Convexity, symmetry, the L^2 method and moment maps: a discussion following the solution to the Log-Brunn–Minkowski conjecture by OpenAI

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Abstract

These lecture notes follow up the recent OpenAI proof [131] of the Log-Brunn–Minkowski conjecture posed by Böröczky, Lutwak, Yang, and Zhang [22].

The main analytic result presented here is Theorem C, a variance inequality extending the homogeneous statement used in [131]. It is equivalent to the functional dilation inequality in Theorem B and the weighted intersection inequality in Theorem A. These formulations yield the Log-Brunn–Minkowski inequality and several related geometric results.

We outline the general framework: Hörmander's L^2 duality principle, together with a suitable Bochner identity, leads to Poincaré-type inequalities. We first discuss the "baby setting" of the classical weighted Laplacian, obtaining the Brascamp–Lieb and Klartag H^{-1} inequalities, and then extrapolate to the general operator setting.

We then discuss "the other baby" case: the operator L_μ^m associated with a moment map, building on work of Klartag, Cordero-Erausquin, Fathi, Rotem, and others [42, 84, 83, 58, 139, 89]. The key object is the iterated moment-map operator $P = L_\mu^m(L_\mu^m + I)$. This fourth-order differential operator has affine kernel, and moreover possesses optimality properties among such operators with affine kernels. The resulting second-order L^2 argument for P gives strong isoperimetric consequences in the even setting. We also discuss a semigroup associated with P (which

converges to an affine function), prove hypercontractivity and discuss semigroup proofs of the main results, aiming to create a better geometric intuition.

A separate section explains the connections with E. Milman's centroaffine geometry. Lastly, we discuss the rigidity and optimality properties of moment coordinates, functional cost inequalities, and pose questions about convexity, symmetry, and transport.

The AI-produced analysis is technically involving. We aimed to organize the presentation so that the main argument is easier to follow. The notes are intended for the course at the CIRM research school *Functional inequalities and applications*, Marseille, November 2026; the topic of the Log-Brunn-Minkowski inequality was explicitly requested by the organizers in the Fall of 2025.

Contents

1 Introduction **4**
- 1.1 Geometric phenomena in the presence of convexity and symmetry . . 6
- 1.2 A functional inequality 12
- 1.3 The key variance inequality 15
- 1.4 Convexity of entropy along an exponential transport 18
- 1.5 Proof overview, notes organization and acknowledgements 19

2 Preliminaries: second-order operators and existence results **23**

3 L^2 method: the baby case **29**
- 3.1 Integration by parts and Bochner formula 29
- 3.2 Hörmander duality and its two consequences 31
- 3.3 What evenness improves in the classical setting 32

4 L^2 method: the general (grownup) case **34**
- 4.1 The general form of our operator 34
- 4.2 Bochner with a variable matrix 34
- 4.3 Duality and its two consequences 36
- 4.4 Evenness, low modes, and the choice of a Stein kernel 37

5 L^2 method: the other baby – the iterated moment-map operator **39**
- 5.1 Moment measures: an introductory discussion 42
- 5.2 Moment map operator: first order framework 48
- 5.3 The iterated operator and its bookkeeping identities 53
- 5.4 Elementary matrix inequalities 57
- 5.5 Combining integration by parts with linear algebra into a crucial estimate 59
- 5.6 The variance theorem modulo affine functions 63

5.7 Proof of Theorem C 65

6 Entropy, convex potentials, and interpolation 66
6.1 The moment map as the optimizer 66
6.2 Entropy second variation and the coercivity estimate 68
6.3 Finite entropy interpolation and Theorem B 70
6.4 A simpler interpolation in the unconditional case 72

7 Hypercontractivity and a semigroup proof of Theorem B 73
7.1 The evolution and its affine limit 74
7.2 The moment diffusion and Gross's argument 76
7.3 Hypercontractivity modulo affine functions 77

8 Rigidity and optimality of moment coordinates 79
8.1 An optimal spectral gap, with a fixed normalization 80
8.2 Compatibility with Hessian coordinates 81
8.3 The variance estimate for a general Stein kernel 83

9 Moment geometry and E. Milman's centroaffine geometry 85
9.1 The angular part of a moment map 85
9.2 A short glossary 87

10 Concluding remarks and open questions 88
10.1 A cost inequality and its radial specialization 88
10.2 Some unsolved conjectures 89
10.3 Vague questions 91

A Removing the bounded-Hessian assumption and reducing regularity 94

B Approximation and passage to convex sets: a detailed deduction of Theorem A from Theorem C 99

C Analytic details for the moment semigroups 101
C.1 The closed Hessian energy and the gap 101
C.2 The diffusion inequality and the norm comparison 102
C.3 Relaxation and integration in time 103
C.4 Compact tests and the dilation argument 103

References 105

1 Introduction

The isoperimetric inequality, familiar to the ancient Greeks, states that among plane figures of prescribed perimeter, the circle encloses the largest area. More generally, a bounded smooth domain $K \subset \mathbb{R}^n$ satisfies

$$|\partial K| \geq n|B_2^n|^{1/n}|K|^{(n-1)/n}, \tag{1.1}$$

where B_2^n is the Euclidean unit ball. Here $|\partial K|$ stands for the perimeter and $|\cdot|$ stands for Lebesgue measure. Equality holds when K is a ball. The passage from the ancient problem, through Steiner's geometric arguments, to the modern theory illustrates how a simple geometric question can acquire many analytic formulations; see [133, 68, 145] for history and further references.

A related powerful fact is the Brunn–Minkowski inequality: for nonempty Borel measurable sets $K, L \subset \mathbb{R}^n$,

$$|K + L|^{1/n} \geq |K|^{1/n} + |L|^{1/n}, \tag{1.2}$$

where $K + L = \{x + y : x \in K,\ y \in L\}$ is the Minkowski sum; Lebesgue measure of the sum is understood in the completed measure space. Brunn and Minkowski established the convex-geometric theory in the late nineteenth century [68, 145]; Lusternik extended the inequality beyond convex sets [68]. The ensuing theory of mixed volumes, developed further by Aleksandrov and Fenchel [145], has become a meeting point of convex geometry, analysis, and PDE; we refer to [68, 145] and the L_p extensions of Firey and Lutwak [63, 108, 109, 110]. For affine Sobolev connections, see [112]. To deduce the isoperimetric inequality from Brunn–Minkowski, apply (1.2) to K and tB_2^n:

$$|K + tB_2^n|^{1/n} \geq |K|^{1/n} + t|B_2^n|^{1/n}.$$

For a smooth domain, differentiating at $t = 0+$, using $\frac{d}{dt}|K + tB_2^n|\Big|_{0+} = |\partial K|$, gives (1.1). Due to the homogeneity of Lebesgue measure, the Brunn–Minkowski inequality is equivalent to

$$|(1-\lambda)K + \lambda L|^{1/n} \geq (1-\lambda)|K|^{1/n} + \lambda|L|^{1/n}, \qquad \lambda \in [0,1]. \tag{1.3}$$

Thus Brunn–Minkowski expresses a concavity property of Lebesgue measure under addition of sets [68]. In view of the fact that

$$((1-\lambda)a^p + \lambda b^p)^{1/p} \geq a^{1-\lambda}b^\lambda$$

for all $p, a, b > 0$ and $\lambda \in [0,1]$, (1.3) implies

$$|(1-\lambda)K + \lambda L| \geq |K|^{1-\lambda}|L|^\lambda.$$

Moreover, homogeneity makes this multiplicative form equivalent to (1.3) (see, e.g., [68]). Borell's theorem (a consequence of the Prékopa–Leindler theorem) extends this phenomenon further: a measure with a log-concave density e^{-V}, where V is convex and finite on its convex effective domain, is a *log-concave measure*, and satisfies

$$\mu((1-\lambda)K + \lambda L) \geq \mu(K)^{1-\lambda}\mu(L)^{\lambda}$$

for Borel measurable sets [19, 134, 28]. This is one of the key manifestations of the fact that convexity and isoperimetry go hand in hand.

On the other hand, isoperimetry manifests itself in spectral properties of linear differential operators. For instance, the planar isoperimetric inequality is equivalent to the Poincaré, or Wirtinger, inequality on the circle [69]. For a smooth strictly convex curve with support function $h(\theta)$, its length and enclosed area are

$$L = \int_0^{2\pi} h\, d\theta, \qquad A = \frac{1}{2}\int_0^{2\pi} \left(h^2 - (h')^2\right) d\theta.$$

Write $h = \overline{h} + f$, where $\int_0^{2\pi} f = 0$. Then

$$L^2 - 4\pi A = 2\pi \int_0^{2\pi} \left((f')^2 - f^2\right) d\theta.$$

Consequently, the planar isoperimetric inequality for convex curves is equivalent to the inequality

$$\int_0^{2\pi} f^2\, d\theta \leq \int_0^{2\pi} (f')^2\, d\theta, \qquad \int_0^{2\pi} f\, d\theta = 0. \tag{1.4}$$

Conversely, any smooth mean-zero f gives a strictly convex support function $h = R + f$ for sufficiently large R, so the implication goes both ways. Convexification and approximation connect this calculation with the usual planar isoperimetric problem; see [69].

One of the most natural ways to prove (1.4) is via a Fourier series. If

$$f(\theta) = \sum_{k\geq 1} (a_k \cos k\theta + b_k \sin k\theta),$$

then Parseval's identity gives

$$\begin{aligned}\int_0^{2\pi} f^2\, d\theta &= \pi \sum_{k\geq 1} (a_k^2 + b_k^2) \\ &\leq \pi \sum_{k\geq 1} k^2 (a_k^2 + b_k^2) = \int_0^{2\pi} (f')^2\, d\theta,\end{aligned}$$

and (1.4) follows.

What happens if K is origin-symmetric in the plane? This corresponds to the situation when f is π-periodic, that is, $f(\theta+\pi)=f(\theta)$. In this case, all odd-frequency terms disappear, and the constant improves:

$$\int_0^{2\pi} f^2\,d\theta \le \frac{1}{4}\int_0^{2\pi}(f')^2\,d\theta. \tag{1.5}$$

In these notes we attempt to capture a much more general manifestation of the same phenomenon. *How does symmetry improve isoperimetry? More precisely, what isoperimetric phenomena appear in the presence of both convexity and symmetry?*

These questions have fueled a rich research direction in convex geometry. The logarithmic Brunn–Minkowski conjecture of Böröczky, Lutwak, Yang, and Zhang [22], the dimensional Brunn–Minkowski conjecture of Gardner–Zvavitch and its log-concave extension [67, 40], Banaszczyk's B-conjecture, and Dar's conjecture (which is still open) [51, 151] are a few examples which explore different aspects of this phenomenon.

The recent moment-coordinate proof in [131] and the centered extension in [103] provided the positive answer to the Log-Brunn-Minkowski conjecture, and hence to many other questions. These notes aim for a clear exposition of this topic while placing things in context.

1.1 Geometric phenomena in the presence of convexity and symmetry

Throughout this subsection the measure has an even log-concave density:

$$d\mu(x)=e^{-V(x)}\,dx, \qquad V:\mathbb{R}^n\longrightarrow(-\infty,+\infty], \tag{1.6}$$

where V is proper, lower semicontinuous, even, and convex. Its effective domain is assumed to have nonempty interior. Symmetry of a set means symmetry about the origin. A convex body is compact and has nonempty interior.

We begin with a weighted correlation formulation of the Log-Brunn–Minkowski inequality, following Saroglou [143, 144]: an inequality involving intersections of dilated convex sets.

Theorem A (Weighted correlation inequality)**.** *Let $K_1,\dots,K_m$ be closed symmetric convex sets, and suppose*

$$0<\mu\left(\bigcap_{i=1}^m K_i\right)<\infty.$$

For positive numbers r_i, s_i and $\lambda \in [0,1]$,

$$\mu\left(\bigcap_{i=1}^{m} r_i^{1-\lambda} s_i^{\lambda} K_i\right) \geq \mu\left(\bigcap_{i=1}^{m} r_i K_i\right)^{1-\lambda} \mu\left(\bigcap_{i=1}^{m} s_i K_i\right)^{\lambda}. \tag{1.7}$$

Equivalently, the following function is concave on $\mathbb{R}^m$:

$$(t_1, \ldots, t_m) \longmapsto \log \mu\left(\bigcap_{i=1}^{m} e^{t_i} K_i\right).$$

Theorem A yields the geometric results below. The support function and logarithmic sum are defined by

$$h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle, \qquad \theta \in \mathbb{S}^{n-1}, \tag{1.8}$$

$$\begin{aligned} &(1-\lambda) \cdot K +_0 \lambda \cdot L \\ &\quad = \bigcap_{\theta \in \mathbb{S}^{n-1}} \{x : \langle x, \theta \rangle \leq h_K(\theta)^{1-\lambda} h_L(\theta)^{\lambda}\}. \end{aligned} \tag{1.9}$$

In van Handel's formulation [149, Section 1.2], the logarithmic sum is "the largest convex body whose support function is dominated by" $h_K^{1-\lambda} h_L^{\lambda}$; the latter need not itself be a support function. Weighted arithmetic–geometric mean gives

$$h_K(\theta)^{1-\lambda} h_L(\theta)^{\lambda} \leq (1-\lambda) h_K(\theta) + \lambda h_L(\theta) = h_{(1-\lambda)K + \lambda L}(\theta).$$

Consequently, the logarithmic sum is contained in the corresponding Minkowski sum:

$$(1-\lambda) \cdot K +_0 \lambda \cdot L \subseteq (1-\lambda)K + \lambda L.$$

The inclusion can be strict, as illustrated in Figure 1.

From Theorem A we deduce the validity of the Log-Brunn–Minkowski conjecture of Böröczky, Lutwak, Yang, Zhang [22, 23], following the argument from Saroglou [143, 144]:

Theorem 1.1 (Logarithmic Brunn–Minkowski for even log-concave measures)**.** *For symmetric convex bodies K, L and $\lambda \in [0,1]$,*

$$\mu\Big((1-\lambda) \cdot K +_0 \lambda \cdot L\Big) \geq \mu(K)^{1-\lambda} \mu(L)^{\lambda}. \tag{1.10}$$

Proof via Theorem A. Choose a dense sequence $(\theta_j)_{j \geq 1}$ in $\mathbb{S}^{n-1}$ whose first n vectors form a basis of $\mathbb{R}^n$. For $r > 0$, define the symmetric strip of half-width r by

$$S_j(r) = \{x \in \mathbb{R}^n : |\langle x, \theta_j \rangle| \leq r\} = r S_j(1), \qquad r_j = h_K(\theta_j), \quad s_j = h_L(\theta_j).$$

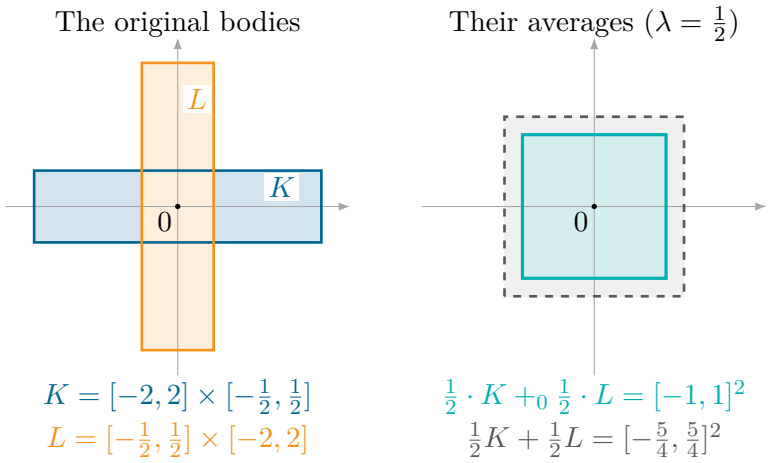


Figure 1: The logarithmic average (solid teal boundary) lies strictly inside the Minkowski average (dashed boundary). Both panels use the same scale.

For $N \geq n$, put

$$K_N = \bigcap_{j=1}^{N} S_j(r_j), \qquad L_N = \bigcap_{j=1}^{N} S_j(s_j),$$
$$C_N = \bigcap_{j=1}^{N} S_j(r_j^{1-\lambda} s_j^{\lambda}) = \bigcap_{j=1}^{N} r_j^{1-\lambda} s_j^{\lambda} S_j(1).$$

The first n directions ensure boundedness, and all widths are positive. Apply Theorem A to the strips $S_1(1), \ldots, S_N(1)$ with dilation factors r_j, s_j:

$$\begin{aligned}\mu(C_N) &= \mu\left(\bigcap_{j=1}^{N} r_j^{1-\lambda} s_j^{\lambda} S_j(1)\right) \\ &\geq \mu\left(\bigcap_{j=1}^{N} r_j S_j(1)\right)^{1-\lambda} \mu\left(\bigcap_{j=1}^{N} s_j S_j(1)\right)^{\lambda} = \mu(K_N)^{1-\lambda} \mu(L_N)^{\lambda}.\end{aligned}$$

Continuity and evenness of the support functions, together with density of the directions, give

$$K_N \downarrow K, \qquad L_N \downarrow L,$$
$$C_N \downarrow \bigcap_{\theta \in \mathbb{S}^{n-1}} \{x : |\langle x, \theta \rangle| \leq h_K(\theta)^{1-\lambda} h_L(\theta)^{\lambda}\} = (1-\lambda) \cdot K +_0 \lambda \cdot L.$$

All three decreasing sequences start with bounded sets of finite measure. Continuity of measure from above therefore yields

$$\begin{aligned}\mu\Big((1-\lambda)\cdot K +_0 \lambda\cdot L\Big) &= \lim_{N\to\infty}\mu(C_N)\\ &\geq \lim_{N\to\infty}\mu(K_N)^{1-\lambda}\mu(L_N)^{\lambda} = \mu(K)^{1-\lambda}\mu(L)^{\lambda}.\end{aligned}$$

□

Corollary 1.2 (Lebesgue logarithmic Brunn–Minkowski)**.** *For symmetric convex bodies and* $\lambda\in[0,1]$*,*

$$\Big|(1-\lambda)\cdot K +_0 \lambda\cdot L\Big| \geq |K|^{1-\lambda}|L|^{\lambda}.$$

Proof. Take $V=0$ in Theorem 1.1. □

Saroglou [143, 144] proved unconditional cases of the Log-Brunn–Minkowski conjecture and established the transfer from Lebesgue LogBM to even log-concave measures. Colesanti–Livshyts–Marsiglietti [40], Kolesnikov–Milman [92, 90, 91, 94], and Kolesnikov–Livshyts [96] develop local inequalities and their geometric interpretations. Chen–Huang–Li–Liu [34] obtained global L_p results for p close to one, and Putterman [135] established the local-to-global equivalence. Other important cases include complex bodies and reflection symmetries [138, 21]. Van Handel [149, Theorem 1.4] proved the local logarithmic Brunn–Minkowski inequality at every symmetric zonoid, with arbitrary symmetric convex perturbations. His argument uses a variant of the Bochner method developed with Shenfeld [149, Section 2.3]. Stancu [146] proposed a PDE approach. Log-Brunn–Minkowski theory relates closely to existence, uniqueness, and regularity for Monge–Ampère equations, see [23, 111, 75, 114, 37, 152, 27, 20, 34, 92, 77, 78].

