# On a version of the slicing problem for the surface area of convex bodies

Silouanos Brazitikos (joint work with Dimitris Liakopoulos)
University of Athens

AGA Seminar, March 2022

# Classical Slicing Problem

The classical slicing problem asks if there exists an absolute constant $C_2 > 0$ such that for every $n \geqslant 2$ and every centered convex body $K$ in $\mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leqslant C_2 \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.$$

# Classical Slicing Problem

The classical slicing problem asks if there exists an absolute constant $C_2 > 0$ such that for every $n \geqslant 2$ and every centered convex body $K$ in $\mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leqslant C_2 \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}|.$$

It is well-known that this problem is equivalent to the question if there exists an absolute constant $C_3 > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C_3,$$

for all $n \geqslant 1$, where $L_K$ is the isotropic constant of $K$.

# Classical Slicing Problem

The classical slicing problem asks if there exists an absolute constant $C_2 > 0$ such that for every $n \geqslant 2$ and every centered convex body $K$ in $\mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leqslant C_2 \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}|.$$

It is well-known that this problem is equivalent to the question if there exists an absolute constant $C_3 > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C_3,$$

for all $n \geqslant 1$, where $L_K$ is the isotropic constant of $K$.
Koldobsky proved the following variants for the surface area. If $K$ is an intersection body in $\mathbb{R}^n$, then

$$as(K) \leq c_n \max_{\xi \in S^{n-1}} as(K \cap \xi^{\perp}) |K|^{1/n},$$

The classical slicing problem asks if there exists an absolute constant $C_2 > 0$ such that for every $n \geqslant 2$ and every centered convex body $K$ in $\mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leqslant C_2 \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|.$$

It is well-known that this problem is equivalent to the question if there exists an absolute constant $C_3 > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C_3,$$

for all $n \geqslant 1$, where $L_K$ is the isotropic constant of $K$.
Koldobsky proved the following variants for the surface area. If $K$ is an intersection body in $\mathbb{R}^n$, then

$$as(K) \leq c_n \max_{\xi \in S^{n-1}} as(K \cap \xi^\perp)|K|^{1/n},$$

and

$$S(K) \geq c_n \min_{\xi \in S^{n-1}} S(P_{\xi^\perp} K)|K|^{1/n},$$

where $as(K) = \int_{S^{n-1}} |K \cap \xi^\perp| \, d\sigma(\xi)$.

Koldobsky asked if this is true if we replace projections with sections.

Koldobsky asked if this is true if we replace projections with sections.

**Question**

Is it true that there exists a constant $\alpha_n$ depending (or not) on the dimension $n$ so that

$$S(K) \leqslant \alpha_n |K|^{\frac{1}{n}} \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

for every centrally symmetric convex body $K$ in $\mathbb{R}^n$?

Koldobsky asked if this is true if we replace projections with sections.

**Question**

Is it true that there exists a constant $\alpha_n$ depending (or not) on the dimension $n$ so that

$$S(K) \leqslant \alpha_n |K|^{\frac{1}{n}} \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

for every centrally symmetric convex body $K$ in $\mathbb{R}^n$?

In general, for any $2 \leqslant k \leqslant n-1$, one may ask for a constant $\alpha_{n,k}$ such that

$$S(K) \leqslant \alpha_{n,k}^k |K|^{\frac{k}{n}} \max_{H \in G_{n,n-k}} S(K \cap H).$$

Koldobsky asked if this is true if we replace projections with sections.

## Question

Is it true that there exists a constant $\alpha_n$ depending (or not) on the dimension $n$ so that

$$S(K) \leqslant \alpha_n |K|^{\frac{1}{n}} \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

for every centrally symmetric convex body $K$ in $\mathbb{R}^n$?

In general, for any $2 \leqslant k \leqslant n-1$, one may ask for a constant $\alpha_{n,k}$ such that

$$S(K) \leqslant \alpha_{n,k}^{k} |K|^{\frac{k}{n}} \max_{H \in G_{n,n-k}} S(K \cap H).$$

## Theorem. B.-Liakopoulos

The answer to both questions is negative.

Our examples will be given by ellipsoids. They will based on

## Theorem

Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$ and write $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ for the lengths and $e_1, e_2, \ldots, e_n$ for the corresponding directions of its semi-axes. If $1 \leqslant k \leqslant n-1$ then for any $H \in G_{n,k}$ and any $0 \leqslant j < k$ we have that

$$W_j(\mathcal{E} \cap F_k) \leqslant W_j(\mathcal{E} \cap H) \leqslant W_j(P_H(\mathcal{E})) \leqslant W_j(\mathcal{E} \cap E_k),$$

where $F_k = \mathrm{span}\{e_1, \ldots, e_k\}$ and $E_k = \mathrm{span}\{e_{n-k+1}, \ldots, e_n\}$. In particular, for every $\xi \in S^{n-1}$,

$$S(\mathcal{E} \cap \xi^\perp) \leqslant S(P_{\xi^\perp}(\mathcal{E})) \leqslant S(\mathcal{E} \cap e_1^\perp).$$

Our examples will be given by ellipsoids. They will based on

## Theorem

Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$ and write $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ for the lengths and $e_1, e_2, \ldots, e_n$ for the corresponding directions of its semi-axes. If $1 \leqslant k \leqslant n-1$ then for any $H \in G_{n,k}$ and any $0 \leqslant j < k$ we have that

$$W_j(\mathcal{E} \cap F_k) \leqslant W_j(\mathcal{E} \cap H) \leqslant W_j(P_H(\mathcal{E})) \leqslant W_j(\mathcal{E} \cap E_k),$$

where $F_k = \mathrm{span}\{e_1, \ldots, e_k\}$ and $E_k = \mathrm{span}\{e_{n-k+1}, \ldots, e_n\}$. In particular, for every $\xi \in S^{n-1}$,

$$S(\mathcal{E} \cap \xi^\perp) \leqslant S(P_{\xi^\perp}(\mathcal{E})) \leqslant S(\mathcal{E} \cap e_1^\perp).$$

The proof relies on Cauchy interlacing Theorem and comparison with a spheroid.

