu_{\partial K} - \int_{\partial K} (n - 1 - hH) f^2 d\mu_{\partial K}.$$

On the other hand, recalling that $g = hf$ the left-hand side of (39) is

$$\int_{\partial K} (H - \langle \nabla V, n_K \rangle) f^2 h^2 e^{-V} = \int_{\partial K} (H - \langle \nabla V, n_K \rangle) f^2 h d\mu_{\partial K}.$$

The term $\int_{\partial K} H f^2 h d\mu_{\partial K}$ cancels, and thus (39) is equivalent to

$$\int_{\partial K} [(n-1) - \langle \nabla_{\partial K} V, Px \rangle - \langle \nabla V, n_K \rangle h] f^2 d\mu_{\partial K} \leq \int_{\partial K} \Pi^{-1} (\nabla_{\partial K} f, \nabla_{\partial K} f) h d\mu_{\partial K}.$$

Finally, we observe that

$$\langle \nabla_{\partial K} V, Px \rangle + \langle \nabla V, n_K \rangle h = \langle P(\nabla V), Px \rangle + \langle \nabla V, n_K \rangle \langle x, n_K \rangle = \langle \nabla V, x \rangle,$$

so the proof is complete. $\square$

The last ingredient we need is the following one-dimensional lemma, which is precisely [9, Lemma 8]:

Lemma 8.4. *Let $v, w : [0, \infty) \rightarrow \mathbb{R}$ be continuous functions and C^1 on $(0, \infty)$. Let f be a C^1 function on $[0, \infty)$ which is compactly supported (for simplicity) and satisfies $f(0) = 0$. Then for every $\alpha \geq 0$ we have*

$$\int_0^\infty \frac{v'}{t} f^2 t^\alpha e^{-v-w} dt \leq \int_0^\infty \left((f')^2 + \alpha \cdot \left(\frac{f}{t} \right)^2 - w' \frac{f^2}{t} \right) t^\alpha e^{-v-w} dt.$$

With all our lemmas stated, we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Since ν is even and more log-concave than μ , we may write

$$d\nu(x) = e^{-W(x)} d\mu(x) = e^{-v(\|x\|) - W(x)} dx,$$

where W is even and convex.

Using the standard approximation arguments one may assume that f is compactly supported and that the function W is smooth. The co-area formula implies that for all integrable functions ϕ we have

$$\int_{\mathbb{R}^n} \phi(x) dx = \int_{\partial K} \int_0^\infty \phi(rx) r^{n-1} dr d\sigma_K(x)$$

where σ_K is the measure on ∂K with density $h(x) = \langle x, n_K \rangle$. In particular, we have

$$\int \frac{v'(\|x\|)}{\|x\|} f^2 d\nu = \int_{\partial K} \int_0^\infty \frac{v'(r)}{r} f(rx)^2 r^{n-1} e^{-v(r) - W(rx)} dr d\sigma_K(x).$$

Choosing $f_x(r) = f(rx)$ and $w_x(r) = W(rx)$, we have $f_x(0) = f(0) = 0$ as f is odd. Therefore we may use Lemma 8.4 and bound our integral by

$$\int_{\partial K} \int_0^\infty \left[\underbrace{\langle \nabla f(rx), x \rangle^2}_{(A)} + \underbrace{(n-1) \left(\frac{f(rx)}{r} \right)^2 - \langle \nabla W(rx), x \rangle \frac{f^2(rx)}{r}}_{(B)} \right] r^{n-1} e^{-v(r) - W(rx)} dr d\sigma_K(x).$$

For term (B) we will exchange the order of the integrals, and use Lemma 8.2 for the functions $f_r(x) = \frac{f(rx)}{r}$ and $W_r(x) = W(rx)$. Since f_r is odd and he^{-W_r} is even we have $\int_{\partial K} f_r h e^{-W_r} = 0$, and so by the lemma

$$\int_{\partial K} (n-1 - \langle \nabla W(rx), rx \rangle) \left(\frac{f(rx)}{r} \right)^2 h e^{-W(rx)} \leq \int_{\partial K} \Pi^{-1}(\nabla_{\partial K} f_r(x), \nabla_{\partial K} f_r(x)) h^2 e^{-W(rx)}.$$

Note that $\nabla_{\partial K} f_r(x) = P_x(\nabla f(rx))$, where P_x denotes the projection to $T_x \partial K$. Therefore we see that our integral $\int \frac{v'(\|x\|)}{\|x\|} f^2 d\nu$ is bounded by

$$\int_{\partial K} \int_0^\infty [\langle \nabla f(rx), x \rangle^2 + \Pi^{-1}(P_x \nabla f(rx), P_x \nabla f(rx)) h] r^{n-1} e^{-v(r)-W(rx)} dr d\sigma_K(x).$$

Using Lemma 8.1 the right-hand side above is equal to

$$\int_{\partial K} \int_0^\infty \left\langle (\nabla^2 \alpha(rx))^{-1} \nabla f(rx), \nabla f(rx) \right\rangle r^{n-1} e^{-v(r)-W(rx)} dr d\sigma_K(x),$$

where we also used the fact that $\nabla^2 \alpha$ is 0-homogeneous and therefore $\nabla^2 \alpha(rx) = \nabla^2 \alpha(x)$. Again by the co-area formula this expression is nothing more than

$$\int_{\mathbb{R}^n} \left\langle (\nabla^2 \alpha)^{-1} \nabla f, \nabla f \right\rangle d\nu,$$

so the proof is complete. $\square$

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