The logarithmic Minkowski problem asks which measures on the sphere arise as cone-volume measures, $dV_K=\frac{1}{n}h_K\,dS_K$, and to what extent this measure determines K. Its uniqueness question is closely related to the equality case in LogBM [22, 23]. Some qualification is necessary: independent dilations of complementary summands can preserve the cone-volume measure when their total volume factor is one. Firey's work on worn stones [62] established the symmetric uniqueness result for smooth bodies with constant cone-volume density, an early instance of this problem. Kolesnikov [87] developed a variational approach to the logarithmic Minkowski problem using optimal transport on the sphere. For curvature-flow developments, see [70, 29]. Related problems concerning geometric measures and affine normalization are studied in [24, 76, 113, 25, 26].

E. Milman's work on the centroaffine interpretation of the Hilbert–Brunn–Minkowski operator [121] and sharp centroaffine spectral inequality [123] connect

this analytic viewpoint directly to the L_p-Minkowski problem. Further uniqueness results under curvature assumptions were obtained with Ivaki [77, 78]. The work of Kolesnikov and Milman on Riemannian metrics associated with convex sets [93] is particularly relevant to the choice of metric and operator in these notes. For spectral comparison and hypercontractivity, see also Milman [124].

Armeniakos, Kolesnikov, and Ulivelli [155, Theorem 1.1] give a boundary characterization of Theorem 1.1 for smooth strictly convex even density potentials. Weighted LogBM corresponds to testing every even variation of the logarithm of the support function. Testing only the constant variation characterizes the B-property, since a dilation adds a constant to $\log h_K$. We give the boundary inequality in Section 9.

Corollary 1.3 (The B-conjecture)**.** *For an even log-concave measure and a symmetric convex body, the function*

$$t \longmapsto \mu(e^t K)$$

is log-concave. Explicitly,

$$\mu(e^{(1-\lambda)s+\lambda t}K) \geq \mu(e^s K)^{1-\lambda}\mu(e^t K)^{\lambda}.$$

Proof. The support functions of two dilates are proportional, and hence

$$(1-\lambda)\cdot(e^s K) +_0 \lambda\cdot(e^t K) = e^{(1-\lambda)s+\lambda t}K.$$

Apply Theorem 1.1 to $e^s K$ and $e^t K$. □

The B-conjecture was posed by Banaszczyk and popularized by Latała [100, 46]. The Gaussian B-conjecture was proved by Cordero-Erausquin, Fradelizi, and Maurey [46], and the rotation-invariant case by Cordero-Erausquin and Rotem [43] (in fact, their result goes beyond the log-concave setting). Livne Bar-on [104] proved the uniform planar case; Saroglou [144] obtained the planar log-concave case and further implications between the B-property and LogBM. Eskenazis, Nayar, and Tkocz [55] established extensions for Gaussian mixtures. Applications of the Gaussian B-theorem include the small-ball estimates and random-section results of Klartag and Vershynin [80], and Bobkov's study of positions of convex bodies maximizing Gaussian measure and their relation to Milman's ellipsoids [16].

Lastly, we deduce the dimensional Brunn–Minkowski conjecture, proposed in the Gaussian case by Gardner and Zvavitch [67] and in general by Colesanti, Livshyts, and Marsiglietti [40].

Corollary 1.4 (Dimensional Brunn–Minkowski)**.** *For an even log-concave measure on $\mathbb{R}^n$, symmetric convex bodies K, L, and $\lambda \in [0,1]$,*

$$\mu((1-\lambda)K + \lambda L)^{1/n} \geq (1-\lambda)\mu(K)^{1/n} + \lambda \mu(L)^{1/n}. \tag{1.11}$$

Proof. We outline the transfer argument of [105, Proposition 1]. Put $a = \mu(K)^{1/n}$, $b = \mu(L)^{1/n}$, $c = (1-\lambda)a + \lambda b$, and $\theta = \lambda b/c$; we may assume $0 < \lambda < 1$. Define

$$\begin{aligned} C &= (1-\theta)\cdot K +_0 \theta \cdot L, \\ t &= \left(\frac{1-\lambda}{1-\theta}\right)^{1-\theta} \left(\frac{\lambda}{\theta}\right)^{\theta} \leq 1. \end{aligned} \tag{1.12}$$

Weighted arithmetic–geometric mean, applied to the defining support bounds, gives $tC \subset (1-\lambda)K + \lambda L$. Even convex V satisfies $V(tx) \leq V(x)$ for $0 < t \leq 1$, so $\mu(tC) \geq t^n \mu(C)$. Theorem 1.1 therefore gives

$$\mu((1-\lambda)K + \lambda L)^{1/n} \geq t\, a^{1-\theta} b^{\theta} = c.$$

This proves (1.11). □

Livshyts, Marsiglietti, Nayar, and Zvavitch [105] connected Log-Brunn–Minkowski with dimensional concavity; further work includes Kolesnikov–Livshyts [95] where some bounds were obtained in the Gaussian case under weaker assumptions, Hosle–Kolesnikov–Livshyts [74], and Livshyts [106]. Eskenazis and Moschidis [54] settled the Gaussian dimensional inequality. Cordero-Erausquin and Rotem [43] obtained dimensional concavity for a class of rotationally invariant measures. Cordero-Erausquin and Eskenazis [41] developed functional versions through concavity of weighted marginals. Aishwarya and Rotem [2] used displacement convexity of entropy to obtain dimensional inequalities for centered star-shaped sets and related functional inequalities; Aishwarya and Li [1] proved entropic and functional forms of the Gaussian dimensional inequality. Armeniakos and Ulivelli [3, 4] studied infinitesimal formulations of Prékopa–Leindler and showed that a strengthened dimensional inequality implies the B-property. See also more recent works [153, 154] on the sharp exponent for the Gaussian case under the assumption of containing the origin, and the references therein, including Tkocz, Nayar [130].

Armeniakos, Kolesnikov, and Ulivelli [155, Theorems 1.4–1.5] also give direct L^2 proofs of dimensional concavity. For a smooth symmetric body C of positive curvature, put $r = \|\cdot\|_C$, $\Phi = r^2/2$, and suppose $\alpha I \leq D^2\Phi \leq \beta I$ away from zero, with $\gamma = \alpha/\beta > 0$. For the finite measure $e^{-w(r)}dx$ they obtain the exponent γ/n in (1.11) whenever $w \in C^2((0,\infty))$ is increasing and $w''(s) + \gamma w'(s)/s \geq 0$.

This includes $w(s) = s^a$ with $0 < a < 1$ and $a \geq 1 - \gamma$, for which the density fails to be log-concave. For a smooth even potential V satisfying $D^2V \geq k_1 I$ and $\Delta V \leq nk_2$, with $0 < k_1 \leq k_2$, their second estimate gives the exponent $k_1/(nk_2)$ for $e^{-V}dx$.

Gaussian correlation, proved by Royen [141], arguably belongs to the same broad landscape, although it is a different assertion from Theorem A. E. Milman [125] gave a proof through inverse Brascamp–Lieb inequalities. Milman, Nakamura, and Tsuji [126] obtained the Gaussian conjugate Rogers–Shephard inequality, combining intersection and Minkowski addition. Their heterovariant Gaussian Brunn–Minkowski inequality [127] unifies the dimensional Gaussian inequality and the strong Gaussian B-theorem by allowing different covariance matrices.

1.2 A functional inequality

The intersection inequality Theorem A has a functional version in which convex potentials replace the indicators of convex sets.

Theorem B (Functional dilation inequality)**.** *Let $V_0, V_1, \dots, V_m$ be proper lower semicontinuous even convex potentials on $\mathbb{R}^n$, and put $d\rho(x) = e^{-V_0(x)}dx$. Assume that*

$$0 < \int_{\mathbb{R}^n} e^{-\sum_{i=1}^m V_i(x)}\, d\rho(x) < \infty. \tag{1.13}$$

Then all the following integrals are positive and finite, and

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} e^{-\sum_{i=1}^m V_i(e^{s_i}x)}\, d\rho(x)\right)\left(\int_{\mathbb{R}^n} e^{-\sum_{i=1}^m V_i(e^{-s_i}x)}\, d\rho(x)\right) \\ &\qquad \leq \left(\int_{\mathbb{R}^n} e^{-\sum_{i=1}^m V_i(x)}\, d\rho(x)\right)^2, \qquad s \in \mathbb{R}^m. \end{aligned} \tag{1.14}$$

More generally, for $s, t \in \mathbb{R}^m$ and $\lambda \in [0,1]$,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-\sum_i V_i(e^{(1-\lambda)s_i + \lambda t_i}x)}\, d\rho(x) \\ &\quad \geq \left(\int_{\mathbb{R}^n} e^{-\sum_i V_i(e^{s_i}x)}\, d\rho(x)\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} e^{-\sum_i V_i(e^{t_i}x)}\, d\rho(x)\right)^{\lambda}. \end{aligned} \tag{1.15}$$

Claim 1.5. *Theorem A and Theorem B are equivalent.*

Proof. Take $V_i = I_{K_i}$ in Theorem B, where I_K is zero on K and $+\infty$ outside. Then

$$\int e^{-\sum_i I_{K_i}(e^{s_i}x)}\,d\rho(x) = \rho\left(\bigcap_i e^{-s_i}K_i\right),$$

so changing the signs of the parameters gives Theorem A.

Conversely, assume Theorem A. Subtracting the minima, we may assume that $V_i(0) = 0$ and $V_i \geq 0$. We present the argument for bounded positive sublevel sets; all approximation details are deferred to Appendix B. For $s \in \mathbb{R}^m$ and $r \in (0,\infty)^m$, put

$$K_i(r_i) = \{x : V_i(x) \leq r_i\}, \qquad C(s,r) = \bigcap_{i=1}^m e^{-s_i}K_i(r_i),$$
$$F_s(r) = \rho(C(s,r)).$$

Extend F_s by zero outside $(0,\infty)^m$.

Fix $s,t \in \mathbb{R}^m$, $r,u \in (0,\infty)^m$ and $0 < \lambda < 1$. Convexity of V_i gives

$$(1-\lambda)K_i(r_i) + \lambda K_i(u_i) \subseteq K_i((1-\lambda)r_i + \lambda u_i),$$

since $V_i((1-\lambda)x + \lambda y) \leq (1-\lambda)r_i + \lambda u_i$ whenever $V_i(x) \leq r_i$ and $V_i(y) \leq u_i$. Thus, for every $\theta \in \mathbb{S}^{n-1}$, arithmetic–geometric mean and the support-function identity for Minkowski addition imply

$$\begin{aligned}
&(e^{-s_i}h_{K_i(r_i)}(\theta))^{1-\lambda}(e^{-t_i}h_{K_i(u_i)}(\theta))^{\lambda} \\
&\quad = e^{-[(1-\lambda)s_i+\lambda t_i]}h_{K_i(r_i)}(\theta)^{1-\lambda}h_{K_i(u_i)}(\theta)^{\lambda} \\
&\quad \leq e^{-[(1-\lambda)s_i+\lambda t_i]}\Big((1-\lambda)h_{K_i(r_i)}(\theta) + \lambda h_{K_i(u_i)}(\theta)\Big) \\
&\quad \leq e^{-[(1-\lambda)s_i+\lambda t_i]}h_{K_i((1-\lambda)r_i+\lambda u_i)}(\theta).
\end{aligned}$$

Apply Theorem A to the symmetric strips $\{x : |\langle x,\theta\rangle| \leq 1\}$, indexed by i and θ, with endpoint widths

$$e^{-s_i}h_{K_i(r_i)}(\theta), \qquad e^{-t_i}h_{K_i(u_i)}(\theta).$$

Their endpoint intersections are $C(s,r)$ and $C(t,u)$. The preceding bound places the intersection with the geometrically interpolated widths inside $C((1-\lambda)s + \lambda t, (1-\lambda)r + \lambda u)$. Theorem A and monotonicity of ρ therefore give

$$\begin{aligned}
&\rho\left(\bigcap_{i=1}^m e^{-[(1-\lambda)s_i+\lambda t_i]}K_i((1-\lambda)r_i + \lambda u_i)\right) \\
&\qquad \geq \rho\left(\bigcap_{i=1}^m e^{-s_i}K_i(r_i)\right)^{1-\lambda}\rho\left(\bigcap_{i=1}^m e^{-t_i}K_i(u_i)\right)^{\lambda}.
\end{aligned} \tag{$*$}$$

For nonnegative measurable functions $f_1, \dots, f_m$, the layer-cake formula and Tonelli's theorem give

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(x)\, d\rho(x) = \int_{(0,\infty)^m} \rho\left(\bigcap_{i=1}^m \{f_i \ge q_i\} \right) dq_1 \cdots dq_m.$$

Apply this with $f_i(x) = e^{-V_i(e^{s_i} x)} \le 1$ and make the change of variables $r_i = -\log q_i$, so $q_i = e^{-r_i}$:

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-\sum_i V_i(e^{s_i} x)}\, d\rho(x) \\ &\quad = \int_{(0,1)^m} \rho\left(\bigcap_{i=1}^m \{x : e^{-V_i(e^{s_i} x)} \ge q_i\} \right) dq_1 \cdots dq_m \\ &\quad = \int_{(0,1)^m} \rho\left(\bigcap_{i=1}^m e^{-s_i} K_i(-\log q_i) \right) dq_1 \cdots dq_m \\ &\quad = \int_{\mathbb{R}^m} F_s(r) e^{-\sum_i r_i}\, dr. \end{aligned} \tag{1.16}$$

In our notation, (∗) reads

$$F_{(1-\lambda)s+\lambda t}((1-\lambda)r + \lambda u) \ge F_s(r)^{1-\lambda} F_t(u)^{\lambda}.$$

The zero extension makes this valid for all $r, u \in \mathbb{R}^m$. Multiplying by $e^{-\sum_i ((1-\lambda)r_i + \lambda u_i)}$ gives

$$\begin{aligned} &F_{(1-\lambda)s+\lambda t}((1-\lambda)r + \lambda u) e^{-\sum_i ((1-\lambda)r_i + \lambda u_i)} \\ &\qquad \ge \left(F_s(r) e^{-\sum_i r_i} \right)^{1-\lambda} \left(F_t(u) e^{-\sum_i u_i} \right)^{\lambda}. \end{aligned}$$

This is precisely the pointwise hypothesis of Prékopa–Leindler on $\mathbb{R}^m$ for the three functions

$$r \longmapsto F_s(r) e^{-\sum_i r_i}, \qquad r \longmapsto F_t(r) e^{-\sum_i r_i},$$
$$r \longmapsto F_{(1-\lambda)s+\lambda t}(r) e^{-\sum_i r_i}.$$

They are measurable and have finite positive integrals by (1.16) and (B.3). Prékopa–Leindler [134, 28], applied in the sublevel variables r, therefore yields

$$\begin{aligned} &\int_{\mathbb{R}^m} F_{(1-\lambda)s+\lambda t}(r) e^{-\sum_i r_i}\, dr \\ &\qquad \ge \left(\int_{\mathbb{R}^m} F_s(r) e^{-\sum_i r_i}\, dr \right)^{1-\lambda} \left(\int_{\mathbb{R}^m} F_t(r) e^{-\sum_i r_i}\, dr \right)^{\lambda}. \end{aligned}$$

By (1.16), this is exactly (1.15). Taking $t = -s$ and $\lambda = 1/2$ and squaring gives (1.14). □

Remark 1.6. *For homogeneous potentials the parameters become reciprocal coefficients: if $V_i(rx) = r^{p_i} V_i(x)$ with $p_i > 0$, then*

$$\begin{aligned}\left(\int e^{-\sum_i q_i V_i}\, d\rho\right)\left(\int e^{-\sum_i q_i^{-1} V_i}\, d\rho\right) \\ \leq \left(\int e^{-\sum_i V_i}\, d\rho\right)^2, \qquad q_i > 0.\end{aligned} \tag{1.17}$$

The degrees p_i need not agree: take $s_i = p_i^{-1} \log q_i$. The reciprocal product suggests a Blaschke–Santaló interpretation. We return in Section 10 to a full-pair cost inequality and its radial specialization.

1.3 The key variance inequality

The main analytic ingredient is the following variance estimate, equivalent to the two preceding theorems. It is the second-derivative form of the functional dilation inequality.

Theorem C (General dilation variance inequality)**.** *Let $V_0, V_1, \dots, V_m$ be smooth even convex functions. Assume that the derivatives occurring below have at most polynomial growth and that the normalizing integral is positive and finite. Define*

$$d\mu(x) = \frac{e^{-V_0(x) - \sum_{i=1}^m V_i(x)}}{\int_{\mathbb{R}^n} e^{-V_0 - \sum_{i=1}^m V_i}\, dx}\, dx. \tag{1.18}$$

For every $a_1, \dots, a_m \in \mathbb{R}$,

$$\operatorname{Var}_\mu\left(\sum_{i=1}^m a_i \langle x, \nabla V_i\rangle\right) \leq \sum_{i=1}^m a_i^2 \int \Big(\langle x, \nabla V_i\rangle + \langle D^2 V_i x, x\rangle\Big)\, d\mu. \tag{1.19}$$

The associated integral statement extends to proper lower semicontinuous even convex potentials whenever the integral is positive and finite.

Claim 1.7. *Theorem B and Theorem C are equivalent.*

Proof. For smooth potentials satisfying the growth assumptions of Theorem C, put

$$\begin{aligned}W_{i,s}(x) = V_i(e^{s_i} x), \qquad Z(s) = \int e^{-V_0(x) - \sum_i W_{i,s}(x)} dx, \\ d\mu_s = Z(s)^{-1} e^{-V_0 - \sum_i W_{i,s}} dx.\end{aligned} \tag{1.20}$$

Differentiating in a direction $a \in \mathbb{R}^m$ gives

$$\frac{d^2}{dt^2} \log Z(s+ta)\Big|_{t=0} = \operatorname{Var}_{\mu_s}\Big(\sum_i a_i \langle x, \nabla W_{i,s}\rangle\Big) - \sum_i a_i^2 \int \Big(\langle x, \nabla W_{i,s}\rangle + \langle D^2 W_{i,s} x, x\rangle\Big)\, d\mu_s. \tag{1.21}$$

Concavity in Theorem B, differentiated at $s = 0$, gives Theorem C. Conversely, apply Theorem C to $W_{i,s}$ at every s: the displayed second derivative is nonpositive in every direction, so $\log Z$ is concave. Section B justifies the approximation to extended-valued potentials.