We shall use the next formula of Rivin:

We shall use the next formula of Rivin:
If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \ldots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}| \int_{S^{n-1}} \Big( \sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2} \Big)^{1/2} d\sigma(\xi).$$

We shall use the next formula of Rivin:
If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \dots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}|\,\int_{S^{n-1}} \Big(\sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2}\Big)^{1/2} d\sigma(\xi).$$

Now assume that there exists a constant $\alpha_n > 0$ such that we have the following inequality for ellipsoids:

$$S(\mathcal{E}) \leqslant \alpha_n |\mathcal{E}|^{1/n} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp}).$$

We know that the maximum is attained for the section $\mathcal{E} \cap e_1^{\perp}$. Then we have

$$\max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp}) = S(\mathcal{E} \cap e_1^{\perp}) = (n-1)\,|\mathcal{E} \cap e_1^{\perp}|\,\int_{S^{n-2}} \Big(\sum_{i=2}^{n} \frac{\xi_i^2}{a_i^2}\Big)^{1/2} d\sigma(\xi).$$

We may assume that $\prod_{i=1}^{n} a_i = 1$. Then, we can rewrite

$$S(\mathcal{E}) \leqslant \alpha_n |\mathcal{E}|^{1/n} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

as

We may assume that $\prod_{i=1}^n a_i = 1$. Then, we can rewrite

$$S(\mathcal{E}) \leqslant \alpha_n |\mathcal{E}|^{1/n} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

as

$$n\omega_n \cdot \frac{1}{d_n} \mathbb{E}\Big[ \Big( \sum_{i=1}^n \frac{g_i^2}{a_i^2} \Big)^{1/2} \Big] \leqslant \alpha_n \omega_n^{1/n} \cdot (n-1)\omega_{n-1} \frac{1}{a_1} \cdot \frac{1}{d_{n-1}} \mathbb{E}\Big[ \Big( \sum_{i=2}^n \frac{g_i^2}{a_i^2} \Big)^{1/2} \Big].$$

We may assume that $\prod_{i=1}^n a_i = 1$. Then, we can rewrite

$$S(\mathcal{E}) \leqslant \alpha_n |\mathcal{E}|^{1/n} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

as

$$n\omega_n \cdot \frac{1}{d_n} \mathbb{E}\Big[ \Big( \sum_{i=1}^n \frac{g_i^2}{a_i^2} \Big)^{1/2} \Big] \leqslant \alpha_n \omega_n^{1/n} \cdot (n-1)\omega_{n-1} \frac{1}{a_1} \cdot \frac{1}{d_{n-1}} \mathbb{E}\Big[ \Big( \sum_{i=2}^n \frac{g_i^2}{a_i^2} \Big)^{1/2} \Big].$$

Then,

$$\alpha_n \geqslant C_n a_1 \frac{\mathbb{E}\left[ \left( \sum_{i=1}^n \frac{g_i^2}{a_i^2} \right)^{1/2} \right]}{\mathbb{E}\left[ \left( \sum_{i=2}^n \frac{g_i^2}{a_i^2} \right)^{1/2} \right]}.$$

Since $x \mapsto \left(\sum_{i=1}^n \frac{x_i^2}{a_i^2}\right)^{1/2}$ is a seminorm, using Hölder and Khintchine's inequality for this seminorm in Gauss space we get

$$\frac{\mathbb{E}\left[\left(\sum_{i=1}^n \frac{g_i^2}{a_i^2}\right)^{1/2}\right]}{\mathbb{E}\left[\left(\sum_{i=2}^n \frac{g_i^2}{a_i^2}\right)^{1/2}\right]} \geqslant c\left(\frac{\mathbb{E}\left(\sum_{i=1}^n \frac{g_i^2}{a_i^2}\right)}{\mathbb{E}\left(\sum_{i=2}^n \frac{g_i^2}{a_i^2}\right)}\right)^{1/2} = c\left(\frac{\sum_{i=1}^n \frac{1}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2},$$

Since $x \mapsto \left(\sum_{i=1}^n \frac{x_i^2}{a_i^2}\right)^{1/2}$ is a seminorm, using Hölder and Khintchine's inequality for this seminorm in Gauss space we get

$$\frac{\mathbb{E}\left[\left(\sum_{i=1}^n \frac{g_i^2}{a_i^2}\right)^{1/2}\right]}{\mathbb{E}\left[\left(\sum_{i=2}^n \frac{g_i^2}{a_i^2}\right)^{1/2}\right]} \geqslant c\left(\frac{\mathbb{E}\left(\sum_{i=1}^n \frac{g_i^2}{a_i^2}\right)}{\mathbb{E}\left(\sum_{i=2}^n \frac{g_i^2}{a_i^2}\right)}\right)^{1/2} = c\left(\frac{\sum_{i=1}^n \frac{1}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2},$$

and hence

$$\alpha_n \geqslant c \cdot C_n a_1 \left(\frac{\sum_{i=1}^n \frac{1}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2} = c \cdot C_n \left(\frac{1+\sum_{i=2}^n \frac{a_1^2}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2}.$$

Now choose $a_2 = \cdots = a_n = r$ and $a_1 = r^{-(n-1)}$. Then,

$$\left(\frac{1+\sum_{i=2}^n \frac{a_1^2}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2} = \left(\frac{1+\frac{n-1}{r^{2n}}}{\frac{n-1}{r^2}}\right)^{1/2} = \left(\frac{1}{r^{2n-2}} + \frac{r^2}{n-1}\right)^{1/2} \to \infty$$

as $r \to \infty$. So, we arrive at a contradiction, i.e. there can be no upper bound for $\alpha_n$.