The reciprocal form (1.14) also suffices: applying it to rescaled potentials gives midpoint concavity at every center. The radial comparison (B.3) makes $\log Z$ locally bounded, and hence midpoint concavity gives full concavity. □

Remark 1.8 (The direct route to the B-conjecture). *For $m = 1$ and $a_1 = 1$, Theorem C says*

$$\operatorname{Var}_\mu \langle x, \nabla V_1\rangle \le \int \Big(\langle x, \nabla V_1\rangle + \langle D^2 V_1 x, x\rangle\Big)\, d\mu, \qquad d\mu \propto e^{-V_0 - V_1} dx.$$

If $H(t) = \log \int e^{-V_0(x) - V_1(e^t x)} dx$, this is exactly $H''(0) \le 0$. Apply the same assertion to $V_1(e^s \cdot)$ for each s to obtain concavity of H. Convex-indicator approximation then gives the B-conjecture (Corollary 1.3) directly. Equivalently, for $d\mu = e^{-V} dx$ and a symmetric body K, take $V_0 = I_K$ and $V_1 = V$. The change of variables $y = e^t x$ gives

$$\log \int_K e^{-V(e^t x)}\, dx = \log \mu(e^t K) - nt.$$

The scalar variance bound above is then exactly [155, equation (27)] for the probability $\mu(\cdot \cap K)/\mu(K)$. The authors identify it by integration by parts with their boundary test for the constant function.

We point out the following consequence of Theorem C (it could also be deduced directly from Theorem B).

Corollary 1.9 (A logarithmic Sobolev inequality for dilation perturbations)**.** *Under the assumptions of Theorem C, let μ be as in (1.18). For $s \in \mathbb{R}^m$, put*

$$f_s(x) = \sum_i \Big(V_i(x) - V_i(e^{s_i} x)\Big), \qquad \operatorname{Ent}_\mu h = \int h \log h\, d\mu - \Big(\int h\, d\mu\Big) \log \int h\, d\mu.$$

Then

$$\operatorname{Ent}_\mu(e^{f_s}) \le \int e^{f_s} \sum_i \Big(V_i(x) - V_i(e^{s_i} x) + s_i \langle e^{s_i} x, \nabla V_i(e^{s_i} x)\rangle\Big)\, d\mu(x). \tag{1.22}$$

In particular, if V_i is p_i-homogeneous with $p_i > 0$, then for $q_i > 0$,

$$\begin{aligned}&\mathrm{Ent}_\mu\Big(e^{\sum_i (1-q_i)V_i}\Big)\\ &\qquad\le \int e^{\sum_i (1-q_i)V_i} \sum_i (q_i \log q_i - q_i + 1) V_i \, d\mu.\end{aligned} \tag{1.23}$$

Proof. Write $Z(s) = \int e^{-V_0(x) - \sum_i V_i(e^{s_i} x)} dx$ and $H = \log Z$. Differentiating twice and applying Theorem C to the rescaled potentials gives concavity of H; this is the calculation in (1.21). Consequently

$$F(t) = H((1-t)s) - (1-t)H(s) - tH(0) \ge 0, \qquad 0 \le t \le 1,$$

with $F(0) = 0$, so differentiation at zero gives $H(s) - H(0) \ge s \cdot \nabla H(s)$. Let $M_s = Z(s)/Z(0) = \int e^{f_s} d\mu$ and $d\mu_s = M_s^{-1} e^{f_s} d\mu$. Since

$$\begin{aligned}s \cdot \nabla H(s) &= -\int \sum_i s_i \langle e^{s_i} x, \nabla V_i(e^{s_i} x)\rangle \, d\mu_s,\\ \frac{\mathrm{Ent}_\mu(e^{f_s})}{M_s} &= \int f_s \, d\mu_s - H(s) + H(0),\end{aligned}$$

we obtain (1.22). The growth assumptions and the exponential tails justify these differentiations. For the homogeneous form, take $s_i = p_i^{-1} \log q_i$ and use Euler's identity. □

The following special case of Theorem C is especially neat:

Corollary 1.10 (Homogeneous potentials and powers of linear functions)**.** *Let V_0 be proper lower semicontinuous even and convex, and let $V_1, \dots, V_m$ be finite, nonnegative, even, and convex, with*

$$V_i(rx) = r^{p_i} V_i(x), \qquad r > 0, \qquad p_i \ge 1.$$

Assume that the normalizing integral defining μ is positive and finite. Then, for $b_1, \dots, b_m \in \mathbb{R}$,

$$\mathrm{Var}_\mu\left(\sum_{i=1}^m b_i V_i\right) \le \sum_{i=1}^m b_i^2 \int V_i \, d\mu. \tag{1.24}$$

In particular this applies to $V_i(x) = c_i |\langle v_i, x\rangle|^{p_i}$, $c_i \ge 0$.

Proof. For smooth homogeneous potentials Euler's identity gives

$$\langle x, \nabla V_i\rangle = p_i V_i, \qquad \langle x, \nabla V_i\rangle + \langle D^2 V_i x, x\rangle = p_i^2 V_i.$$

Use $a_i = b_i/p_i$ in (1.19). The nonsmooth assertion follows from integral concavity and differentiation in positive coefficients, as justified in Section B. The total convex potential has an exponential tail, and each finite homogeneous potential has polynomial growth, so the required moments are finite. □

Remark 1.11 (Reduction of LogBM to the homogeneous case). *Corollary 1.10 extends [131, Proposition 2.1], whose potentials are powers of linear forms of a common degree, to arbitrary finite convex homogeneous summands, possibly of different degrees, and an even convex background. That powers-of-linear-forms case is already sufficient for LogBM. Indeed, supporting slabs approximate symmetric convex bodies, and*

$$\exp\left(-\sum_i \left|\frac{\langle x,\theta_i\rangle}{r_i}\right|^q\right) \longrightarrow \mathbf{1}_{\{\,|\langle x,\theta_i\rangle|\le r_i \ \text{for all } i\,\}} \quad a.e., \qquad q\to\infty.$$

The variance bound gives concavity in the logarithms of the widths r_i before passage to the limit. For Lebesgue LogBM, $V_0 = 0$ and the approximating total potentials are homogeneous, and the equivalence of the Lebesgue case to the general log-concave case was established by Saroglou [144, Theorem 3.1].

1.4 Convexity of entropy along an exponential transport

For a probability density ρ, write $\mathrm{Ent}(\rho\,dx) = \int \rho\log\rho\,dx$. For $d\mu = Z^{-1}e^{-V}dx$, define the weighted row divergence by

$$(\delta_\mu M)_i = \sum_j \Big(M_{ij}\partial_j V - \partial_j M_{ij}\Big).$$

Theorem D (Infinitesimal convexity of entropy)**.** *Let $d\mu = Z^{-1}e^{-V}dx$ be a centered probability measure on $\mathbb{R}^n$, where V is smooth and $D^2V \ge \varepsilon I$ for some $\varepsilon > 0$. There exists a smooth strictly convex function φ, with $\nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism, such that*

$$\delta_\mu\Big((D_x^2\varphi^*)^{-1}\Big) = x, \tag{1.25}$$

and for every $u \in C_c^\infty(\mathbb{R}^n)$, defining

$$\boxed{T_t(x) = \delta_\mu\Big[\exp\Big(t(D_x^2\varphi^*)^{-1}D_x^2u\Big)\,(D_x^2\varphi^*)^{-1}\Big]\,(x),} \tag{1.26}$$

we have $T_0(x) = x$, the maps T_t are orientation-preserving diffeomorphisms for sufficiently small $|t|$, and

$$\boxed{\frac{d^2}{dt^2}\,\mathrm{Ent}((T_t)_\#\mu)\Big|_{t=0} \ge 0.} \tag{1.27}$$

In other words, the entropy is convex along this interpolation.

Theorem D expresses the coercivity estimate as entropy convexity at the initial parameter. The corresponding interval statement is Question 1 below.

Claim 1.12 (Relation with the functional dilation inequality)**.** *Theorem D, with its displayed normalization, is equivalent, under the common regularity hypotheses, to the coercivity estimate in Proposition 5.17. It therefore implies Theorem B. The finite entropy-and-energy interpolation property in Proposition 6.2 is equivalent to Theorem B.*

Proof. The exact second-variation identity (6.5) proves the first equivalence. The duality argument in Section 5 then gives Theorem C, and the equivalence already proved in Claim 1.7 gives Theorem B. Proposition 6.2 proves the last assertion by the Gibbs variational principle. □

Question 1. Is $t \mapsto \mathrm{Ent}((T_t)_{\#}\mu)$ convex on every interval containing zero on which (1.26) consists of orientation-preserving diffeomorphisms? Can this global statement be shown equivalent to Theorem B by constructing an interpolation between its endpoint Gibbs measures?

McCann's displacement convexity [116] concerns a different interpolation. If $\nabla\psi$ is the optimal quadratic transport from μ_0 to μ_1, then

$$t \longmapsto \mathrm{Ent}\Big(((1-t)\,\mathrm{Id} + t\nabla\psi)_{\#}\mu_0\Big)$$

is convex on $[0,1]$. In the smooth case, its Jacobian is $(1-t)I + tD^2\psi$, and concavity of $\log\det$ proves the assertion pointwise before integration. Both endpoints are prescribed, and no evenness is required. Formula (1.26) instead exponentiates an inverse-Hessian-weighted Hessian and then takes its weighted divergence. Its initial velocity encodes the fourth-order operator P; its entropy second variation is exactly the compensated estimate needed in the L^2 argument. Prescribing its second endpoint is an additional problem. Entropy methods for geometric and functional inequalities are also developed in [2, 1].

For unconditional potentials, logarithmic coordinates provide a simpler interpolation: ordinary displacement interpolation in those coordinates becomes coordinatewise geometric interpolation in the positive orthant. Subsection 6.4 proves entropy convexity along that interpolation and gives a direct proof of Theorem B in the unconditional case.

1.5 Proof overview, notes organization and acknowledgements

In view of the theorem equivalences outlined above, our remaining goal is to prove Theorem C. This relies on the L^2 method, whose foundations are Hörmander's

duality estimates, the Brascamp–Lieb inequality, Bochner's identity, and the diffusion approach of Bakry and Émery [72, 28, 17, 6]. Namely, we formulate a more general result concerning bounding a variance of an even function F, which simplifies to (1.19) for the particular choice $F = \sum_{i=1}^m a_i \langle x, \nabla V_i \rangle$ (see Corollary 5.2). In order to estimate $Var_\mu(F)$, we approximately solve the PDE $F - \int F \, d\mu = Pu$, for the appropriate choice of the differential operator P, and use integration by parts in conjunction with several non-trivial pointwise matrix inequalities, in order to bound $\int (Pu)^2 \, d\mu$ from below and thereby estimate the variational expression from above for an arbitrary function u.

The main magic here occurs because we consider moment maps and moment measures, introduced by Klartag [84, 42] and studied in many subsequent works including [14, 38, 83, 139, 86, 58, 142, 85, 88, 89]. The moment source associated with μ is a measure $d\nu = e^{-\varphi(y)} dy$ such that $\nabla \varphi$ pushes ν forward to μ. Existence and uniqueness in the essentially continuous class were proved by Cordero-Erausquin and Klartag [42], following Berman and Berndtsson [14]; Santambrogio [142] gave an entropy-and-transport proof. Falah and Rotem [56] solved the general functional Minkowski problem, building on the first-variation theory of Colesanti–Fragalà and Rotem [38, 139, 140]. Fathi [58] identified the moment Hessian as a Stein kernel. The natural operator L_μ^m which arises in this framework is associated with the Stein kernel $(D^2 \varphi^*)^{-1}$ and has unique optimal properties among operators whose first nonconstant eigenspace consists of linear functions. Kolesnikov, Livshyts and Rotem [89] previously explored connections of L_μ^m with the Log-Brunn-Minkowski conjecture. However, the main object of study is the operator $P = L_\mu^m (L_\mu^m + I)$ – the iterated version of L_μ^m. Its kernel consists of affine functions and thus it is well suited for estimating variance of an even function via L_2 method. In addition, optimality properties of moment geometry allow for convenient cancellations in the computations, which makes this choice of operator work especially well. The associated analysis produced by the OpenAI internal model is quite involving, substantially more so than the previous explorations of this topic. In broad terms, it involves integration by parts and completing the squares – activities which were familiar to the experts in the field also before the OpenAI announcement – but in practice this means second order integration by parts for a fourth order elliptic operator intermitted with several applications of rather involving matrix inequalities.

Section 2 collects the analytic preliminaries. For the reader's convenience, Section 3 presents the classical "baby case" of the L^2 method, deriving the Brascamp–Lieb and Klartag H^{-1} inequalities. Section 4 explains the same framework for a general family of operators, followed by a discussion of the additional structure needed for the symmetric case.

Section 5, the main analytic section, treats "the other baby": elliptic operators associated with moment coordinates. Following a general discussion of moment maps and the first-order analysis, we establish the iterated operator's identities, prove the variance estimate modulo affine functions, and deduce Theorem C.

Section 6 proves Theorem D, identifies the optimizing convex potential, and compares the entropy interpolation with McCann's displacement convexity. It also gives the finite entropy formulation equivalent to Theorem B and a direct unconditional interpolation.

Section 7 gives an evolution, or semigroup, formulation of that argument. We define a semigroup associated with the iterated moment operator P and prove hypercontractivity for it.

Section 8 discusses the rigidity and optimality of moment coordinates, attempting to explain why this particular choice works especially well, and Section 9 develops the connections with centroaffine geometry. Section 10 records the concluding remarks, including a Blaschke–Santaló-type cost inequality and its radial case, and further questions.

An expert reader may skip straight to Section 5 for the proof of Theorem C; the implications leading to the introductory geometric results have already been outlined above. Figure 2 records the dependencies used in the main argument.

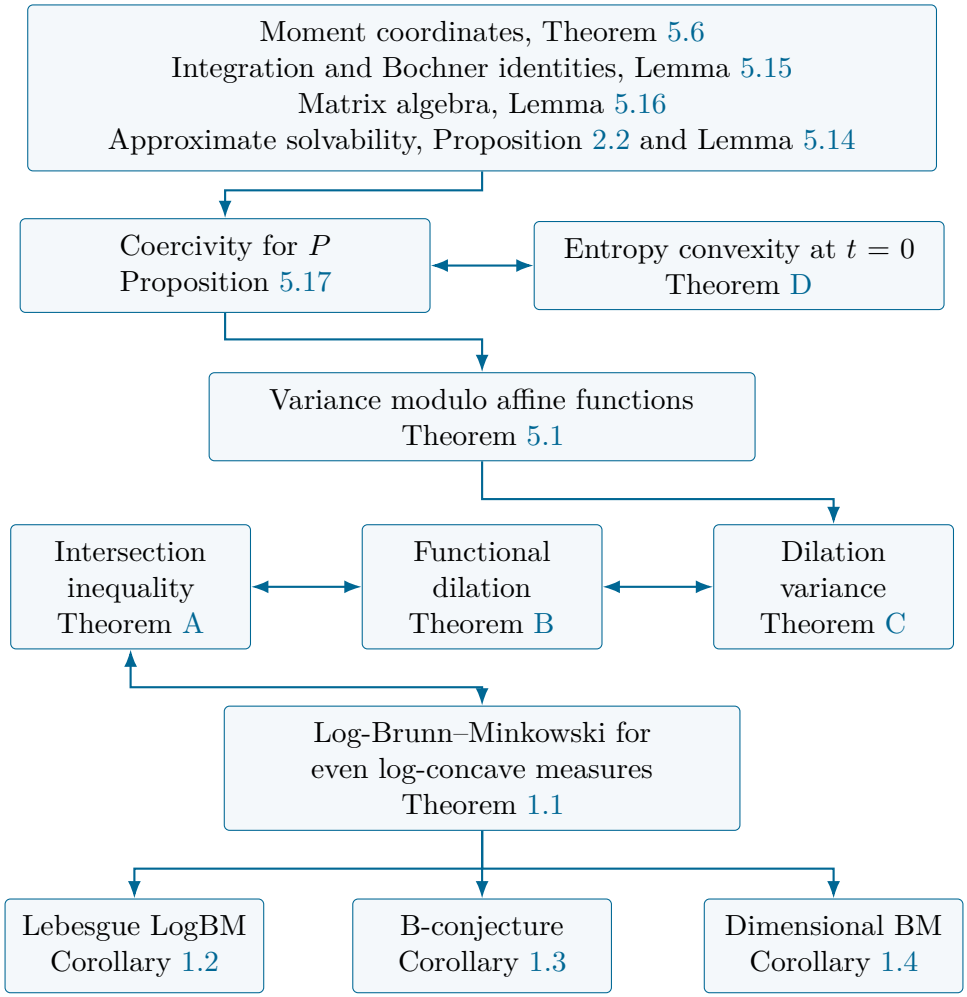


Figure 2: The analytic argument and its geometric consequences. Theorem D is the infinitesimal entropy statement, equivalent to coercivity by (6.5).

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Statement of novelty and comparison with the OpenAI preprint. An internal OpenAI model produced the proof of the symmetric Log-Brunn–Minkowski conjecture in [131]; see also the account of the model-produced manuscripts in [132]. That preprint states and proves, in Proposition 2.1, the common-degree powers-of-linear-forms case of Corollary 1.10, and deduces LogBM.

By comparison, the present lecture notes contain Theorem C, which extends the homogeneous variance statement to general dilation derivatives and is interesting in its own right; we also present the equivalent functional inequality in Theorem B and its geometric version Theorem A.

We also unpack the proof of Theorem C and place it in the context of the previously understood L^2 method and the theory of moment maps. We attempt to motivate the approach and its relevance to the problem, human-style, explaining how the special setting of moment measures provides the analytic tools for understanding the effect of symmetry on isoperimetric phenomena, and which compatibility and rigidity properties make moment coordinates particularly suitable. Beyond that, we provide a semigroup argument and prove hypercontractivity. We also draw connections with E. Milman's centroaffine geometry. Importantly, we discuss open questions and directions in the end.

Statement of use of AI. ChatGPT Astra was used for collaboration and joint brainstorming, for providing the author with illuminating explanations, as well as for writing, checking and proofreading these notes. The choice of the statements, the intended mathematical direction, the organization, and the editorial decisions are the author's.

2 Preliminaries: second-order operators and existence results

In the analytic sections, *smooth* means C^∞ on the relevant open set. Unless a domain is specified, it means C^∞ on all of $\mathbb{R}^n$. The measure is a probability measure with smooth positive density:

$$d\mu(x) = Z^{-1}e^{-V(x)}\,dx, \qquad Z = \int_{\mathbb{R}^n} e^{-V(x)}\,dx < \infty. \tag{2.1}$$

For a probability measure we write

$$\operatorname{Var}_\mu f = \int f^2\,d\mu - \left(\int f\,d\mu\right)^2. \tag{2.2}$$

We use ordinary Euclidean derivatives, with conventions

$$\begin{aligned} (DF)_{ij} &= \partial_j F_i, \qquad & D^2 u &= (\partial_{ij} u)_{ij}, \\ \langle A, C\rangle_{\mathrm{HS}} &= \operatorname{Tr}(A^T C), \qquad & \|A\|_{\mathrm{HS}}^2 &= \operatorname{Tr}(A^T A). \end{aligned} \tag{2.3}$$

Repeated coordinate indices are summed when an index formula is used. The Hessians of the density and moment potentials are always written explicitly: they are different matrices. Source and target variables are indicated until their roles have been fixed. We follow the sign convention of [107, Sections 7.4–7.9].

A second-order operator with no zero-order term has the form

$$\mathcal{L}u = a_{ij}(x)\partial_{ij} u + b_i(x)\partial_i u. \tag{2.4}$$

Local ellipticity means that on every compact set there are positive finite constants such that

$$\lambda_K |\xi|^2 \le a_{ij}(x)\xi_i \xi_j \le \Lambda_K |\xi|^2, \qquad x \in K.$$

We work with smooth coefficients. Interior elliptic regularity makes distributional solutions smooth when the right side is smooth; the corresponding local statement also applies to the elliptic fourth-order operator used later. Standard references are [60, 52, 73].

For a smooth symmetric positive-definite matrix field M, define

$$\begin{aligned} \mathcal{E}_M(u, v) &= \int \langle M\nabla u, \nabla v\rangle \, d\mu, \\ \|u\|_{W^{1,2}(\mu;M)}^2 &= \int u^2 \, d\mu + \mathcal{E}_M(u, u). \end{aligned} \tag{2.5}$$

We use the closed form obtained from smooth functions and its nonpositive self-adjoint generator [79, 66]. For a general M, we assume that constants belong to the form domain with zero energy, the realization is conservative, and the compact calculations extend by a core or cutoff approximation.

Proposition 2.1 (Resolvents, the Poisson equation, and dense range)**.** *Let L be the self-adjoint generator of this form. For every $a > 0$ and $f \in L^2(\mu)$ there is a unique weak solution of*

$$au - Lu = f.$$

If f has mean zero, so does the solution. If

$$\int v^2 \, d\mu \le C\mathcal{E}_M(v, v), \qquad \int v \, d\mu = 0,$$

then each mean-zero $f \in L^2(\mu)$ has a unique mean-zero weak solution of $-Lu = f$. Without assuming a spectral gap, one still has

$$L_0^2(\mu) = \left\{ f \in L^2(\mu) : \int f \, d\mu = 0 \right\}, \tag{2.6}$$

$$\overline{\{Lu : u \in \mathcal{D}(L)\}}^{L^2(\mu)} = L_0^2(\mu). \tag{2.7}$$

Proof. The resolvent bilinear form

$$a \int uv \, d\mu + \mathcal{E}_M(u, v)$$

is coercive on the form domain. Lax–Milgram gives existence and uniqueness [30, Chapter 5]. Testing against 1 gives the assertion about the mean. Under the Poincaré inequality, the energy alone is coercive on the mean-zero subspace and gives the Poisson solution. Local regularity supplies smoothness for smooth data.

If $Lu = 0$, the energy identity implies $\mathcal{E}_M(u, u) = 0$. Positivity of M and connectedness of $\mathbb{R}^n$ make u constant. The closure of the range of a self-adjoint operator is the orthogonal complement of its kernel, proving (2.7) [79, Chapter V]. Equivalently, for mean-zero data the resolvent solutions satisfy

$$-Lu_a = f - au_a \longrightarrow f \quad \text{in } L^2(\mu), \qquad a \downarrow 0,$$

by the spectral theorem [79, Chapter V]. □

In particular, for $f \in L^2(\mu)$,

$$\operatorname{Var}_\mu f = \sup_u \left\{ 2 \int f Lu \, d\mu - \int (Lu)^2 \, d\mu \right\}. \tag{2.8}$$

For vector fields the pairing and dual norm are

$$\begin{aligned} \langle F, G \rangle_{L^2(\mu)} &= \int \langle F(x), G(x) \rangle \, d\mu(x), \\ \|G\|_* &= \sup_{\|F\| \le 1} \left| \int \langle F, G \rangle \, d\mu \right|. \end{aligned} \tag{2.9}$$

The same definition applies to seminorms, with value $+\infty$ allowed. Finiteness requires that the pairing vanish on every zero-energy field.

Approximate solutions for an iterated operator. The next proposition records the needed analytic fact in a form independent of moment coordinates. Its hypotheses will follow from the moment integration-by-parts and Bochner formulas. The cutoff argument is the one used in [131, Lemma 5.1], with the affine obstruction retained before restricting to even functions.

Proposition 2.2 (Approximate solvability modulo affine functions)**.** *Suppose that μ is a probability measure with smooth positive density and finite second moment. Let M be smooth and symmetric, with $0 < M(x) \le CI$ for every x, and put*

$$\begin{gathered} Lu = \operatorname{Tr}(MD^2u) - \langle x, \nabla u\rangle, \qquad \mathcal{P} = L(L+I), \\ \operatorname{Aff} = \{x \mapsto a + \langle b, x\rangle : a \in \mathbb{R},\ b \in \mathbb{R}^n\}. \end{gathered} \tag{2.10}$$

Assume that, for compact smooth tests, integration by parts and the following Hessian identity hold:

$$\int vLu\, d\mu = -\int \langle M\nabla u, \nabla v\rangle\, d\mu, \tag{2.11}$$

$$\int (Lu)^2\, d\mu = \int \|M^{1/2}D^2uM^{1/2}\|_{\mathrm{HS}}^2\, d\mu + \int \langle M\nabla u, \nabla u\rangle\, d\mu. \tag{2.12}$$

Then for every $f \in L^2(\mu)$ orthogonal to Aff *and every $\varepsilon > 0$, there is $u \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\|f - \mathcal{P}u\|_{L^2(\mu)} < \varepsilon. \tag{2.13}$$

Every $\mathcal{P}u$ is orthogonal to Aff*. If both μ and M are even, u may be chosen even whenever f is even.*

Proof. Since $L1 = 0$ and $Lx_i = -x_i$, we have $\mathcal{P}\ell = 0$ for every affine ℓ. Formal symmetry and compact support give $\int \ell\mathcal{P}u\, d\mu = \int u\mathcal{P}\ell\, d\mu = 0$. It remains to show that no further obstruction exists.

Let $g \in L^2(\mu)$ annihilate the compact-test range. Then $L(L+I)g = 0$ distributionally. Its principal symbol is $(\xi^T M(x)\xi)^2$, positive for $\xi \neq 0$, so interior elliptic regularity makes g smooth [52, Section 14]; see also [73]. We verify the global integrability needed to use (2.12) on this solution. Choose smooth cutoffs with

$$\begin{gathered} 0 \le \eta_r \le 1, \quad \eta_r = 1 \text{ on } B_r, \quad \operatorname{supp}\eta_r \subset B_{2r}, \\ |\nabla\eta_r| \le C/r, \quad |D^2\eta_r| \le C/r^2, \qquad r \ge 1. \end{gathered}$$

Write

$$\begin{gathered} \Gamma(a,b) = \langle M\nabla a, \nabla b\rangle, \quad \Gamma(a) = \Gamma(a,a), \quad v = -Lg, \\ G = \|g\|_2, \quad X_r = \|\eta_r^2 v\|_2, \quad Y_r^2 = \int \eta_r^2\Gamma(g)\, d\mu. \end{gathered} \tag{2.14}$$

Boundedness of M and the drift $-x$ give $\Gamma(\eta_r) \le C/r^2$ and $|L\eta_r| \le C$; both vanish outside the annulus where the cutoff varies. Here and below C can change but is independent of r.

Testing $L(L+I)g = 0$ against $\eta_r^4 g$ means pairing v with

$$\begin{aligned}(-L-I)(\eta_r^4 g) = \eta_r^4 v + g\{-4\eta_r^3 L\eta_r - 12\eta_r^2\Gamma(\eta_r) - \eta_r^4\} \\ - 8\eta_r^3\Gamma(\eta_r, g).\end{aligned}$$

Using $2ab \le \delta a^2 + \delta^{-1}b^2$ pointwise to absorb the factors $\eta_r^2 v$ gives

$$X_r^2 \le CG^2 + \frac{C}{r^2}Y_r^2.$$

Similarly, integration by parts (2.11), applied to g and $\eta_r^2 g$, gives

$$Y_r^2 = \int \eta_r^2 g v \, d\mu - 2\int \eta_r g\Gamma(\eta_r, g)\, d\mu \le X_r^2 + CG^2.$$

The same elementary inequality absorbs half of Y_r^2 in the last step. For sufficiently large r, the two bounds absorb the $r^{-2}X_r^2$ term and bound X_r, Y_r uniformly. Exhaustion therefore proves $v \in L^2(\mu)$ and $\int \Gamma(g)\, d\mu < \infty$.

The equation now splits into $L(g-v) = 0$ and $Lv = -v$. Testing the first with $\eta_r^2(g-v)$ gives the usual cutoff estimate

$$\int \eta_r^2\Gamma(g-v)\, d\mu \le \frac{C}{r^2}\|g-v\|_2^2.$$

Letting $r \to \infty$ shows that $g - v$ is constant. Testing $Lv = -v$ against $\eta_r^2 v$ similarly gives $\int \Gamma(v)\, d\mu < \infty$.

The product rule yields

$$L(\eta_r v) = -\eta_r v + vL\eta_r + 2\Gamma(\eta_r, v).$$

The middle error tends to zero in $L^2(\mu)$ by the tail of v, and

$$\|\Gamma(\eta_r, v)\|_2^2 \le \frac{C}{r^2}\int \Gamma(v)\, d\mu \longrightarrow 0.$$

Consequently $\eta_r v \to v$ and $L(\eta_r v) \to -v$ in $L^2(\mu)$. Apply (2.12) (the assumed Hessian identity) to the compact smooth test $\eta_r v$, and use (2.11) to replace its gradient energy:

$$\begin{aligned}\int \|M^{1/2}D^2(\eta_r v)M^{1/2}\|_{\mathrm{HS}}^2\, d\mu &= \|L(\eta_r v)\|_2^2 + \int \eta_r v L(\eta_r v)\, d\mu \\ &\longrightarrow \|v\|_2^2 - \|v\|_2^2 = 0.\end{aligned}$$

On each fixed ball $\eta_r v = v$ eventually, so positive definiteness of M forces $D^2 v = 0$. The equation $Lv = -v$ removes its constant term. Hence g is affine. Taking orthogonal complements proves the approximation assertion. In the even case, replace an approximating u by $(u(x) + u(-x))/2$; the operator commutes with this orthogonal projection, so the error cannot increase. □

This proposition concerns approximate solutions with compact smooth tests. It requires no bounded inverse for $L(L+I)$ and does not assume that its compact tests are a fourth-order operator core. Its two input identities will be proved directly for the moment operator in Section 5.

Remark 2.3 (Regularity convention). *For the sake of presentation, differential calculations are first carried out for smooth coefficients and compact tests, where every integration by parts is justified. Core approximation extends the first-order statements; Proposition 2.2 (approximate solvability modulo affine functions) supplies the global input for the iterated operator. For geometric applications, even convex smoothing, Gaussian regularization, and passage to the multiplicative integral inequality remove the initial regularity assumptions; see Section B.*

Legendre duality and the Hessian. For a proper lower semicontinuous convex function $U : \mathbb{R}^n \to (-\infty, +\infty]$, its *Legendre transform* is

$$U^*(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - U(y)\}. \tag{2.15}$$

It is again lower semicontinuous and convex, and $U^{**} = U$ [59, 136]. The defining inequality is Fenchel's inequality:

$$U(y) + U^*(x) \geq \langle x, y\rangle, \tag{2.16}$$

with equality precisely when $x \in \partial U(y)$, where $\partial U(y)$ is the set of slopes of supporting affine functions at y. If U is differentiable, this means $x = \nabla U(y)$ [136, Section 23]. For example, $(|y|^2/2)^* = |x|^2/2$, and the convex indicator I_K, equal to zero on a closed convex set K and $+\infty$ outside, satisfies $I_K^* = h_K$ (with the support function extended to all of $\mathbb{R}^n$). Adding a constant reverses its sign under duality: $(U + c)^* = U^* - c$.

For a smooth strictly convex potential φ with positive Hessian, the gradient is a diffeomorphism onto its image. On that image its inverse is $\nabla\varphi^*$ [136, Section 26]; no assertion of surjectivity onto all of $\mathbb{R}^n$ is implicit. The source and target coordinates satisfy the inverse-Hessian identity

$$\begin{aligned} x = \nabla_y \varphi(y), \qquad & y = \nabla_x \varphi^*(x), \\ D_y^2 \varphi\Big(\nabla_x \varphi^*(x)\Big) = & \Big(D_x^2 \varphi^*(x)\Big)^{-1}. \end{aligned} \tag{2.17}$$

Indeed, differentiating $\nabla_y\varphi(\nabla_x\varphi^*(x)) = x$ gives

$$D_y^2\varphi\Big(\nabla_x\varphi^*(x)\Big)D_x^2\varphi^*(x) = I. \tag{2.18}$$

As a metric, the source Hessian $D_y^2\varphi(y)$ becomes $D_x^2\varphi^*(x)$ in target coordinates, since $dx = D_y^2\varphi(y)\,dy$. Its inverse $(D_x^2\varphi^*(x))^{-1}$ is the diffusion matrix and Stein kernel on the target.

In formulas written in x-coordinates, including integrals against μ, we write this matrix as $(D^2\varphi^*)^{-1}$; all derivatives in such formulas are ordinary x-derivatives. In source calculations we write $D_y^2\varphi(y)$ explicitly. Subscripts on φ, such as φ_{ij} and φ_{ijk}, denote ordinary y-derivatives at the stated source point. Compositions with the moment map or its inverse are displayed when the variables change.

3 L^2 method: the baby case

For the reader's convenience, we begin with the classical setting. The same argument will appear twice more, with different operators and energies. An expert may skip this section; its purpose is to make the mechanism, and the place where convexity and evenness enter, transparent. We follow the sign convention and the order of the duality argument in [107, Sections 7.1–7.9].

Throughout this section,

$$Lu = \Delta u - \langle \nabla V, \nabla u\rangle, \qquad d\mu = Z^{-1}e^{-V}\,dx. \tag{3.1}$$

Initially, all differentiations are for smooth functions and all integrations by parts have a compactly supported factor. The domain conventions of Section 2 give the corresponding closed-form statements.

3.1 Integration by parts and Bochner formula

Direct integration by parts gives

$$\int vLu\,d\mu = -\int \langle \nabla u, \nabla v\rangle\,d\mu. \tag{3.2}$$

Thus L is nonpositive and symmetric. There is also a useful way to obtain both this formula and the integrated Bochner formula by differentiating a change of variables, as in [107, Sections 3.15 and 7.1].

Take $u \in C_c^\infty(\mathbb{R}^n)$ and, for sufficiently small $|t|$, set

$$T_t(x) = x + t\nabla u(x). \tag{3.3}$$

This is a diffeomorphism: $I + tD^2u$ is positive definite for small $|t|$, and T_t is the identity outside a compact set. For a smooth integrable test function v, the expression

$$I_v(t) = \frac{1}{Z}\int v(T_t(x))e^{-V(T_t(x))}\det(I + tD^2u(x))\,dx = \int v\,d\mu \tag{3.4}$$

is independent of t. The change of variables is $z = T_t(x)$. Differentiating once at zero gives

$$0 = I_v'(0) = \int \Big(\langle \nabla v, \nabla u\rangle + vLu\Big)\,d\mu,$$

which is precisely (3.2).

Proposition 3.1 (Classical Bochner formula [17, 7])**.** *The pointwise and integrated formulas are*

$$\frac{1}{2}L|\nabla u|^2 - \langle \nabla u, \nabla Lu\rangle = \|D^2u\|_{\mathrm{HS}}^2 + \langle D^2V\nabla u, \nabla u\rangle, \tag{3.5}$$

$$\int (Lu)^2\,d\mu = \int \|D^2u\|_{\mathrm{HS}}^2\,d\mu + \int \langle D^2V\nabla u, \nabla u\rangle\,d\mu. \tag{3.6}$$

Two proofs. For the pointwise identity, differentiate L and use its product rule:

$$\partial_i Lu = L(\partial_i u) - \sum_j V_{ij}\partial_j u, \qquad \frac{1}{2}L|\nabla u|^2 = \sum_i (\partial_i u)L(\partial_i u) + \sum_{i,j} u_{ij}^2.$$

Subtracting proves (3.5). Its integral, together with (3.2), proves (3.6).

Alternatively, to see the integrated identity directly by differentiation, take $v = 1$ in (3.4). Write the logarithm of its density ratio as

$$\begin{aligned} q_t(x) = \; & -V(x + t\nabla u(x)) + V(x) \\ & + \log\det(I + tD^2u(x)). \end{aligned} \tag{3.7}$$

Then $I_1(t) = \int e^{q_t}\,d\mu = 1$ and

$$q_0' = Lu, \qquad q_0'' = -\langle D^2V\nabla u, \nabla u\rangle - \mathrm{Tr}((D^2u)^2).$$

The elementary identity $(e^q)'' = e^q((q')^2 + q'')$ now gives

$$0 = I_1''(0) = \int \Big((Lu)^2 - \|D^2u\|_{\mathrm{HS}}^2 - \langle D^2V\nabla u, \nabla u\rangle\Big)\,d\mu.$$

Here D^2u is symmetric, so $\mathrm{Tr}((D^2u)^2) = \|D^2u\|_{\mathrm{HS}}^2$. □

Thus integration by parts is a first variation of a constant integral, and integrated Bochner is its second variation. Convexity of V makes the second term on the right of (3.6) nonnegative.