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true.

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true. Then we can show that the reverse inequality holds.

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true. Then we can show that the reverse inequality holds.In GHP it was prove that that for any convex body $K$ in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$ we have

$$\frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leqslant \frac{2(n-1)}{n} \frac{S(K)}{|K|},$$

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true. Then we can show that the reverse inequality holds.In GHP it was prove that that for any convex body $K$ in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$ we have

$$\frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leqslant \frac{2(n-1)}{n} \frac{S(K)}{|K|},$$

therefore

$$|K| \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{2(n-1)}{n} S(K) \max_{\xi \in S^{n-1}} |P_{\xi^\perp}(K)|.$$

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true. Then we can show that the reverse inequality holds.In GHP it was prove that that for any convex body $K$ in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$ we have

$$\frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leqslant \frac{2(n-1)}{n} \frac{S(K)}{|K|},$$

therefore

$$|K| \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{2(n-1)}{n} S(K) \max_{\xi \in S^{n-1}} |P_{\xi^\perp}(K)|.$$

Since we trivially have

$$|P_{\xi^\perp}(K)| = \frac{1}{2} \int_{S^{n-1}} |\langle \xi, \theta \rangle| \, d\sigma_K(\theta) \leqslant \frac{1}{2} S(K),$$

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true. Then we can show that the reverse inequality holds.In GHP it was prove that that for any convex body $K$ in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$ we have

$$\frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leqslant \frac{2(n-1)}{n} \frac{S(K)}{|K|},$$

therefore

$$|K| \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{2(n-1)}{n} S(K) \max_{\xi \in S^{n-1}} |P_{\xi^\perp}(K)|.$$

Since we trivially have

$$|P_{\xi^\perp}(K)| = \frac{1}{2} \int_{S^{n-1}} |\langle \xi, \theta \rangle| \, d\sigma_K(\theta) \leqslant \frac{1}{2} S(K),$$

we see that

$$|K| \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{n-1}{n} S(K)^2.$$

On the other hand, if $K$ is in some classical position (e.g. isotropic or John's position or minimal surface area or minimal mean width position) then we know that a reverse isoperimetric inequality of the form

$$S(K) \leqslant c_n |K|^{\frac{n-1}{n}}$$

holds true. Then we can show that the reverse inequality holds.In GHP it was prove that that for any convex body $K$ in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$ we have

$$\frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leqslant \frac{2(n-1)}{n} \frac{S(K)}{|K|},$$

therefore

$$|K| \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{2(n-1)}{n} S(K) \max_{\xi \in S^{n-1}} |P_{\xi^\perp}(K)|.$$

Since we trivially have

$$|P_{\xi^\perp}(K)| = \frac{1}{2} \int_{S^{n-1}} |\langle \xi, \theta \rangle| \, d\sigma_K(\theta) \leqslant \frac{1}{2} S(K),$$

we see that

$$|K| \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{n-1}{n} S(K)^2.$$

Using the reserve isoperimetric

$$|K|^{\frac{1}{n}} \max_{\xi \in S^{n-1}} S(K \cap \xi^\perp) \leqslant |K|^{\frac{1}{n}} \max_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant C_n \, S(K)$$

for a constant $C_n$.

If we are not in a classical position then the inequality holds with a parameter.

If we are not in a classical position then the inequality holds with a parameter. One such parameter is

$$t(K) := \left( \frac{|K|}{|r(K)B_2^n|} \right)^{1/n},$$

where $r(K)$ is the inradius.

If we are not in a classical position then the inequality holds with a parameter. One such parameter is

$$t(K) := \left(\frac{|K|}{|r(K)B_2^n|}\right)^{1/n},$$

where $r(K)$ is the inradius.

**Theorem**

Let $K$ be a convex body with barycenter at the origin in $\mathbb{R}^n$. Then, for every $1 \leqslant j \leqslant n-k-1 \leqslant n-1$ we have that

$$W_j(K) \leqslant \alpha_{n,k,j} L_K^{\frac{k(n-k-j)}{n-k}} t(K)^j |K|^{\frac{k}{n}} \max_{H \in G_{n,n-k}} W_j(K \cap H),$$

for some constant $\alpha_{n,k,j}$.

Using the monotonicity of mixed volumes we may write

$$W_j(K) = V((K, n-j), (B_2^n, j)) \leqslant V\left((K, n-j), \left(\frac{K}{r(K)}, j\right)\right) = \frac{|K|}{r(K)^j}.$$

Using the monotonicity of mixed volumes we may write

$$W_j(K) = V((K, n-j), (B_2^n, j)) \leqslant V\left((K, n-j), \left(\frac{K}{r(K)}, j\right)\right) = \frac{|K|}{r(K)^j}.$$

We rewrite this inequality in the form

$$W_j(K) \leqslant \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{n-j}{n}} = \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{k}{n}} |K|^{\frac{n-k-j}{n}}. \tag{1}$$

Using the monotonicity of mixed volumes we may write

$$W_j(K) = V((K, n-j), (B_2^n, j)) \leqslant V\left((K, n-j), \left(\frac{K}{r(K)}, j\right)\right) = \frac{|K|}{r(K)^j}.$$

We rewrite this inequality in the form

$$W_j(K) \leqslant \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{n-j}{n}} = \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{k}{n}} |K|^{\frac{n-k-j}{n}}. \tag{1}$$

Now, we use the estimate (Dafnis-Paouris)

$$\frac{c_0}{L_K} \leq \widetilde{\Phi}_{[k]}(K) := \frac{1}{|K|^{\frac{n-k}{nk}}} \left( \int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k} \right)^{\frac{1}{nk}}.$$

Using the monotonicity of mixed volumes we may write

$$W_j(K) = V((K, n-j), (B_2^n, j)) \leqslant V\left((K, n-j), \left(\frac{K}{r(K)}, j\right)\right) = \frac{|K|}{r(K)^j}.$$