3.2 Hörmander duality and its two consequences

Lemma 3.2 (Hörmander duality, classical form)**.** *Let a norm $\|\cdot\|$, or a seminorm, on vector fields satisfy*

$$\int (Lu)^2 \, d\mu \geq \|\nabla u\|^2 \tag{3.8}$$

on an operator core. With duality taken under the ordinary integrated Euclidean pairing, one has

$$\operatorname{Var}_\mu f \leq \|\nabla f\|_*^2. \tag{3.9}$$

The assertion holds for $f \in L^2(\mu)$ whenever integration by parts with f is valid and the displayed dual norm is defined; in particular it holds for smooth f with the stated finite energies, by approximation.

Proof. Here is the completion-of-squares argument of the L^2 method. If one can solve $Lu = f - \int f \, d\mu$, then

$$\operatorname{Var}_\mu f = 2\int f Lu \, d\mu - \int (Lu)^2 \, d\mu \leq -2\int \langle \nabla f, \nabla u \rangle \, d\mu - \|\nabla u\|^2 \leq \|\nabla f\|_*^2.$$

For the general case, apply the same estimate to each core function u and take the supremum in (2.8). The last step uses the definition of a dual norm

$$\sup_F \left\{ 2\int \langle G, F \rangle \, d\mu - \|F\|^2 \right\} = \|G\|_*^2.$$

□

As a corollary, we deduce the inequality of Brascamp and Lieb [28] (for this proof, see [72]; see also [107, Sections 7.4–7.9]).

Corollary 3.3 (The Brascamp–Lieb inequality)**.** *If D^2V is positive definite, then*

$$\operatorname{Var}_\mu f \leq \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle \, d\mu. \tag{3.10}$$

Proof. Discard the Hessian term in (3.6) and apply Lemma 3.2 to the norm

$$\|F\|^2 = \int \langle D^2V F, F \rangle \, d\mu. \tag{3.11}$$

Its squared dual norm is the integral with $(D^2V)^{-1}$. □

Define the homogeneous negative Sobolev seminorm by

$$\|g\|_{H^{-1}(\mu)} = \sup \left\{ \left| \int gh \, d\mu \right| : \int |\nabla h|^2 \, d\mu \leq 1 \right\}. \tag{3.12}$$

The test class contains constants and the appropriate form-domain limits of smooth functions. Thus the value is infinite unless $\int g \, d\mu = 0$. We deduce a result by Klartag [82, Lemma 1] and Barthe, Klartag [12, Proposition 10]:

Corollary 3.4 (Klartag's H^{-1} inequality)**.** *If V is convex, then*

$$\operatorname{Var}_\mu f \le \sum_{i=1}^{n} \|\partial_i f\|^2_{H^{-1}(\mu)}. \tag{3.13}$$

Proof. This time discard the D^2V term in the same Bochner formula and use

$$\|F\|^2 = \sum_i \int |\nabla F_i|^2 \, d\mu. \tag{3.14}$$

The squared dual seminorm is the sum of the squared scalar dual seminorms in (3.12). □

3.3 What evenness improves in the classical setting

Suppose V is even. The operator preserves parity, so the variational formula for an even f may be restricted to even u. The derivatives $\partial_i u$ are then odd. Define the odd spectral constant by

$$\lambda_{\text{odd}} = \inf_{\substack{h \neq 0 \\ h \text{ odd}}} \frac{\int |\nabla h|^2 \, d\mu}{\int h^2 \, d\mu}, \tag{3.15}$$

where the infimum is over the form domain. Its definition, applied to each $\partial_i u$, gives

$$\int \|D^2 u\|^2_{\text{HS}} \, d\mu \ge \lambda_{\text{odd}} \int |\nabla u|^2 \, d\mu. \tag{3.16}$$

We can therefore keep both Bochner terms and use duality once more.

Corollary 3.5 (An even improvement)**.** *For even μ and even f, provided $D^2V + \lambda_{\text{odd}} I$ is positive definite,*

$$\operatorname{Var}_\mu f \le \int \langle (D^2V + \lambda_{\text{odd}} I)^{-1} \nabla f, \nabla f \rangle \, d\mu. \tag{3.17}$$

In particular, if $D^2V \ge \rho I$ with $\rho > 0$, then

$$\operatorname{Var}_\mu f \le \frac{1}{2\rho} \int |\nabla f|^2 \, d\mu \qquad (f \text{ even}). \tag{3.18}$$

Proof. The estimate (3.16) and (3.6) give

$$\int (Lu)^2 \, d\mu \ge \int \langle (D^2V + \lambda_{\text{odd}} I) \nabla u, \nabla u \rangle \, d\mu \qquad (u \text{ even}).$$

The second assertion follows from the first. □

Further refinements use the geometry of the reference potential. Cordero-Erausquin and Rotem [43, Theorem 3] obtain a sharp matrix weight for even convex perturbations of rotationally invariant measures. Colesanti, Kolesnikov, Livshyts, and Rotem [39, Theorem C] prove strong Brascamp–Lieb inequalities for several homogeneous potentials, while Kolesnikov, Livshyts, and Rotem [89] relate homogeneous refinements to local L_p-Brunn–Minkowski inequalities. Related curvature and symmetry improvements appear in [92, 77].

There is an anisotropic version of the odd estimate used in (3.16). Let C be a smooth symmetric convex body of positive curvature, $r = \|\cdot\|_C$, $\Phi = r^2/2$, and $d\eta = Z^{-1}e^{-U-w(r)}dx$, with U even and convex and w increasing and C^1 on $(0,\infty)$. For every smooth odd h with finite integrals, Armeniakos, Kolesnikov, and Ulivelli [155, Theorem 1.2] obtain

$$\int \frac{w'(r)}{r}\, h^2 \, d\eta \le \int \langle (D^2\Phi)^{-1}\nabla h, \nabla h\rangle \, d\eta.$$

For $w \in C^1([0,\infty))$ this inequality was obtained independently by Cordero-Erausquin, Livshyts, and Rotem [156, Theorem 1.8]. Suppose now $\alpha I \le D^2\Phi \le \beta I$ away from zero, with $\gamma = \alpha/\beta > 0$. Apply the odd estimate to $h = \partial_i u$ for even u and sum over i. Since $\gamma D^2\Phi \le \alpha I$ and $\alpha(D^2\Phi)^{-1} \le I$, this gives

$$\int \|D^2u\|_{\mathrm{HS}}^2 \, d\eta \ge \gamma \int \frac{w'(r)}{r} \langle D^2\Phi \nabla u, \nabla u\rangle \, d\eta.$$

If U, w are C^2 and $w''(s) + \gamma w'(s)/s \ge 0$ for $s > 0$, keeping this term in (3.6) gives the estimate of [155, Theorem 1.3] for $W = w(r)$ and even f,

$$\mathrm{Var}_\eta f \le \int \langle \left(D^2W + \gamma \frac{w'(r)}{r} D^2\Phi \right)^{-1} \nabla f, \nabla f\rangle \, d\eta.$$

At a singular matrix the inverse denotes the dual quadratic form. For $C = B_2^n$ one can take $\gamma = 1$, recovering the rotationally invariant estimate of Cordero-Erausquin and Rotem [43].

For the standard Gaussian, Corollary 3.5 provides the sharp factor $1/2$, with equality for centered quadratic polynomials. The same argument applies to an even convex perturbation of the Gaussian, and by approximation to restriction to a symmetric convex set. This improvement is one of the analytic ingredients of the Gaussian B-theorem [46]; see also [107, Section 7.9]. It is the same basic phenomenon as the removal of the first harmonics in the introductory circle example.

For a general even log-concave measure, this calculation does not yet supply the dilation estimate of Theorem C. We need an operator whose low modes and

second-derivative energies fit the Euclidean convexity of the potentials. In the next section we outline a general version of these explorations, for a general elliptic operator. Later we will see a particular choice of the operator which fits our needs.

4 L^2 method: the general (grownup) case

4.1 The general form of our operator

As before, let $d\mu = Z^{-1}e^{-V}dx$. Prescribe a smooth symmetric positive definite matrix field M and define the operator by

$$\int vLu\,d\mu = -\int \langle M\nabla u, \nabla v\rangle\,d\mu. \tag{4.1}$$

Initially u, v are compactly supported smooth functions; in the closed realization, u belongs to the operator domain and v to the form domain $W^{1,2}(\mu; M)$ from Section 2. For an unbounded matrix field, ordinary $H^1(\mu)$ alone need not make the right side finite.

Symmetry and positivity of M make the energy symmetric and nonnegative. Testing against compactly supported v determines the differential expression uniquely:

$$Lu = e^V \operatorname{div}(e^{-V}M\nabla u) = \sum_{i,j} M_{ij}u_{ij} + \sum_i b_i u_i, \qquad b_i = \sum_j \partial_j M_{ij} - \sum_j M_{ij}V_j. \tag{4.2}$$

Indeed, the distributional divergence is the adjoint of the right side of (4.1).

4.2 Bochner with a variable matrix

Define

$$\Gamma_M(u,v) = \langle M\nabla u, \nabla v\rangle, \qquad \Gamma_M(u) = \Gamma_M(u,u), \tag{4.3}$$

$$\Gamma_{2,M}(u) = \frac{1}{2}L\Gamma_M(u) - \Gamma_M(u, Lu). \tag{4.4}$$

Integration by parts gives

$$\int \Gamma_{2,M}(u)\,d\mu = \int (Lu)^2\,d\mu. \tag{4.5}$$

The following coefficients are the Christoffel symbols of the metric M^{-1}, written without introducing a second name for that metric:

$$C^k_{ij} = \frac{1}{2}\sum_\ell M_{k\ell}\left(\partial_i (M^{-1})_{j\ell} + \partial_j (M^{-1})_{i\ell} - \partial_\ell (M^{-1})_{ij}\right), \qquad (C^k)_{ij} = C^k_{ij}. \tag{4.6}$$

For a vector field, viewed as a covector by the Euclidean pairing, define its corrected derivative by

$$(\mathcal{D}_M F)_{ij} = \partial_j F_i - \sum_k C_{ij}^k F_k. \tag{4.7}$$

In particular, $\mathcal{D}_M \nabla u$ is symmetric. Finally, define the symmetric curvature matrix by

$$(\mathcal{R}_M)_{k\ell} = \frac{1}{2} L M_{k\ell} - \frac{1}{2} \sum_i \left(M_{ki} \partial_i b_\ell + M_{\ell i} \partial_i b_k \right) - \operatorname{Tr}(M C^k M C^\ell). \tag{4.8}$$

Proposition 4.1 (Matrix Bochner formula)**.** *Pointwise,*

$$\Gamma_{2,M}(u) = \| M^{1/2} (\mathcal{D}_M \nabla u) M^{1/2} \|_{\mathrm{HS}}^2 + \langle \mathcal{R}_M \nabla u, \nabla u \rangle. \tag{4.9}$$

Consequently,

$$\int (Lu)^2 \, d\mu = \int \| M^{1/2} (\mathcal{D}_M \nabla u) M^{1/2} \|_{\mathrm{HS}}^2 \, d\mu + \int \langle \mathcal{R}_M \nabla u, \nabla u \rangle \, d\mu. \tag{4.10}$$

Proof. Repeated indices are summed in this calculation. Expanding the product rule in $\Gamma_{2,M}$ cancels the third derivatives of u and leaves

$$\begin{aligned} \Gamma_{2,M}(u) = {} & M_{ij} M_{k\ell} u_{ik} u_{j\ell} + 2 M_{ij} (\partial_i M_{k\ell}) u_{kj} u_\ell - M_{k\ell} u_k (\partial_\ell M_{ij}) u_{ij} \\ & + \left[\frac{1}{2} L M_{k\ell} - \frac{1}{2} \left(M_{ki} \partial_i b_\ell + M_{\ell i} \partial_i b_k \right) \right] u_k u_\ell. \end{aligned} \tag{4.11}$$

Differentiate $M^{-1} M = I$ in (4.6). The terms with one Hessian and one gradient combine as

$$2 M_{ij} (\partial_i M_{k\ell}) u_{kj} u_\ell - M_{k\ell} u_k (\partial_\ell M_{ij}) u_{ij} = -2 \sum_k u_k \operatorname{Tr}(M D^2 u M C^k).$$

Completing the square gives

$$\begin{aligned} \| M^{1/2} (\mathcal{D}_M \nabla u) M^{1/2} \|_{\mathrm{HS}}^2 = {} & \operatorname{Tr}(M D^2 u M D^2 u) - 2 \sum_k u_k \operatorname{Tr}(M D^2 u M C^k) \\ & + \sum_{k,\ell} u_k u_\ell \operatorname{Tr}(M C^k M C^\ell). \end{aligned}$$

The remaining quadratic term is (4.8). Integrating and using (4.5) proves the second formula. □

This is the weighted Riemannian Bochner identity in Euclidean coordinates. The Γ_2 framework is due to Bakry and Émery [6, 7, 101]; the corresponding geometric Brascamp–Lieb theory, including boundary terms, is developed in [90].

4.3 Duality and its two consequences

Lemma 4.2 (Hörmander duality for M)**.** *Assume the self-adjoint realization of L satisfies the domain conventions of Section 2, with* $\ker L = \operatorname{span}\{1\}$*. Suppose a norm or seminorm on vector fields satisfies*

$$\int (Lu)^2 \, d\mu \geq \|\nabla u\|^2$$

on an operator core. Define the dual by the ordinary integrated pairing,

$$\|G\|_* = \sup_{\|F\| \leq 1} \left| \int \langle G, F \rangle \, d\mu \right|. \tag{4.12}$$

Then, for $f \in L^2(\mu)$ whenever integration by parts and the dual pairing are defined,

$$\operatorname{Var}_\mu f \leq \|M \nabla f\|_*^2. \tag{4.13}$$

The right side is allowed to be $+\infty$.

Proof. The case of an infinite dual value is immediate. Otherwise, for each core function u, integration by parts gives

$$\begin{aligned} \left| \int f L u \, d\mu \right| &= \left| \int \langle M \nabla f, \nabla u \rangle \, d\mu \right| \\ &\leq \|M \nabla f\|_* \|\nabla u\| \leq \|M \nabla f\|_* \|Lu\|_{L^2(\mu)}. \end{aligned}$$

For a seminorm, finiteness of the dual value forces the pairing to vanish on every zero-energy field, so the same argument applies. By Proposition 2.1, the range of L on its core is dense in $L_0^2(\mu)$. Choose $Lu_k \to f - \int f \, d\mu$ in $L^2(\mu)$ and pass to the limit:

$$\operatorname{Var}_\mu f \leq \|M \nabla f\|_* \sqrt{\operatorname{Var}_\mu f}.$$

Cancellation proves the assertion when the variance is nonzero; the zero case is immediate. Equivalently, one can complete the square in (2.8), exactly as in the classical proof. □

Corollary 4.3 (A Brascamp–Lieb inequality for M)**.** *If $\mathcal{R}_M$ is positive definite, then*

$$\operatorname{Var}_\mu f \leq \int \langle M \mathcal{R}_M^{-1} M \nabla f, \nabla f \rangle \, d\mu, \tag{4.14}$$

where, explicitly,

$$(\mathcal{R}_M)_{k\ell} = \frac{1}{2} L M_{k\ell} - \frac{1}{2} \sum_i \left(M_{ki} \partial_i b_\ell + M_{\ell i} \partial_i b_k \right) - \operatorname{Tr}(M C^k M C^\ell),$$

$$b_i = \sum_j \partial_j M_{ij} - \sum_j M_{ij}\partial_j V,$$
$$(C^k)_{ij} = \frac{1}{2}\sum_r M_{kr}\left(\partial_i (M^{-1})_{jr} + \partial_j (M^{-1})_{ir} - \partial_r (M^{-1})_{ij}\right).$$

Proof. Discard the nonnegative square in (4.10). For

$$\|F\|^2 = \int \langle \mathcal{R}_M F, F\rangle \, d\mu \tag{4.15}$$

we obtain $\int (Lu)^2 d\mu \geq \|\nabla u\|^2$. The squared dual norm is $\|G\|_*^2 = \int \langle \mathcal{R}_M^{-1} G, G\rangle \, d\mu$. Lemma 4.2, with $G = M\nabla f$, proves (4.14). □

For the other consequence define

$$\begin{aligned} \|F\|^2_{\dot{H}^1_M(\mu)} &= \int \|M^{1/2}(\mathcal{D}_M F) M^{1/2}\|^2_{\mathrm{HS}} \, d\mu, \\ \|G\|_{\dot{H}^{-1}_M(\mu)} &= \sup_{\|F\|_{\dot{H}^1_M(\mu)} \leq 1} \left| \int \langle G, F\rangle \, d\mu \right|. \end{aligned} \tag{4.16}$$

Corollary 4.4 (An H^{-1} inequality for M)**.** *If $\mathcal{R}_M$ is positive semidefinite, then*

$$\mathrm{Var}_\mu f \leq \|M\nabla f\|^2_{\dot{H}^{-1}_M(\mu)}. \tag{4.17}$$

Proof. This time discard the curvature term of (4.10). The remaining inequality is

$$\int (Lu)^2 \, d\mu \geq \|\nabla u\|^2_{\dot{H}^1_M(\mu)}.$$

Apply Lemma 4.2 with the seminorm (4.16). □

4.4 Evenness, low modes, and the choice of a Stein kernel

Suppose $V(-x) = V(x)$ and $M(-x) = M(x)$. Then L preserves parity, and the variational formula for even f may be restricted to even u, whose gradients are odd. To see what should be removed by a stronger argument, start with the identity

$$\mathrm{Var}_\mu f = \inf_{a\in\mathbb{R}} \int (f-a)^2 \, d\mu.$$

Let $f_{\mathrm{ev}}(x) = (f(x) + f(-x))/2$ and let $\mathcal{H}_{\mathrm{odd}} \subset L^2(\mu)$ be the closed subspace of odd functions. Orthogonality of even and odd functions gives

$$\inf_{\substack{a\in\mathbb{R} \\ g\in\mathcal{H}_{\mathrm{odd}}}} \int (f-a-g)^2 \, d\mu = \mathrm{Var}_\mu(f_{\mathrm{ev}}). \tag{4.18}$$

The infimum over odd functions alone would retain the squared mean of the even part. For even f, (4.18) is exactly $\mathrm{Var}_\mu f$. This suggests proving an estimate after removing selected odd modes, which are automatically absent from an even test function.