We rewrite this inequality in the form

$$W_j(K) \leqslant \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{n-j}{n}} = \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{k}{n}} |K|^{\frac{n-k-j}{n}}. \tag{1}$$

Now, we use the estimate (Dafnis-Paouris)

$$\frac{c_0}{L_K} \leq \widetilde{\Phi}_{[k]}(K) := \frac{1}{|K|^{\frac{n-k}{nk}}} \left(\int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k}\right)^{\frac{1}{nk}}.$$

This gives

$$|K|^{\frac{n-k}{nk}} \leqslant \frac{L_K}{c_0} \left(\int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k}\right)^{\frac{1}{nk}} \leqslant c_1 L_K \max_{H \in G_{n,n-k}} |K \cap H|^{\frac{1}{k}},$$

Using the monotonicity of mixed volumes we may write

$$W_j(K) = V((K, n-j), (B_2^n, j)) \leqslant V\left((K, n-j), \left(\frac{K}{r(K)}, j\right)\right) = \frac{|K|}{r(K)^j}.$$

We rewrite this inequality in the form

$$W_j(K) \leqslant \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{n-j}{n}} = \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{k}{n}} |K|^{\frac{n-k-j}{n}}. \qquad (1)$$

Now, we use the estimate (Dafnis-Paouris)

$$\frac{c_0}{L_K} \leq \widetilde{\Phi}_{[k]}(K) := \frac{1}{|K|^{\frac{n-k}{nk}}} \left(\int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k}\right)^{\frac{1}{nk}}.$$

This gives

$$|K|^{\frac{n-k}{nk}} \leqslant \frac{L_K}{c_0} \left(\int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k}\right)^{\frac{1}{nk}} \leqslant c_1 L_K \max_{H \in G_{n,n-k}} |K \cap H|^{\frac{1}{k}},$$

and hence,

$$|K|^{\frac{n-k-j}{n}} \leqslant (c_1 L_K)^{\frac{k(n-k-j)}{n-k}} \max_{H \in G_{n,n-k}} |K \cap H|^{\frac{n-k-j}{n-k}}.$$

Using the monotonicity of mixed volumes we may write

$$W_j(K) = V((K, n-j), (B_2^n, j)) \leqslant V\left((K, n-j), \left(\frac{K}{r(K)}, j\right)\right) = \frac{|K|}{r(K)^j}.$$

We rewrite this inequality in the form

$$W_j(K) \leqslant \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{n-j}{n}} = \omega_n^{\frac{j}{n}} t(K)^j |K|^{\frac{k}{n}} |K|^{\frac{n-k-j}{n}}. \tag{1}$$

Now, we use the estimate (Dafnis-Paouris)

$$\frac{c_0}{L_K} \leq \widetilde{\Phi}_{[k]}(K) := \frac{1}{|K|^{\frac{n-k}{nk}}} \left(\int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k}\right)^{\frac{1}{nk}}.$$

This gives

$$|K|^{\frac{n-k}{nk}} \leqslant \frac{L_K}{c_0} \left(\int_{G_{n,n-k}} |K \cap H|^n d\nu_{n,n-k}\right)^{\frac{1}{nk}} \leqslant c_1 L_K \max_{H \in G_{n,n-k}} |K \cap H|^{\frac{1}{k}},$$

and hence,

$$|K|^{\frac{n-k-j}{n}} \leqslant (c_1 L_K)^{\frac{k(n-k-j)}{n-k}} \max_{H \in G_{n,n-k}} |K \cap H|^{\frac{n-k-j}{n-k}}.$$

On the other hand, applying Aleksandrov's inequalities for $K \cap H$ we get

$$|K \cap H|^{\frac{n-k-j}{n-k}} \leqslant \omega_{n-k}^{-\frac{j}{n-k}} W_j(K \cap H)$$

for every $H \in G_{n,n-k}$.

Combining the above we see that

$$|K|^{\frac{n-k-j}{n}} \leqslant \frac{1}{\omega_{n-k}^{\frac{j}{n-k}}}(c_1 L_K)^{\frac{k(n-k-j)}{n-k}} \max_{H\in G_{n,n-k}} W_j(K\cap H),$$

and then (1) takes the form

$$W_j(K) \leqslant (\omega_n^{\frac{j}{n}}/\omega_{n-k}^{\frac{j}{n-k}})(c_1 L_K)^{\frac{k(n-k-j)}{n-k}} t(K)^j |K|^{\frac{k}{n}} \max_{H\in G_{n,n-k}} W_j(K\cap H).$$

Setting $\alpha_{n,k,j} = (\omega_n^{\frac{j}{n}}/\omega_{n-k}^{\frac{j}{n-k}})c_1^{\frac{k(n-k-j)}{n-k}}$ we conclude the proof.

## Question-Koldobsky and König

If $K$ and $D$ are two convex bodies in $\mathbb{R}^n$ such that $S(K\cap\xi^\perp)\leqslant S(D\cap\xi^\perp)$ for all $\xi\in S^{n-1}$ does it then follow that $S(K)\leqslant S(D)$?

## Question-Koldobsky and König

If $K$ and $D$ are two convex bodies in $\mathbb{R}^n$ such that $S(K \cap \xi^\perp) \leqslant S(D \cap \xi^\perp)$ for all $\xi \in S^{n-1}$ does it then follow that $S(K) \leqslant S(D)$?

Answering a question of Pełczynski, they prove that the central $(n-1)$-dimensional section of the cube $B_\infty^n = [-1,1]^n$ that has maximal surface area is the one that corresponds to the unit vector $\xi_0 = \frac{1}{\sqrt{2}}(1,1,0,\dots,0)$ (exactly as in the case of volume) i.e.

$$\max_{\xi \in S^{n-1}} S(B_\infty^n \cap \xi^\perp) = S(B_\infty^n \cap \xi_0^\perp) = 2((n-2)\sqrt{2}+1).$$

# Connection with Busemann-Petty

## Question-Koldobsky and König

If $K$ and $D$ are two convex bodies in $\mathbb{R}^n$ such that $S(K \cap \xi^\perp) \leqslant S(D \cap \xi^\perp)$ for all $\xi \in S^{n-1}$ does it then follow that $S(K) \leqslant S(D)$?

Answering a question of Pełczynski, they prove that the central $(n-1)$-dimensional section of the cube $B_\infty^n = [-1,1]^n$ that has maximal surface area is the one that corresponds to the unit vector $\xi_0 = \frac{1}{\sqrt{2}}(1,1,0,\dots,0)$ (exactly as in the case of volume) i.e.

$$\max_{\xi \in S^{n-1}} S(B_\infty^n \cap \xi^\perp) = S(B_\infty^n \cap \xi_0^\perp) = 2((n-2)\sqrt{2}+1).$$

Comparing with a ball of suitable radius one gets that the answer to the Busemann-Petty problem for surface area is negative in dimensions $n \geqslant 14$.

# Connection with Busemann-Petty

## Question-Koldobsky and König

If $K$ and $D$ are two convex bodies in $\mathbb{R}^n$ such that $S(K \cap \xi^\perp) \leqslant S(D \cap \xi^\perp)$ for all $\xi \in S^{n-1}$ does it then follow that $S(K) \leqslant S(D)$?

Answering a question of Pełczynski, they prove that the central $(n-1)$-dimensional section of the cube $B_\infty^n = [-1,1]^n$ that has maximal surface area is the one that corresponds to the unit vector $\xi_0 = \frac{1}{\sqrt{2}}(1,1,0,\dots,0)$ (exactly as in the case of volume) i.e.

$$\max_{\xi \in S^{n-1}} S(B_\infty^n \cap \xi^\perp) = S(B_\infty^n \cap \xi_0^\perp) = 2((n-2)\sqrt{2}+1).$$

Comparing with a ball of suitable radius one gets that the answer to the Busemann-Petty problem for surface area is negative in dimensions $n \geqslant 14$.

## Isomorphic version

Is there a constant $\beta_n$ such that if $K$ and $D$ are two convex bodies in $\mathbb{R}^n$ with $S(K \cap \xi^\perp) \leqslant S(D \cap \xi^\perp)$ for all $\xi \in S^{n-1}$ then $S(K) \leqslant \beta_n S(D)$?

Suppose that the isomorphic version holds, i.e. there is a constant $\beta_n$ such that if $K$ and $D$ are centrally symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$S(K \cap \xi^\perp) \leqslant S(D \cap \xi^\perp),$$

for all $\xi \in S^{n-1}$, then $S(K) \leqslant \beta_n S(D)$.

Suppose that the isomorphic version holds, i.e. there is a constant $\beta_n$ such that if $K$ and $D$ are centrally symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$S(K \cap \xi^{\perp}) \leqslant S(D \cap \xi^{\perp}),$$

for all $\xi \in S^{n-1}$, then $S(K) \leqslant \beta_n S(D)$.

Now, let $K$ be a convex body in $\mathbb{R}^n$ and choose $\xi_0 \in S^{n-1}$ such that

$$S(K \cap \xi_0^{\perp}) = \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

Suppose that the isomorphic version holds, i.e. there is a constant $\beta_n$ such that if $K$ and $D$ are centrally symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$S(K \cap \xi^{\perp}) \leqslant S(D \cap \xi^{\perp}),$$

for all $\xi \in S^{n-1}$, then $S(K) \leqslant \beta_n S(D)$.
Now, let $K$ be a convex body in $\mathbb{R}^n$ and choose $\xi_0 \in S^{n-1}$ such that

$$S(K \cap \xi_0^{\perp}) = \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

and $r > 0$ such that $r^{n-2} S(B_2^{n-1}) = S(rB_2^{n-1}) = S(K \cap \xi_0^{\perp})$.

Suppose that the isomorphic version holds, i.e. there is a constant $\beta_n$ such that if $K$ and $D$ are centrally symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$S(K \cap \xi^\perp) \leqslant S(D \cap \xi^\perp),$$

for all $\xi \in S^{n-1}$, then $S(K) \leqslant \beta_n S(D)$.
Now, let $K$ be a convex body in $\mathbb{R}^n$ and choose $\xi_0 \in S^{n-1}$ such that

$$S(K \cap \xi_0^\perp) = \max_{\xi \in S^{n-1}} S(K \cap \xi^\perp)$$

and $r > 0$ such that $r^{n-2} S(B_2^{n-1}) = S(rB_2^{n-1}) = S(K \cap \xi_0^\perp)$.Then,

$$S(K \cap \xi^\perp) \leqslant S(rB_2^n \cap \xi^\perp),$$

for all $\xi \in S^{n-1}$.

Suppose that the isomorphic version holds, i.e. there is a constant $\beta_n$ such that if $K$ and $D$ are centrally symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$S(K \cap \xi^{\perp}) \leqslant S(D \cap \xi^{\perp}),$$

for all $\xi \in S^{n-1}$, then $S(K) \leqslant \beta_n S(D)$.
Now, let $K$ be a convex body in $\mathbb{R}^n$ and choose $\xi_0 \in S^{n-1}$ such that

$$S(K \cap \xi_0^{\perp}) = \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

and $r > 0$ such that $r^{n-2} S(B_2^{n-1}) = S(rB_2^{n-1}) = S(K \cap \xi_0^{\perp})$.Then,

$$S(K \cap \xi^{\perp}) \leqslant S(rB_2^n \cap \xi^{\perp}),$$

for all $\xi \in S^{n-1}$. Therefore,

$$S(K)^{\frac{n-2}{n-1}} \leqslant \beta_n^{\frac{n-2}{n-1}} S(rB_2^n)^{\frac{n-2}{n-1}} = \beta_n^{\frac{n-2}{n-1}} \frac{S(B_2^n)^{\frac{n-2}{n-1}}}{S(B_2^{n-1})} \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp}).$$