The first modes to try are the lowest odd eigenfunctions. More precisely, if $-L$ has a positive eigenvalue λ and $L\psi = -\lambda\psi$, then the self-adjoint polynomial operator

$$P_\lambda = L(L + \lambda I) \tag{4.19}$$

annihilates both constants and ψ. Spectral calculus [79, Chapter V] gives

$$\ker P_\lambda = \ker L \oplus \ker(L + \lambda I), \qquad \overline{\mathrm{Ran}\, P_\lambda} = (\ker P_\lambda)^\perp.$$

If λ is the first positive spectral value and its eigenspace is odd, this removes the lowest nonconstant modes without removing any centered even function (assuming the existence of the first eigenfunctions).

For our problem, linear functions are particularly suitable: they are odd and their Hessians vanish. Removing affine functions is therefore compatible with both the Euclidean structure and second derivatives of convex potentials. After normalization, we would like

$$Lx_i = -x_i, \qquad P = L(L + I).$$

By (4.2), the first requirement is

$$M\nabla V - \operatorname{div} M = x. \tag{4.20}$$

Equivalently, integration by parts gives, for every smooth compactly supported vector field F,

$$\int \langle x, F\rangle \, d\mu = \int \mathrm{Tr}(MDF)\, d\mu.$$

This is the Stein-kernel identity for a centered measure. We seek a symmetric positive definite Stein kernel; general Stein kernels need not have either property and need not be unique. Fathi's moment-map construction [58], based on the moment-measure theory [42, 83], provides a distinguished choice:

$$\begin{aligned} M(x) &= (D_x^2 \varphi^*(x))^{-1}, \\ (\nabla\varphi)_\# (e^{-\varphi} dy) &= \mu. \end{aligned} \tag{4.21}$$

One precise optimality property can already be seen. Under the usual integrability assumptions, the Stein identity gives

$$\int \langle Mb, b\rangle \, d\mu = \int \langle x, b\rangle^2 \, d\mu = \mathrm{Var}_\mu(\langle x, b\rangle).$$

Thus the spectral gap of a positive Stein-kernel energy is at most one. The moment kernel attains one, by Corollary 5.11. This optimality alone does not assert uniqueness among all positive Stein kernels. The decisive additional property is compatibility with Hessian coordinates: it gives an exact factorization of $L_\mu^m(L_\mu^m + I)$ through the Euclidean Hessian and makes the second-order estimate available. We prove these statements in Section 5, and discuss the rigidity and the precise sense of optimality in Section 8.

5 L^2 method: the other baby – the iterated moment-map operator

The moment-map approach was developed by Klartag [84]. The existence and uniqueness theory for essentially continuous moment potentials is due to Cordero-Erausquin and Klartag [42]; in parallel, Colesanti and Fragalà [38] introduced the related functional surface-area measure through first variation. These works were followed by Rotem [139, 140] and the general functional Minkowski theorem of Falah and Rotem [56]. Various analytic and geometric aspects were studied by Klartag, Fathi, Rotem, Kolesnikov and Santambrogio [83, 58, 142, 86, 85, 88]. Moment measures played a crucial role in the recent advances regarding the sharp thin shell bound and the MM^* estimate, see Chen, Klartag [35] and Bizeul [15]. E. Milman's centroaffine geometry [121] bears some analogies with the moment measure theory, as was vaguely pointed out in [89, Remark 1.3 and Section 4.3]; we shall discuss this in more detail in Section 9.

Let μ be a nonzero finite Borel measure on $\mathbb{R}^n$. A proper lower semicontinuous convex function $\varphi : \mathbb{R}^n \to (-\infty, +\infty]$ is a *moment potential* for μ if

$$\begin{gathered} d\nu(y) = e^{-\varphi(y)}\, dy, \qquad \nu(\mathbb{R}^n) = \mu(\mathbb{R}^n) \in (0, \infty), \\ (\nabla\varphi)_{\#}\nu = \mu. \end{gathered} \tag{5.1}$$

The gradient is understood ν-almost everywhere. The target μ is called the *moment measure* of the "source" ν, and $\nabla\varphi$ is the *moment map*. When μ is a probability measure, we call φ a *normalized moment potential*. The same function determines both the source density and the map: if Y has probability density $e^{-\varphi}$, then $\nabla\varphi(Y)$ has distribution μ.

For $d\mu(x) = Z^{-1}e^{-V(x)}dx$, where $Z = \int e^{-V}dx$, and smooth φ with positive Hessian, the change-of-variables formula gives the *transport equation*

$$\begin{aligned} e^{-\varphi(y)} &= Z^{-1}e^{-V(\nabla\varphi(y))} \det D_y^2\varphi(y), \\ \log\det D_y^2\varphi(y) &= V(\nabla\varphi(y)) + \log Z - \varphi(y). \end{aligned} \tag{5.2}$$

Thus prescribing μ gives a Monge–Ampère equation for φ [14, 42]. When boundary terms vanish, integration by parts yields

$$\int x \, d\mu(x) = \int \nabla\varphi(y) e^{-\varphi(y)} \, dy = 0.$$

Source and target derivatives. Recall the inverse-Hessian identity (2.17):

$$x = \nabla_y \varphi(y), \qquad y = \nabla_x \varphi^*(x), \qquad D_y^2 \varphi(y) = \left(D_x^2 \varphi^*(x)\right)^{-1}.$$

Here y is the source variable, with density $e^{-\varphi(y)}$, and x is the target variable, with law μ. The equality holds at corresponding points of the two coordinate systems. In the theorem below and in all target-coordinate formulas, we therefore use $(D^2\varphi^*)^{-1}$. Thus $D^2\varphi^* = D_x^2\varphi^*(x)$, $D^2V = D_x^2V(x)$, and $D^2g = D_x^2g(x)$ are all ordinary Hessians in the same variable. The exponent -1 denotes the matrix inverse.

In source calculations, $D_y^2\varphi(y)$ and $\varphi_{ij}(y) = \partial_{y_i y_j}\varphi(y)$ retain their usual meaning. For example, the chain rule reads

$$\nabla_y(g \circ \nabla_y\varphi)(y) = D_y^2\varphi(y)\,(\nabla_x g)(\nabla_y\varphi(y)).$$

This makes explicit both the differentiation variable and the point at which each derivative is evaluated.

Equipped with this powerful tool, we shall prove the following:

Theorem 5.1 (Variance modulo affine functions)**.** *Let $d\mu = Z^{-1}e^{-V(x)}dx$ be a centered probability measure on $\mathbb{R}^n$, where $V \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $0 < \alpha < 1$ and $D^2V(x) > 0$ for every x. Let $\varphi \in C^2(\mathbb{R}^n)$ be its moment potential. Then for any $f \in C^1(\mathbb{R}^n) \cap L^2(\mu)$ and any auxiliary function $g \in C^2(\mathbb{R}^n)$, we have:*

$$\inf_{a\in\mathbb{R},\, b\in\mathbb{R}^n} \int (f - a - \langle b, x\rangle)^2 \, d\mu \le \int \mathrm{Tr}\left((D^2\varphi^*)^{-1} D^2 g (D^2V)^{-1} D^2 g\right) d\mu + \int \langle (D^2V)^{-1}\nabla(f-g), \nabla(f-g)\rangle \, d\mu. \tag{5.3}$$

All derivatives in these integrals are ordinary derivatives with respect to x.

Corollary 5.2 (Even variance inequality)**.** *Under the assumptions of Theorem 5.1, suppose in addition that μ and f are even. Then*

$$\mathrm{Var}_\mu f \le \int \mathrm{Tr}\left((D^2\varphi^*)^{-1} D^2 g (D^2V)^{-1} D^2 g\right) d\mu + \int \langle (D^2V)^{-1}\nabla(f-g), \nabla(f-g)\rangle \, d\mu$$

for every admissible g.

Proof. The centered even function $f - \int f\,d\mu$ is orthogonal to every linear function, and thus Theorem 5.1 implies the assertion. □

Corollary 5.3 (Variance of an Euler (radial) derivative)**.** *Under the assumptions of Theorem 5.1, suppose that μ is even. For an even function $F \in C^2(\mathbb{R}^n)$, with $\langle x, \nabla F\rangle \in L^2(\mu)$ and finite right-hand side, one has*

$$\mathrm{Var}_\mu(\langle x, \nabla F\rangle) \leq \int \mathrm{Tr}\left(((D^2\varphi^*)^{-1} + xx^T)D^2F(D^2V)^{-1}D^2F\right) d\mu. \tag{5.4}$$

Proof. For an even function $F \in C^2(\mathbb{R}^n)$, set

$$f = \langle x, \nabla F\rangle, \qquad g = F, \qquad \nabla(f-g) = D^2F\, x. \tag{5.5}$$

Corollary 5.2 then reads

$$\begin{aligned}\mathrm{Var}_\mu(\langle x, \nabla F\rangle) \leq & \int \mathrm{Tr}\left((D^2\varphi^*)^{-1}D^2F(D^2V)^{-1}D^2F\right) d\mu \\ & + \int \langle (D^2V)^{-1}D^2F\, x, D^2F\, x\rangle\, d\mu.\end{aligned}$$

Since $\langle Ax, x\rangle = \mathrm{Tr}(xx^TA)$ for a symmetric matrix A, this is (5.4). □

The rest of this section is organized as follows. In subsection 5.1 we discuss moment measures, some intuition, their existence and basic properties. In subsection 5.2 we discuss a second order differential operator L_μ^m associated with $(D^2\varphi^*)^{-1}$ (which is a Stein kernel), derive a Bochner formula for it and a version of Hörmander's L_2 principle, and deduce a moment Brascamp-Lieb inequality and a moment Klartag's H^{-1}-inequality; the content of the subsections 5.1 and 5.2 was largely understood by the experts before the OpenAI preprint [131], see [84, 83, 86, 93, 58, 89].

By contrast, subsections 5.3–5.7 draw on the novel analytic input from the OpenAI preprint [131]. In subsection 5.3 we study the *iterated moment operator* $P = L_\mu^m(L_\mu^m + I)$ (a fourth order linear elliptic differential operator) and its associated integration by parts and Bochner's identities. In subsection 5.4 we outline several elementary linear algebraic identities and inequalities which are crucial in the proof of our main results. In subsection 5.5 we combine the bookkeeping from subsections 5.3 and 5.4 into a (technically challenging) proof of the integral inequality involving the operator P. In subsection 5.6 we prove Theorem 5.1: just like in the familiar L_2 scheme of proving the classical Brascamp-Lieb inequality, we approximately solve the PDE $f = Pu$ for the appropriate affine shift of the function f from Theorem 5.1, and use the analytic advances from subsections 5.3, 5.4 and 5.5 to complete the proof for Theorem 5.1 and hence for the Corollary 5.3. Finally, in subsection 5.7 we deduce Theorem C from Corollary 5.3. The regularity reduction is in Appendix A, and the detailed passage to convex sets is in Appendix B.

5.1 Moment measures: an introductory discussion

A functional Gauss map and functional surface area. For a convex body K, the Gauss map $N_K : \partial K \to \mathbb{S}^{n-1}$ assigns to a smooth boundary point its outer unit normal. For a smooth convex potential, $\nabla\varphi(y)$ records a supporting slope, and $\nabla\varphi(y)/|\nabla\varphi(y)|$ is the outer unit normal to the level set through y, wherever the gradient is nonzero. The moment map retains the length of this normal as well as its direction. In this sense, the moment map is a "functional Gauss map".

There is also a direct link between the moment measure of a log-concave function and the surface-area measure of a convex body. If $\varphi_t = (\varphi^* + tq)^*$ is an admissible smooth variation, then differentiating the Legendre transform gives $\dot{\varphi}_0 = -q \circ \nabla\varphi$ [136, 38, 139]. Hence

$$\frac{d}{dt}\int e^{-\varphi_t(y)}\,dy\bigg|_{t=0} = \int q(\nabla\varphi(y))e^{-\varphi(y)}\,dy = \int q\,d\mu. \tag{5.6}$$

This is the analogue of the geometric formula $\frac{d}{dt}|K_t|\big|_{t=0} = \int \dot{h}_0\,dS_K$ for a variation of support functions. Thus the moment measure can be viewed as functional surface area, as noted by Colesanti–Fragalà and Rotem [38, 139].

The moment-map viewpoint also connects with E. Milman's centroaffine geometry [121], as was noted in [89, Remark 1.3 and Section 4.3]. We expand upon this connection in Section 9.

Examples.

- For the standard Gaussian, both source and moment measure are the standard Gaussian.
- In dimension one there is an explicit inversion formula [158, p. 14]. Let $d\mu(x) = \rho(x)\,dx$ be a centered probability on an interval (a, b), with positive continuous density and finite first moment. Put

  $$H(x) = \int_x^b t\rho(t)\,dt > 0, \qquad y(x) = \int_0^x \frac{\rho(t)}{H(t)}\,dt, \qquad a < x < b.$$

  Then the normalized moment potential is given parametrically by

  $$(\varphi')^{-1}(x) = y(x), \qquad \varphi(y(x)) = -\log H(x).$$

  The choice $y(0) = 0$ fixes the source translation. Indeed, $H'(x) = -x\rho(x)$ gives

  $$\varphi'(y(x)) = x, \qquad \varphi''(y(x)) = \frac{H(x)}{\rho(x)}, \qquad e^{-\varphi(y(x))}y'(x) = \rho(x).$$

On the source interval $y((a,b))$, the measure $e^{-\varphi}dy$ has mass one and moment measure μ. Extend φ by $+\infty$ outside that interval. Equivalently, on the corresponding monotone branch, $(\varphi^{-1})'(-\log H(x)) = 1/x$ for $x \neq 0$, which is the formula in Klartag's slides.

- A useful bounded-target example is the uniform probability measure

$$d\mu = \frac{1}{2}\mathbf{1}_{(-1,1)}dx.$$

Its moment potential and map are

$$\begin{aligned} \varphi(y) &= 2\log\cosh\left(\frac{y}{2}\right) + \log 4, \\ x = \varphi'(y) &= \tanh\left(\frac{y}{2}\right), \qquad ((\varphi^*)''(x))^{-1} = \frac{1-x^2}{2}. \end{aligned} \tag{5.7}$$

Indeed, $e^{-\varphi(y)}dy = dx/2$. This construction extends to any dimension: the moment measure of $Ce^{-\sum_{i=1}^n \varphi(x_i)}dx$ is the uniform distribution on the unit cube in $\mathbb{R}^n$, with φ as above.

- For a geometric example, let K be a smooth strictly convex body containing the origin in its interior, write $r(y) = \|y\|_K$, and let $p > 1$. Take the *source* potential $\varphi_p = r^p/p + \log Z_p$, where $Z_p = \int e^{-r^p/p}$. If $y = t\xi$, $t > 0$ and $\xi \in \partial K$, then

$$\begin{aligned} \nabla\varphi_p(t\xi) &= t^{p-1}\frac{N_K(\xi)}{\langle \xi, N_K(\xi)\rangle}, \\ D_y^2\varphi_p(y) &= r(y)^{p-1}D^2 r(y) + (p-1)r(y)^{p-2}\nabla r(y)\otimes\nabla r(y). \end{aligned} \tag{5.8}$$

Here N_K is the outer unit normal. For $p = 2$ and symmetric K of positive curvature, the inverse of this source Hessian is precisely the weight in the odd estimate of Section 3.3 [155, 156].

- For the symmetric exponential target $d\mu(x) = \frac{1}{2}e^{-|x|}dx$, the normalized moment potential is

$$\begin{aligned} \varphi(y) &= e^{|y|} - 1 - |y| + \log 2, \\ x = \varphi'(y) &= \operatorname{sgn}(y)(e^{|y|} - 1), \qquad ((\varphi^*)''(x))^{-1} = 1 + |x|. \end{aligned}$$

Indeed, $dx/dy = e^{|y|}$ and $e^{-\varphi(y)}dy = \frac{1}{2}e^{-|x|}dx$.

- In the other direction, the exponential *source*

$$\varphi(y) = |y| + \log 2$$

has moment measure

$$\frac{1}{2}(\delta_{-1} + \delta_1),$$

since its gradient is $\operatorname{sgn}(y)$ almost everywhere.

Remark 5.4 (A rigidity property of piece-wise exponential measures). *Moreover, one can characterize log-concave measures whose moment measures are purely atomic: for a full-dimensional source, this happens precisely when the interior of the effective domain is covered, up to a Lebesgue-null set, by countably many convex regions on which φ is affine. The atoms are the distinct slopes, and their masses are the source masses of those regions. Indeed, the set where a given supporting plane agrees with φ is convex; if its gradient level set has positive measure, that agreement set has nonempty interior, and its boundary has measure zero. Applying this observation to each atom proves the assertion, and the converse follows directly from the pushforward definition. If φ is finite on all of $\mathbb{R}^n$, finite atomic support is equivalent to $\varphi(y) = \max_{1\le j\le N}(\langle p_j, y\rangle + c_j)$; countably many atoms need not give a finite polyhedral representation. This is an interesting rigidity property and might be useful for studying extremal properties of exponential log-concave measures, and for passing properties between log-concave measures and finite point configurations (particularly as log-concave measures can be approximated by piece-wise exponential ones). This is a sibling of the fact that polytopes have purely atomic surface area measures, and this also resembles the connection between the behavior of Legendre transform of a function and its growth at infinity.*

Increasing the dual volume. The inverse moment construction has a useful monotonicity property. For a convex potential ψ, consider its *functional volume product*

$$\mathcal{P}(\psi) = \left(\int e^{-\psi}\right)\left(\int e^{-\psi^*}\right). \tag{5.9}$$

For functional volume products and Blaschke–Santaló inequalities, see [8, 5, 64, 102, 65, 81]; transport proofs and weighted extensions are developed in [57, 49, 97, 39, 89, 98].

Suppose $d\mu = e^{-U}dx$ is a centered log-concave probability and φ is its normalized moment potential, chosen so that the source is centered. Then

$$\int e^{-U^*} \le \int e^{-\varphi^*}, \qquad \text{equivalently} \qquad \mathcal{P}(U) \le \mathcal{P}(\varphi). \tag{5.10}$$

Thus passing from the target measure to its moment-potential density increases the functional volume product. Here is the short argument in the smooth setting. Fenchel's inequality, applied to U at $(\nabla\varphi(y), y)$, and the identity $\langle y, \nabla\varphi(y)\rangle = \varphi(y) + \varphi^*(\nabla\varphi(y))$ give

$$\begin{aligned}\int e^{-U^*(y)}\,dy &\le \int e^{U(\nabla\varphi(y))-\langle y,\nabla\varphi(y)\rangle}\,dy\\ &= \int e^{U(\nabla\varphi(y))-\varphi^*(\nabla\varphi(y))}\,d\nu(y)\\ &= \int_{\{U<\infty\}} e^{-\varphi^*(x)}\,dx \le \int e^{-\varphi^*(x)}\,dx.\end{aligned}$$

The third line is exactly the pushforward relation (5.1).

This comparison belongs to the same broad variational and transport approach to functional Blaschke–Santaló inequalities as in Colesanti, Kolesnikov, Livshyts and Rotem [39, Section 7], Kolesnikov, Livshyts and Rotem [89, Section 5], and Kolesnikov, Werner [97, Section 8]. Alternatively, one may deduce the Blaschke-Santaló inequality via an appropriately organized affine dynamical iteration of the inverse moment construction; this discussion goes beyond the present lecture notes.

Existence of moment potentials. Smooth compact-target existence was obtained by Berman–Berndtsson [14]. Cordero-Erausquin and Klartag [42] established the general essentially continuous case, and Santambrogio [142] gave an alternative proof using entropy and optimal transport. The functional first-variation theory was developed by Colesanti–Fragalà [38] and Rotem [139, 140]. Falah and Rotem [56] solved the functional Minkowski problem with both its space and boundary measures prescribed. Prescribing only the measure on $\mathbb{R}^n$ gives the following corollary.

Theorem 5.5 (Existence of moment potential, partial case of [56, Theorem 1.5])**.** *A nonnegative Borel measure μ on $\mathbb{R}^n$ admits a proper lower semicontinuous convex moment potential if and only if*

$$0 < \mu(\mathbb{R}^n) < \infty, \qquad \int_{\mathbb{R}^n} |x|\,d\mu(x) < \infty.$$

In particular, there is such a φ with

$$(\nabla\varphi)_{\#}(e^{-\varphi(y)}dy) = \mu, \qquad \int e^{-\varphi(y)}dy = \mu(\mathbb{R}^n).$$

The measure μ need not be centered, log-concave, or supported on a full-dimensional set.

Sketch of the construction. Put $b = \int x\, d\mu(x)$ and let σ be uniform probability measure on $\mathbb{S}^{n-1}$. Choose the auxiliary spherical measure

$$\eta = \begin{cases} \sigma + |b|\delta_{-b/|b|}, & b \neq 0, \\ \sigma, & b = 0. \end{cases}$$

Then $\int x\, d\mu + \int \theta\, d\eta = 0$, and η spans $\mathbb{R}^n$. Falah–Rotem's theorem realizes μ and η as the space and boundary measures of a log-concave function. The idea is to minimize

$$\int_{\mathbb{R}^n} \varphi^*\, d\mu + \int_{\mathbb{S}^{n-1}} h_{\overline{\{\varphi<\infty\}}}\, d\eta - \mu(\mathbb{R}^n) \log \int_{\mathbb{R}^n} e^{-\varphi(y)} dy.$$

Coercivity and compactness give a minimizer. First variation in Legendre coordinates identifies its space measure with μ after fixing the mass. Necessity follows from $\int |\nabla\varphi| e^{-\varphi} < \infty$ for integrable log-concave densities [140, Proposition 1.6]. □

For the analytic arguments we use the canonical essentially continuous choice. Here *essential continuity* means that $e^{-\varphi} = 0$ at $\mathcal{H}^{n-1}$-almost every point of $\partial\{\varphi < \infty\}$. Cordero-Erausquin and Klartag [42, Theorem 2] proved that such a potential exists precisely when, in addition to the displayed mass and moment conditions, μ is centered and is not supported on a hyperplane; it is unique within this class up to source translation. If μ is even, this translation may be chosen so that φ is even. Without essential continuity, uniqueness can fail: the densities $(2a)^{-1}\mathbf{1}_{[-a,a]}$, $a > 0$, all have moment measure δ_0.

Klartag's transport estimate. Klartag developed the Hessian estimates and associated diffusion [83]; Fathi identified $(D^2\varphi^*)^{-1}$ as a Stein kernel [58]. We state the transport estimate separately, in the smooth setting used below. Smooth global coordinates follow from the compact-target regularity and exhaustion [14, 83]; see also [131, Lemma 3.1].

Theorem 5.6 (Klartag's transport estimate [83, Proposition 3.1 and Remark 3.5])**.** *Let $d\mu = Z^{-1}e^{-V}dx$ be a centered probability measure on $\mathbb{R}^n$, where V is smooth and $D^2V \geq \varepsilon I$ for some $\varepsilon > 0$. Its normalized essentially continuous moment potential φ is smooth on $\mathbb{R}^n$ and satisfies*

$$\begin{aligned} &\nabla\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is a diffeomorphism,} \\ &0 < D_y^2\varphi \leq \frac{1}{\varepsilon} I, \qquad \varphi(y) \geq c|y| - C \end{aligned} \tag{5.11}$$

for some $c > 0$ and $C < \infty$. If V is even, a source translation makes φ even. In particular,

$$|\nabla\varphi(y) - \nabla\varphi(z)| \le \frac{1}{\varepsilon}|y - z|, \qquad y, z \in \mathbb{R}^n.$$

The Hessian is pointwise positive definite; no uniform lower bound is asserted. This is a counterpart of Caffarelli's contraction estimate [32]: the source density and its Brenier transport map are determined by the same potential. For the resulting Stein-kernel bound, see [58, Corollary 2.4].

The following identities involving the moment potential φ are good to keep in mind, along with the transport equation (5.2):

Lemma 5.7 (Stein's identity and the Hessian trace formula)**.** *Let $\varphi \in C^2(\mathbb{R}^n)$ be a normalized strictly convex potential with $(\nabla\varphi)_\#(e^{-\varphi}dy) = \mu$ and with $\nabla\varphi$ a diffeomorphism. For $q \in C^1_c(\mathbb{R}^n)$ and $F \in C^2_c(\mathbb{R}^n)$ one has*

$$\int x_i q(x)\, d\mu(x) = \int \sum_j \left[(D^2\varphi^*)^{-1}\right]_{ij} \partial_{x_j} q(x)\, d\mu(x), \qquad i = 1, \dots, n, \tag{5.12}$$

$$\int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 F)\, d\mu = \int \langle x, \nabla F\rangle\, d\mu. \tag{5.13}$$

If μ has finite second moment and the cutoff passage for linear tests is valid, one also has

$$\int (D^2\varphi^*)^{-1}\, d\mu = \int xx^T\, d\mu = \mathrm{Cov}(\mu). \tag{5.14}$$

The same formulas hold for noncompact tests whenever a cutoff approximation passes the displayed terms and its boundary errors. In particular, this applies to tests with polynomially growing derivatives when $(D^2\varphi^*)^{-1}$ is bounded and μ has an exponential tail.

Proof. The Stein identity. By the pushforward relation, for every integrable test function a,

$$\int a(x)\, d\mu(x) = \int a(\nabla\varphi(y)) e^{-\varphi(y)}\, dy.$$

If $q \in C^1_c(\mathbb{R}^n)$, then $q \circ \nabla\varphi$ is also compactly supported: its support is contained in the image of $\operatorname{supp} q$ under the continuous inverse $(\nabla\varphi)^{-1}$. The chain rule gives

$$\partial_{y_i}\Big(q(\nabla\varphi(y))\Big) = \sum_j \varphi_{ij}(y)(\partial_{x_j} q)(\nabla\varphi(y)).$$

Since $\partial_{y_i} e^{-\varphi} = -\varphi_i e^{-\varphi}$, integration of the derivative of the compactly supported product yields

$$\begin{aligned} 0 &= \int \partial_{y_i}\Big(q(\nabla\varphi(y))e^{-\varphi(y)}\Big)\,dy \\ &= \int \left(\sum_j \varphi_{ij}(y)(\partial_{x_j} q)(\nabla\varphi(y)) - \varphi_i(y) q(\nabla\varphi(y))\right) e^{-\varphi(y)}\,dy. \end{aligned}$$

Move the last term to the other side and use $x = \nabla\varphi(y)$. The two sides become

$$\int x_i q(x)\,d\mu(x) = \int \sum_j \Big[(D_x^2\varphi^*(x))^{-1}\Big]_{ij} \partial_{x_j} q(x)\,d\mu(x).$$

The inverse-Hessian identity (2.17) identifies the matrix in this formula, proving (5.12).

The Hessian trace formula. For $F \in C_c^2(\mathbb{R}^n)$, each $q = \partial_{x_i} F$ belongs to $C_c^1(\mathbb{R}^n)$. Apply (5.12) with this choice for the ith identity, and then sum over i:

$$\begin{aligned} \int \langle x, \nabla F\rangle\,d\mu &= \sum_i \int x_i \partial_{x_i} F\,d\mu \\ &= \sum_{i,j} \int \Big[(D^2\varphi^*)^{-1}\Big]_{ij} \partial_{x_j x_i} F\,d\mu = \int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 F)\,d\mu. \end{aligned}$$

The last equality is the component formula for the trace of a matrix product. This proves (5.13).

Centering and the covariance identity. Apply (5.12) first to $q \equiv 1$ and then to $q(x) = x_j$. This gives

$$\int x_i\,d\mu = 0, \qquad \int x_i x_j\,d\mu = \int \Big[(D^2\varphi^*)^{-1}\Big]_{ij}\,d\mu.$$

The first identity says that μ is centered; the second, for all i, j, is (5.14). □

Remark 5.8. *A* Stein kernel *for a centered measure is a matrix field which satisfies precisely this identity in place of* $(D^2\varphi^*)^{-1}$. *Thus* $(D^2\varphi^*)^{-1}$ *is a symmetric positive definite Stein kernel [58, Theorem 2.3].*

5.2 Moment map operator: first order framework

Fix a log-concave measure $d\mu = e^{-V(x)}dx$ and its moment potential φ. For a vector field w and a matrix field Q, respectively, we fix the following notation:

$$\boxed{\begin{aligned} \delta_\mu w &= \sum_j (V_j w_j - \partial_{x_j} w_j), \\ (\delta_\mu Q)_i &= \sum_j (V_j Q_{ij} - \partial_{x_j} Q_{ij}). \end{aligned}} \tag{5.15}$$

This is the negative weighted divergence, applied row by row in the matrix case. Its defining integration-by-parts rule is

$$\int q\delta_\mu w \, d\mu = \int \langle \nabla q, w \rangle \, d\mu$$

when one factor is compactly supported. The notation might seem heavy, but it is useful for keeping the computations sleeker.

In this notation, Lemma 5.7 says that

$$\delta_\mu((D^2\varphi^*)^{-1}) = x.$$

We will work with the *moment map operator*

$$\boxed{L_\mu^m u = \operatorname{Tr}((D^2\varphi^*)^{-1} D^2 u) - \langle x, \nabla u \rangle, \qquad L_\mu^m x_i = -x_i.} \tag{5.16}$$

For this operator, the Stein identity (5.12) gives

$$\int v L_\mu^m u \, d\mu = -\int \langle (D^2\varphi^*)^{-1} \nabla u, \nabla v \rangle \, d\mu. \tag{5.17}$$

A physical interpretation. If φ is interpreted as an energy, in units where thermal energy equals one, $e^{-\varphi}dy$ is its Gibbs equilibrium [148, Sections 1.3 and 2.1]. The force is $-\nabla\varphi$, so the moment measure is the law of the opposite force at an equilibrium position. The Hessian $D_y^2\varphi(y)$ describes local stiffness. Legendre duality exchanges position y with its conjugate slope $x = \nabla\varphi(y)$; the inverse Hessian $D^2\varphi^*$ describes the response of position to a change in slope.

Next, we state and prove the moment-coordinate version of the transport-Hessian identity in Kolesnikov–Livshyts–Rotem [89, Proposition 4.8]. It is an integrated simplification of Bochner's identity for this operator.

Proposition 5.9 (Integrated moment Bochner formula)**.** *For $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned} \int (L_\mu^m u)^2 \, d\mu = &\int \|(D^2\varphi^*)^{-1/2} D^2 u (D^2\varphi^*)^{-1/2}\|_{\mathrm{HS}}^2 \, d\mu \\ &+ \int \langle (D^2\varphi^*)^{-1} \nabla u, \nabla u \rangle \, d\mu. \end{aligned} \tag{5.18}$$

Proof. For this coordinate calculation only, write $a_{ij}(x) = [(D_x^2\varphi^*(x))^{-1}]_{ij}$ and $u_a = \partial_{x_a} u$. All derivatives in this proof are ordinary x-derivatives, and repeated indices are summed. Differentiating the inverse matrix gives

$$\partial_{x_c} a_{ij} = -\sum_{r,s} a_{ir} (\partial_{x_r x_s x_c} \varphi^*) a_{sj}. \tag{5.19}$$

In particular, differentiating (5.16) gives

$$\partial_{x_a} L_\mu^m u = L_\mu^m(u_a) - u_a + (\partial_{x_a} a_{ij}) u_{ij}.$$

Use (5.17) first with $v = L_\mu^m u$:

$$\int (L_\mu^m u)^2 \, d\mu = -\int a_{ab} u_b \partial_{x_a} L_\mu^m u \, d\mu.$$

Substitute the preceding derivative, then apply the same integration by parts to $L_\mu^m(u_a)$, with test function $a_{ab}u_b$. Expanding its derivative yields

$$\begin{aligned}\int (L_\mu^m u)^2 \, d\mu = &\int a_{ij} a_{ab} u_{bi} u_{aj} \, d\mu + \int a_{ab} u_a u_b \, d\mu \\ &+ \int a_{ij} (\partial_{x_i} a_{ab}) u_b u_{aj} \, d\mu - \int a_{ab} (\partial_{x_a} a_{ij}) u_b u_{ij} \, d\mu.\end{aligned}$$

The two terms on the second line cancel. Indeed, (5.19) gives

$$\sum_i a_{ij} \partial_{x_i} a_{ab} = -\sum_{i,r,s} a_{ij} a_{ar} a_{bs} \partial_{x_r x_s x_i} \varphi^*,$$

which is symmetric in a, b, j. Relabeling the summed indices therefore makes the two integrals equal. Finally,

$$\begin{aligned}\int a_{ij} a_{ab} u_{bi} u_{aj} \, d\mu &= \int \operatorname{Tr}\Big((D^2\varphi^*)^{-1} D^2 u (D^2\varphi^*)^{-1} D^2 u \Big) \, d\mu \\ &= \int \| (D^2\varphi^*)^{-1/2} D^2 u (D^2\varphi^*)^{-1/2} \|_{\mathrm{HS}}^2 \, d\mu,\end{aligned}$$

and the other term is $\int \langle (D^2\varphi^*)^{-1} \nabla u, \nabla u \rangle \, d\mu$. □

Next, we discuss the relevant analogue of the Hörmander principle:

Lemma 5.10 (Moment Hörmander duality)**.** *Suppose a norm or seminorm on vector fields satisfies*

$$\int (L_\mu^m u)^2 \, d\mu \geq \|\nabla u\|^2.$$

Define its dual by

$$\|G\|_* = \sup_{\|F\| \leq 1} \left| \int \langle G, F \rangle \, d\mu \right|. \tag{5.20}$$

For smooth $f \in L^2(\mu)$ for which these pairings are defined,

$$\operatorname{Var}_\mu f \leq \|(D^2\varphi^*)^{-1} \nabla f\|_*^2. \tag{5.21}$$

Proof. Proposition 2.1 (approximate solvability and dense range) applies to the moment operator with energy $\int \langle (D^2\varphi^*)^{-1}\nabla u, \nabla u\rangle \, d\mu$: intuitively speaking, given f, we can approximately solve the PDE $f - \int f \, d\mu = L_\mu^m u$. Writing $f_0 = f - \int f \, d\mu$, the identity

$$f_0^2 - (f_0 - L_\mu^m u)^2 = 2 f_0 L_\mu^m u - (L_\mu^m u)^2$$

shows how the squared error disappears after integration and approximation. More precisely, we have the variational identity

$$\operatorname{Var}_\mu f = \sup_u \left\{ 2 \int f L_\mu^m u \, d\mu - \int (L_\mu^m u)^2 \, d\mu \right\},$$

where u ranges over the operator core. For each such u, use (5.17) (integration by parts), with $v = f$, and then the assumed energy estimate:

$$\begin{aligned} 2 \int f L_\mu^m u \, d\mu - \int (L_\mu^m u)^2 \, d\mu &\le -2 \int \langle (D^2\varphi^*)^{-1}\nabla f, \nabla u\rangle \, d\mu - \|\nabla u\|^2 \\ &\le 2 \|(D^2\varphi^*)^{-1}\nabla f\|_* \|\nabla u\| - \|\nabla u\|^2 \\ &\le \|(D^2\varphi^*)^{-1}\nabla f\|_*^2. \end{aligned}$$

The second inequality is the definition of the dual norm with $F = -\nabla u$; the last is $2ab - b^2 \le a^2$. Take the supremum over u. For a seminorm, a finite dual value forces the pairing to vanish on its zero-energy fields, so the same proof applies. □

In the same way as we derived the Brascamp-Lieb inequality (corollary 3.3) from Hörmander's lemma 3.2, we deduce the following moment Brascamp-Lieb inequality from the preceding Lemma:

Corollary 5.11 (Moment Brascamp–Lieb inequality [83, Section 6])**.** *For smooth $f \in L^2(\mu)$ with finite right side,*

$$\operatorname{Var}_\mu f \le \int \langle (D^2\varphi^*)^{-1}\nabla f, \nabla f\rangle \, d\mu. \tag{5.22}$$

Proof. Discarding the Hessian square in (5.18) (integrated Bochner) gives

$$\int (L_\mu^m u)^2 \, d\mu \ge \int \langle (D^2\varphi^*)^{-1}\nabla u, \nabla u\rangle \, d\mu.$$

Apply Lemma 5.10 with the weighted vector norm and its dual

$$\begin{aligned} \|F\|^2 &= \int \langle (D^2\varphi^*)^{-1} F, F\rangle \, d\mu, \\ \|G\|_*^2 &= \int \langle D^2\varphi^* G, G\rangle \, d\mu. \end{aligned} \tag{5.23}$$

Substitution of $G = (D^2\varphi^*)^{-1}\nabla f$ gives

$$\begin{aligned}\|(D^2\varphi^*)^{-1}\nabla f\|_*^2 &= \int \langle D^2\varphi^*(D^2\varphi^*)^{-1}\nabla f, (D^2\varphi^*)^{-1}\nabla f\rangle \, d\mu \\ &= \int \langle (D^2\varphi^*)^{-1}\nabla f, \nabla f\rangle \, d\mu,\end{aligned}$$

as required. □

Remark 5.12 (Brascamp–Lieb as Poincaré). *The weight here is $(D^2\varphi^*)^{-1}$, rather than $(D^2V)^{-1}$. In the metric $D^2\varphi^*$ on the target, (5.22) is a Poincaré inequality with constant one; all linear functions are equality cases, by (5.14).*

In fact, Corollary 5.11 follows by applying the classical Brascamp–Lieb inequality to $f \circ \nabla\varphi$ under $d\nu = e^{-\varphi}dy$, without the need to analyze the operator L_μ^m separately. Indeed,

$$\nabla_y(f \circ \nabla\varphi) = D_y^2\varphi \, \nabla_x f(\nabla\varphi),$$

so that inequality becomes

$$\begin{aligned}\operatorname{Var}_\mu f &= \operatorname{Var}_\nu(f \circ \nabla\varphi) \\ &\le \int \langle (D_y^2\varphi)^{-1}\nabla_y(f \circ \nabla\varphi), \nabla_y(f \circ \nabla\varphi)\rangle \, d\nu \\ &= \int \langle (D^2\varphi^*)^{-1}\nabla f, \nabla f\rangle \, d\mu.\end{aligned}$$

This is the viewpoint developed by Klartag in [83, Section 6].