Suppose that the isomorphic version holds, i.e. there is a constant $\beta_n$ such that if $K$ and $D$ are centrally symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$S(K\cap\xi^{\perp})\leqslant S(D\cap\xi^{\perp}),$$

for all $\xi\in S^{n-1}$, then $S(K)\leqslant\beta_n S(D)$.
Now, let $K$ be a convex body in $\mathbb{R}^n$ and choose $\xi_0\in S^{n-1}$ such that

$$S(K\cap\xi_0^{\perp})=\max_{\xi\in S^{n-1}}S(K\cap\xi^{\perp})$$

and $r>0$ such that $r^{n-2}S(B_2^{n-1})=S(rB_2^{n-1})=S(K\cap\xi_0^{\perp})$.Then,

$$S(K\cap\xi^{\perp})\leqslant S(rB_2^n\cap\xi^{\perp}),$$

for all $\xi\in S^{n-1}$. Therefore,

$$S(K)^{\frac{n-2}{n-1}}\leqslant\beta_n^{\frac{n-2}{n-1}}S(rB_2^n)^{\frac{n-2}{n-1}}=\beta_n^{\frac{n-2}{n-1}}\frac{S(B_2^n)^{\frac{n-2}{n-1}}}{S(B_2^{n-1})}\max_{\xi\in S^{n-1}}S(K\cap\xi^{\perp}).$$

This implies that there is some constant $c(n)$ such that

$$S(K)\leqslant c(n)S(K)^{\frac{1}{n-1}}\max_{\xi\in S^{n-1}}S(K\cap\xi^{\perp}).$$

The validity of the above is a new question.

We start with an estimate for ellipsoids.

## Proposition

Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$. Then,

$$\frac{S(\mathcal{E})}{\max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})} \leqslant D_n r(\mathcal{E})^{-\frac{1}{n-1}}$$

where $D_n > 0$ is bounded by an absolute constant.

We start with an estimate for ellipsoids.

**Proposition**

Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$. Then,

$$\frac{S(\mathcal{E})}{\max\limits_{\xi\in S^{n-1}} S(\mathcal{E}\cap\xi^{\perp})}\leqslant D_n r(\mathcal{E})^{-\frac{1}{n-1}}$$

where $D_n>0$ is bounded by an absolute constant.

We may assume that $|\mathcal{E}|=1$. Let $a_1\leqslant\cdots\leqslant a_n$ be the lengths of its principal semi-axes of $\mathcal{E}$ in the directions of $e_1,\dots,e_n$. We have seen that

$$\frac{S(\mathcal{E})}{\max\limits_{\xi\in S^{n-1}} S(\mathcal{E}\cap\xi^{\perp})}=C_n a_1\frac{\mathbb{E}\left[\left(\sum_{i=1}^n\frac{g_i^2}{a_i^2}\right)^{1/2}\right]}{\mathbb{E}\left[\left(\sum_{i=2}^n\frac{g_i^2}{a_i^2}\right)^{1/2}\right]},$$

where $C_n$ is bounded by an absolute constant.
Since

$$\frac{\mathbb{E}\left[\left(\sum_{i=1}^n\frac{g_i^2}{a_i^2}\right)^{1/2}\right]}{\mathbb{E}\left[\left(\sum_{i=2}^n\frac{g_i^2}{a_i^2}\right)^{1/2}\right]}\leqslant c\left(\frac{\mathbb{E}\left(\sum_{i=1}^n\frac{g_i^2}{a_i^2}\right)}{\mathbb{E}\left(\sum_{i=2}^n\frac{g_i^2}{a_i^2}\right)}\right)^{1/2}=c\left(\frac{\sum_{i=1}^n\frac{1}{a_i^2}}{\sum_{i=2}^n\frac{1}{a_i^2}}\right)^{1/2},$$

we have that

$$\frac{S(\mathcal{E})}{\max\limits_{\xi\in S^{n-1}} S(\mathcal{E}\cap\xi^{\perp})} \leqslant C_n a_1 \left(\frac{\sum_{i=1}^n \frac{1}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2} = C_n \left(1+\frac{1}{\sum_{i=2}^n \frac{a_1^2}{a_i^2}}\right)^{1/2}.$$

we have that

$$\frac{S(\mathcal{E})}{\max\limits_{\xi\in S^{n-1}} S(\mathcal{E}\cap\xi^{\perp})} \leqslant C_n a_1 \left(\frac{\sum_{i=1}^{n}\frac{1}{a_i^2}}{\sum_{i=2}^{n}\frac{1}{a_i^2}}\right)^{1/2} = C_n \left(1+\frac{1}{\sum_{i=2}^{n}\frac{a_1^2}{a_i^2}}\right)^{1/2}.$$

Using the arithmetic-geometric mean inequality we get

$$\sum_{i=2}^{n}\frac{a_1^2}{a_i^2} \geqslant (n-1)a_1^2\left(\frac{1}{a_2^2\dots a_n^2}\right)^{\frac{1}{n-1}} = (n-1)a_1^2 a_1^{\frac{2}{n-1}} = (n-1)a_1^{\frac{2n}{n-1}}.$$

we have that

$$\frac{S(\mathcal{E})}{\max\limits_{\xi\in S^{n-1}} S(\mathcal{E}\cap\xi^\perp)} \leqslant C_n a_1 \left(\frac{\sum_{i=1}^n \frac{1}{a_i^2}}{\sum_{i=2}^n \frac{1}{a_i^2}}\right)^{1/2} = C_n\left(1+\frac{1}{\sum_{i=2}^n \frac{a_1^2}{a_i^2}}\right)^{1/2}.$$

Using the arithmetic-geometric mean inequality we get

$$\sum_{i=2}^n \frac{a_1^2}{a_i^2} \geqslant (n-1)a_1^2\left(\frac{1}{a_2^2\dots a_n^2}\right)^{\frac{1}{n-1}} = (n-1)a_1^2 a_1^{\frac{2}{n-1}} = (n-1)a_1^{\frac{2n}{n-1}}.$$

Moreover, $1 \leqslant \frac{1}{a_1^{\frac{2n}{n-1}}}$ and adding these two inequalities we get

$$\left(1+\frac{1}{\sum_{i=2}^n \frac{a_1^2}{a_i^2}}\right)^{1/2} \leqslant \left(\frac{1}{a_1^{\frac{2n}{n-1}}}+\frac{1}{(n-1)a_1^{\frac{2n}{n-1}}}\right)^{\frac{1}{2}},$$

therefore

$$\frac{S(\mathcal{E})}{\max\limits_{\xi\in S^{n-1}} S(\mathcal{E}\cap\xi^\perp)} \leqslant D_n\frac{1}{a_1^{\frac{1}{n-1}}} = D_n\frac{1}{r(\mathcal{E})^{\frac{1}{n-1}}},$$

where $D_n$ is bounded by an absolute constant.

The example of an ellipsoid $\mathcal{F}$ with $a_2 = \ldots = a_n = r$ and $a_1 = \frac{1}{r^{n-1}}$ gives that

$$\frac{S(\mathcal{F})}{\max_{\xi \in S^{n-1}} S(\mathcal{F} \cap \xi^{\perp})} \geqslant E_n \frac{1}{r(\mathcal{F})^{\frac{1}{n-1}}},$$

therefore the inequality is sharp.

The example of an ellipsoid $\mathcal{F}$ with $a_2 = \ldots = a_n = r$ and $a_1 = \frac{1}{r^{n-1}}$ gives that

$$\frac{S(\mathcal{F})}{\max_{\xi \in S^{n-1}} S(\mathcal{F} \cap \xi^{\perp})} \geqslant E_n \frac{1}{r(\mathcal{F})^{\frac{1}{n-1}}},$$

therefore the inequality is sharp.
Using the inequality that $\frac{1}{r(K)} \leqslant S(K)$ for every convex body $K$ of volume 1 in $\mathbb{R}^n$ we get

The example of an ellipsoid $\mathcal{F}$ with $a_2 = \ldots = a_n = r$ and $a_1 = \frac{1}{r^{n-1}}$ gives that

$$\frac{S(\mathcal{F})}{\max_{\xi \in S^{n-1}} S(\mathcal{F} \cap \xi^{\perp})} \geqslant E_n \frac{1}{r(\mathcal{F})^{\frac{1}{n-1}}},$$

therefore the inequality is sharp.
Using the inequality that $\frac{1}{r(K)} \leqslant S(K)$ for every convex body $K$ of volume 1 in $\mathbb{R}^n$ we get

## Theorem

*Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$. Then,*

$$S(\mathcal{E}) \leqslant A_n S(\mathcal{E})^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

*where $A_n > 0$ is bounded by an absolute constant.*

The example of an ellipsoid $\mathcal{F}$ with $a_2 = \ldots = a_n = r$ and $a_1 = \frac{1}{r^{n-1}}$ gives that

$$\frac{S(\mathcal{F})}{\max_{\xi \in S^{n-1}} S(\mathcal{F} \cap \xi^{\perp})} \geqslant E_n \frac{1}{r(\mathcal{F})^{\frac{1}{n-1}}},$$

therefore the inequality is sharp.
Using the inequality that $\frac{1}{r(K)} \leqslant S(K)$ for every convex body $K$ of volume 1 in $\mathbb{R}^n$ we get

**Theorem**

*Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$. Then,*

$$S(\mathcal{E}) \leqslant A_n S(\mathcal{E})^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

*where $A_n > 0$ is bounded by an absolute constant.*

Using John's theorem and the monotonicity of surface area one can easily deduce that a similar estimate holds true in full generality:

The example of an ellipsoid $\mathcal{F}$ with $a_2 = \ldots = a_n = r$ and $a_1 = \frac{1}{r^{n-1}}$ gives that

$$\frac{S(\mathcal{F})}{\max_{\xi \in S^{n-1}} S(\mathcal{F} \cap \xi^{\perp})} \geqslant E_n \frac{1}{r(\mathcal{F})^{\frac{1}{n-1}}},$$

therefore the inequality is sharp.
Using the inequality that $\frac{1}{r(K)} \leqslant S(K)$ for every convex body $K$ of volume 1 in $\mathbb{R}^n$ we get

## Theorem

*Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$. Then,*

$$S(\mathcal{E}) \leqslant A_n S(\mathcal{E})^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

*where $A_n > 0$ is bounded by an absolute constant.*

Using John's theorem and the monotonicity of surface area one can easily deduce that a similar estimate holds true in full generality:For any convex body $K$ in $\mathbb{R}^n$ one has

$$S(K) \leqslant A'_n S(K)^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

where $A'_n > 0$ is a constant depending only on $n$.

The example of an ellipsoid $\mathcal{F}$ with $a_2 = \ldots = a_n = r$ and $a_1 = \frac{1}{r^{n-1}}$ gives that

$$\frac{S(\mathcal{F})}{\max_{\xi \in S^{n-1}} S(\mathcal{F} \cap \xi^{\perp})} \geqslant E_n \frac{1}{r(\mathcal{F})^{\frac{1}{n-1}}},$$

therefore the inequality is sharp.
Using the inequality that $\frac{1}{r(K)} \leqslant S(K)$ for every convex body $K$ of volume 1 in $\mathbb{R}^n$ we get

## Theorem

*Let $\mathcal{E}$ be an origin symmetric ellipsoid in $\mathbb{R}^n$. Then,*

$$S(\mathcal{E}) \leqslant A_n S(\mathcal{E})^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(\mathcal{E} \cap \xi^{\perp})$$

*where $A_n > 0$ is bounded by an absolute constant.*

Using John's theorem and the monotonicity of surface area one can easily deduce that a similar estimate holds true in full generality:For any convex body $K$ in $\mathbb{R}^n$ one has

$$S(K) \leqslant A_n' S(K)^{\frac{1}{n-1}} \max_{\xi \in S^{n-1}} S(K \cap \xi^{\perp})$$

where $A_n' > 0$ is a constant depending only on $n$.It is an interesting question to determine the best possible behavior of the constant $A_n'$ with respect to the dimension $n$.

# New Parameters

Recall Rivin's formula.

# New Parameters

Recall Rivin's formula.If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \ldots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}|\,\int_{S^{n-1}} \Big(\sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2}\Big)^{1/2} d\sigma(\xi).$$

# New Parameters

Recall Rivin's formula.If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \ldots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}|\,\int_{S^{n-1}} \Big(\sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2}\Big)^{1/2} d\sigma(\xi).$$

This can be rewritten as

$$S(\mathcal{E}) \approx n\,|\mathcal{E}| M(\mathcal{E})$$

where $M(\mathcal{E}) = \int_{S^{n-1}} \|\xi\|_{\mathcal{E}}\, d\sigma(\xi)$.

# New Parameters

Recall Rivin's formula.If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \ldots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}|\, \int_{S^{n-1}} \Big( \sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2} \Big)^{1/2} d\sigma(\xi).$$

This can be rewritten as

$$S(\mathcal{E}) \approx n\,|\mathcal{E}| M(\mathcal{E})$$

where $M(\mathcal{E}) = \int_{S^{n-1}} \|\xi\|_{\mathcal{E}} d\sigma(\xi)$. It is natural to introduce the parameter

$$p(K) = \frac{S(K)}{|K| M(K)}.$$

Recall Rivin's formula.If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \ldots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}| \int_{S^{n-1}} \Big( \sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2} \Big)^{1/2} d\sigma(\xi).$$

This can be rewritten as

$$S(\mathcal{E}) \approx n\,|\mathcal{E}| M(\mathcal{E})$$

where $M(\mathcal{E}) = \int_{S^{n-1}} \|\xi\|_{\mathcal{E}}\, d\sigma(\xi)$. It is natural to introduce the parameter

$$p(K) = \frac{S(K)}{|K| M(K)}.$$

Our aim is to provide optimal upper and lower bounds for $p(K)$ both in general and in the case where $K$ is in some of the classical positions.
We show the following:

## New Parameters

Recall Rivin's formula.If $\mathcal{E}$ is an ellipsoid in $\mathbb{R}^n$ with semi-axes $a_1 \leqslant \cdots \leqslant a_n$ in the directions of $e_1, \ldots, e_n$ then

$$S(\mathcal{E}) = n\,|\mathcal{E}| \int_{S^{n-1}} \Big(\sum_{i=1}^{n} \frac{\xi_i^2}{a_i^2}\Big)^{1/2} d\sigma(\xi).$$

This can be rewritten as

$$S(\mathcal{E}) \approx n\,|\mathcal{E}| M(\mathcal{E})$$

where $M(\mathcal{E}) = \int_{S^{n-1}} \|\xi\|_{\mathcal{E}}\, d\sigma(\xi)$. It is natural to introduce the parameter

$$p(K) = \frac{S(K)}{|K| M(K)}.$$

Our aim is to provide optimal upper and lower bounds for $p(K)$ both in general and in the case where $K$ is in some of the classical positions.

We show the following:

### Theorem

There exist absolute constants $c_1, c_2 > 0$ such that for every convex body $K \in \mathbb{R}^n$ we have

$$c_1\sqrt{n} \leqslant p(K) \leqslant c_2 n^{3/2}.$$

Moreover, both estimates give the optimal dependence on the dimension.

Proof in the centrally symmetric case.

Proof in the centrally symmetric case.By the simple case $k = 1$ of the Rogers-Shephard inequality we have that

$$\frac{n}{2}\|u\|_K|K| \geqslant |P_{u^\perp}K| \geqslant \frac{\|u\|_K}{2}|K|.$$

Proof in the centrally symmetric case.By the simple case $k = 1$ of the Rogers-Shephard inequality we have that

$$\frac{n}{2}\|u\|_K|K| \geqslant |P_{u^\perp}K| \geqslant \frac{\|u\|_K}{2}|K|.$$

Integrating over the sphere we get

$$\frac{1}{2}M(K)|K| \leqslant \frac{\omega_{n-1}}{n\omega_n}S(K) \approx \frac{1}{\sqrt{n}}S(K) \leqslant \frac{n}{2}M(K)|K|.$$

Proof in the centrally symmetric case.By the simple case $k=1$ of the Rogers-Shephard inequality we have that

$$\frac{n}{2}\|u\|_K|K| \geqslant |P_{u^\perp}K| \geqslant \frac{\|u\|_K}{2}|K|.$$

Integrating over the sphere we get

$$\frac{1}{2}M(K)|K| \leqslant \frac{\omega_{n-1}}{n\omega_n}S(K) \approx \frac{1}{\sqrt{n}}S(K) \leqslant \frac{n}{2}M(K)|K|.$$

The order of the bounds is sharp since they are achieved by

$$P_s = \left\{x \in \mathbb{R}^n\colon\ |x_1| + \frac{1}{s}\sum_{i=2}^{n}|x_i| \leqslant 1\right\}$$

and

$$P_{a,s} = \{x:\ |x_1| \leqslant s, |x_i| \leqslant a \text{ for } i \geqslant 2\}$$

where $0 < s < a$.

Thank you for your attention!!!