Next, we outline the moment analogue of the Klartag H^{-1}-inequality, in the same way as was discussed in Corollary 3.4. We define the following natural moment Sobolev seminorms:

$$\begin{aligned}\|F\|_{\dot H^1_\varphi(\mu)}^2 &= \int \|(D^2\varphi^*)^{-1/2} DF (D^2\varphi^*)^{-1/2}\|_{\mathrm{HS}}^2 \, d\mu, \\ \|G\|_{\dot H^{-1}_\varphi(\mu)} &= \sup_{\|F\|_{\dot H^1_\varphi(\mu)} \le 1} \left| \int \langle G, F\rangle \, d\mu \right|.\end{aligned} \tag{5.24}$$

Here $(DF)_{ij} = \partial_{x_j} F_i$; the test space includes constant fields and the corresponding energy-domain limits.

Corollary 5.13 (Moment H^{-1} inequality)**.** *For smooth $f \in L^2(\mu)$ with the indicated pairings defined,*

$$\operatorname{Var}_\mu f \le \|(D^2\varphi^*)^{-1}\nabla f\|_{\dot H^{-1}_\varphi(\mu)}^2. \tag{5.25}$$

A necessary condition for a finite right side is

$$\int (D^2\varphi^*)^{-1}\nabla f \, d\mu = \int x f \, d\mu = 0. \tag{5.26}$$

Proof. This time keep the Hessian term in (5.18) (integrated Bochner):

$$\int (L_\mu^m u)^2 \, d\mu \geq \int \|(D^2\varphi^*)^{-1/2} D^2 u (D^2\varphi^*)^{-1/2}\|_{\mathrm{HS}}^2 \, d\mu = \|\nabla u\|_{\dot{H}^1_\varphi(\mu)}^2.$$

Lemma 5.10 (moment duality), with the first seminorm in (5.24), gives the asserted inequality.

For the compatibility condition, every constant vector field $F \equiv a$ has zero seminorm. If $\int \langle (D^2\varphi^*)^{-1}\nabla f, a\rangle \, d\mu \neq 0$, its arbitrary scalar multiples make the dual supremum infinite. Consequently finiteness requires $\int (D^2\varphi^*)^{-1}\nabla f \, d\mu = 0$. Apply (5.12) (the Stein identity) with scalar test $F = f$, one coordinate at a time, to obtain $\int (D^2\varphi^*)^{-1}\nabla f \, d\mu = \int x f \, d\mu$. □

5.3 The iterated operator and its bookkeeping identities

In order to obtain sharper inequalities in the even setting we need to "factor out" the eigenspace associated with the first positive eigenvalue of $-L_\mu^m$. Like for other operators associated with Stein kernels, linear functions are eigenfunctions of L_μ^m with eigenvalue -1. Therefore, this setup is especially compatible with studying questions related to convexity, as linear functions are also in the kernel of the Euclidean Hessian operator. We define *the iterated moment operator* together with its kernel:

$$\boxed{\begin{array}{c} P = L_\mu^m (L_\mu^m + I), \\ \mathrm{Aff} = \{x \mapsto a + \langle b, x\rangle : a \in \mathbb{R},\ b \in \mathbb{R}^n\}. \end{array}} \tag{5.27}$$

Since $L_\mu^m 1 = 0$ and $L_\mu^m x_i = -x_i$, the operator P annihilates Aff, and since P is self-adjoint, the range of P is orthogonal to its kernel Aff. Recall that our goal is to estimate

$$\inf_{a\in\mathbb{R},\, b\in\mathbb{R}^n} \int (f - a - \langle b, x\rangle)^2 \, d\mu.$$

Given f, for the optimal a, b in this infimum, the function $\tilde{f} = f - a - \langle b, x\rangle$ is orthogonal to all affine functions. Therefore, the relevant PDE existence results guarantee that for each admissible f there exists a function u such that $Pu \approx \tilde{f}$, and therefore, as before,

$$\inf_{a\in\mathbb{R},\, b\in\mathbb{R}^n} \int (f - a - \langle b, x\rangle)^2 \, d\mu \approx 2\int f P u d\mu - \int (Pu)^2 d\mu. \tag{5.28}$$

Similarly to how the L_2 proof of the Brascamp-Lieb goes (as discussed in the prequel in Section 3, especially Lemma 3.2 and Corollary 3.3), we will now deduce integration-by-parts and Bochner formulas for P, and prove the inequality

in Theorem 5.1 by estimating (5.28) from above, with the help of somewhat involving matrix-algebraic inequalities.

For a compactly supported smooth u, fix the notation

$$\boxed{\begin{aligned} B &= (D^2\varphi^*)^{-1} D^2 u (D^2\varphi^*)^{-1}, \\ w &= \delta_\mu B, \qquad h = \delta_\mu w. \end{aligned}} \tag{5.29}$$

We will be using this notation throughout the rest of this section, and I see no way around this.

Lemma 5.14 (Approximate solvability of $Pu = f$)**.** *Suppose that μ has a smooth positive density and finite second moment, that $(D^2\varphi^*)^{-1}$ is smooth and bounded from above, and that (5.17) and (5.18) hold for compact tests. For every $f \in L^2(\mu)$ satisfying*

$$\int f \, d\mu = 0, \qquad \int x f \, d\mu = 0,$$

and every $\varepsilon > 0$, there exists $u \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|f - Pu\|_{L^2(\mu)} < \varepsilon. \tag{5.30}$$

For an even moment pair and even f, u may be chosen even; then the condition on f is simply $\int f \, d\mu = 0$.

Proof. Since $L_\mu^m 1 = 0$ and $L_\mu^m x_i = -x_i$, $P\ell = 0$ for every $\ell \in \mathrm{Aff}$. Formal symmetry and compact support give

$$\int \ell P u \, d\mu = \int u P \ell \, d\mu = 0.$$

Thus *the image of P is orthogonal to affine functions.* Apply Proposition 2.2 (approximate solvability modulo affine functions) with $L = L_\mu^m$ and $M = (D^2\varphi^*)^{-1}$. Its two input identities are exactly (5.17) (moment integration by parts) and (5.18) (integrated moment Bochner). This gives the stated approximation, including its even version. □

For the calculations that follow, it is useful to record the source derivatives and their adjoints:

$$\boxed{\begin{aligned} \partial_k &= \partial_{y_k}, \qquad \partial_k^* = \varphi_k(y) - \partial_k, \\ \varphi^{ij}(y) &= \left[(D_y^2 \varphi(y))^{-1} \right]_{ij}. \end{aligned}} \tag{5.31}$$

Here φ_{ijk} is an ordinary source derivative, and $u_{ijk} = \partial_{x_i x_j x_k} u$ is an ordinary target derivative. To keep the changes of variables explicit, set

$$\widetilde{w}(y) = w(\nabla\varphi(y)), \qquad \widetilde{B}(y) = B(\nabla\varphi(y)),$$
$$U(y) = (D_x^2 u)(\nabla\varphi(y)).$$

Thus $U_{ij} = u_{ij} \circ \nabla\varphi$ and $\widetilde{B} = (D_y^2\varphi) U (D_y^2\varphi)$. The chain rule gives $\partial_k U_{ij} = \sum_l \varphi_{kl}(u_{ijl} \circ \nabla\varphi)$. Source integration by parts reads

$$\int a\, \partial_k^* b \, d\nu = \int (\partial_k a) b \, d\nu, \qquad d\nu = e^{-\varphi(y)} dy. \tag{5.32}$$

We write these integrals against ν and use $(\nabla\varphi)_{\#}\nu = \mu$ when returning to target coordinates. The product rule gives

$$[\partial_l, \partial_k^*] = \varphi_{kl}. \tag{5.33}$$

Lemma 5.15 (Integration by parts and Bochner for L_μ^m and P)**.** *With the notation of* (2.17), (5.15), (5.27), (5.29), *and* (5.31), *the following identities hold for a smooth moment pair and $u \in C_c^\infty(\mathbb{R}^n)$:*

$$\int v L_\mu^m u \, d\mu = -\int \langle (D^2\varphi^*)^{-1} \nabla u, \nabla v \rangle \, d\mu, \tag{5.34}$$

$$w = -(D^2\varphi^*)^{-1} \nabla (L_\mu^m + I) u, \qquad h = Pu, \tag{5.35}$$

$$\begin{aligned} \int q P u \, d\mu &= \int \langle w, \nabla q \rangle \, d\mu = \int \operatorname{Tr}(B D^2 q) \, d\mu \\ &= \int \operatorname{Tr}\Big((D^2\varphi^*)^{-1} D^2 u (D^2\varphi^*)^{-1} D^2 q \Big) \, d\mu. \end{aligned} \tag{5.36}$$

The integrated Bochner formula for L_μ^m is (5.18) *in Proposition 5.9. For P, one has*

$$\begin{aligned} \int (Pu)^2 \, d\mu &= \int \langle D^2 V w, w \rangle \, d\mu \\ &\quad + \int \left\| (D^2\varphi^*)^{1/2} D_x w (D^2\varphi^*)^{-1/2} \right\|_{\mathrm{HS}}^2 \, d\mu. \end{aligned} \tag{5.37}$$

All derivatives in these formulas are in x. The matrix $(D^2\varphi^*)^{1/2} D_x w (D^2\varphi^*)^{-1/2}$ is symmetric.

Proof. The first identity is (5.17). To compute w, use $\delta_\mu((D^2\varphi^*)^{-1}) = x$ and the source chain rule

$$\partial_k (q \circ \nabla\varphi) = \sum_j \varphi_{kj} ((\partial_{x_j} q) \circ \nabla\varphi).$$

After composing the definition $w = \delta_\mu B$ with $\nabla\varphi$, the inverse-Hessian identity gives, with repeated indices summed,

$$\begin{aligned}\widetilde{w}_a &= \varphi_k\varphi_{ai}U_{ik} - \partial_k(\varphi_{ai}U_{ik}) = \partial_k^*(\varphi_{ai}U_{ik})\\ &= \varphi_k\varphi_{ai}U_{ik} - \varphi_{aik}U_{ik} - \varphi_{ai}\varphi_{jk}(u_{ikj}\circ\nabla\varphi).\end{aligned} \tag{5.38}$$

On differentiating

$$((L_\mu^m + I)u)\circ\nabla\varphi = \varphi_{ij}U_{ij} - \varphi_i(u_i\circ\nabla\varphi) + u\circ\nabla\varphi$$

in y_a, the terms containing first derivatives of u cancel. The remaining terms are exactly $-\widetilde{w}_a$. Hence

$$\begin{aligned}\widetilde{w} &= -\nabla_y\Big(((L_\mu^m + I)u)\circ\nabla\varphi\Big),\\ w &= -(D^2\varphi^*)^{-1}\nabla_x(L_\mu^m + I)u.\end{aligned}$$

The first identity proves that $D_y\widetilde{w}$ is symmetric. Applying δ_μ to the second and using $\delta_\mu((D^2\varphi^*)^{-1}\nabla v) = -L_\mu^m v$ gives $h = Pu$.

The weighted adjoint identities, applied first to w, q and then to $B, \nabla q$, give

$$\int q\delta_\mu w\,d\mu = \int w_i q_i\,d\mu, \qquad \int(\delta_\mu B)_i q_i\,d\mu = \int B_{ij}q_{ij}\,d\mu.$$

Substituting $h = Pu$ and the definition of B proves (5.36).

For the last formula, every compact smooth vector field satisfies

$$\int(\delta_\mu w)^2\,d\mu = \int\langle D^2V w, w\rangle\,d\mu + \int \mathrm{Tr}((D_x w)^2)\,d\mu. \tag{5.39}$$

Indeed, weighted integration by parts gives

$$\begin{aligned}\int(\delta_\mu w)^2\,d\mu &= \int w_i\partial_{x_i}(V_j w_j - \partial_{x_j}w_j)\,d\mu\\ &= \int V_{ij}w_i w_j\,d\mu + \int w_i(V_j - \partial_{x_j})\partial_{x_i}w_j\,d\mu\\ &= \int V_{ij}w_i w_j\,d\mu + \int(\partial_{x_j}w_i)(\partial_{x_i}w_j)\,d\mu.\end{aligned}$$

For a general vector field, the last term need not be nonnegative. For our field,

$$(D_x w)\circ\nabla\varphi = D_y\widetilde{w}\,(D_y^2\varphi)^{-1}.$$

Since $D_y\widetilde{w}$ is symmetric, so is $(D^2\varphi^*)^{1/2}D_x w(D^2\varphi^*)^{-1/2}$. Cyclicity of trace therefore gives

$$\mathrm{Tr}((D_x w)^2) = \left\|(D^2\varphi^*)^{1/2}D_x w(D^2\varphi^*)^{-1/2}\right\|_{\mathrm{HS}}^2.$$

Substituting this and $\delta_\mu w = Pu$ into (5.39) proves (5.37). □

5.4 Elementary matrix inequalities

In this subsection we do some bookkeeping involving linear algebra. Later we shall combine these matrix identities and inequalities with integration by parts, in order to deduce the estimates required for the proof of Theorem 5.1 and subsequently Theorem C. AI seems to think that these identities and inequalities are really simple and he (or she) seamlessly applies them in conjunction with the material of the preceding subsection, but for human consumption it feels important to me to explicitly outline them before diving into the analytic argument in the sequel.

Lemma 5.16 (Matrix algebra)**.** *All matrices are real $n \times n$ matrices. The notation $A \geq 0$ means that A is symmetric positive semidefinite, and $A > 0$ means positive definite. For matrices and third-order arrays we use*

$$\boxed{\begin{aligned} \langle X, Y\rangle_{\mathrm{HS}} &= \mathrm{Tr}(X^T Y), \qquad \|X\|_{\mathrm{HS}}^2 = \mathrm{Tr}(X^T X), \\ \langle S, J\rangle &= \sum_{i,j,k} S_{ijk} J_{ijk}, \qquad |J|^2 = \sum_{i,j,k} J_{ijk}^2. \end{aligned}} \tag{5.40}$$

The following assertions hold.

(A1) *If matrices $C \geq 0$ and $A \leq B$ are symmetric, then*

$$\mathrm{Tr}(CA) \leq \mathrm{Tr}(CB). \tag{A1}$$

In particular, $\mathrm{Tr}(CA) \geq 0$ if $C, A \geq 0$.

(A2) *Let $A, H > 0$ and let B, C be symmetric matrices. Set*

$$\boxed{X = A^{1/2} B H^{-1/2}, \qquad Y = A^{-1/2} C H^{1/2}.} \tag{5.41}$$

Then

$$\begin{aligned} \mathrm{Tr}(BC) &= \langle X, Y\rangle_{\mathrm{HS}}, \\ \|X\|_{\mathrm{HS}}^2 &= \mathrm{Tr}(H^{-1} B A B), \\ \|Y\|_{\mathrm{HS}}^2 &= \mathrm{Tr}(H C A^{-1} C). \end{aligned} \tag{A2}$$

(A3) *With the same matrices, and $w, z \in \mathbb{R}^n$,*

$$\begin{aligned} 2\{\mathrm{Tr}(BC) + \langle w, z\rangle\} - \mathrm{Tr}(H^{-1} B A B) - \langle Aw, w\rangle \\ \leq \mathrm{Tr}(H C A^{-1} C) + \langle A^{-1} z, z\rangle. \end{aligned} \tag{A3}$$

(This is a completion of squares.)

(A4) *Let S be a fully symmetric third-order array and consider a third-order array $J_{ijk} = J_{ikj}$. Define*

$$\boxed{\begin{aligned} J^{(12)}_{ijk} &= J_{jik}, \\ (\operatorname{Sym} J)_{ijk} &= \frac{J_{ijk} + J_{jik} + J_{kij}}{3}. \end{aligned}} \tag{5.42}$$

Then

$$\begin{aligned} &2|S|^2 + 4\langle S, J\rangle + |J|^2 + \langle J, J^{(12)}\rangle \\ &\qquad = 2|S + \operatorname{Sym} J|^2 + \frac{1}{6}|J - J^{(12)}|^2 \geq 0. \end{aligned} \tag{A4}$$

(A5) *If matrices $A_0, A_1, \ldots, A_m \geq 0$ and $A = A_0 + \sum_i A_i > 0$, then, for all real numbers a_i,*

$$\Big(\sum_i a_i A_i\Big) A^{-1} \Big(\sum_i a_i A_i\Big) \leq \sum_i a_i^2 A_i. \tag{A5}$$

Proof. For (A1), cyclicity of trace gives

$$\operatorname{Tr}(C(B - A)) = \operatorname{Tr}(C^{1/2}(B - A)C^{1/2}) \geq 0.$$

The matrix inside the last trace is positive semidefinite, so its trace is the sum of its nonnegative eigenvalues.

For (A2), transpose and multiply the factors in (5.41), then move the last factor cyclically inside the trace. Namely,

$$\begin{aligned} \operatorname{Tr}(X^T Y) &= \operatorname{Tr}\Big(H^{-1/2} B A^{1/2} A^{-1/2} C H^{1/2}\Big) \\ &= \operatorname{Tr}(H^{-1/2} B C H^{1/2}) = \operatorname{Tr}(BC), \\ \operatorname{Tr}(X^T X) &= \operatorname{Tr}(H^{-1/2} B A B H^{-1/2}) = \operatorname{Tr}(H^{-1} B A B). \end{aligned}$$

The formula for $\|Y\|^2_{\mathrm{HS}}$ follows in the same way.

For (A3), use the elementary identity

$$2\langle X, Y\rangle_{\mathrm{HS}} - \|X\|^2_{\mathrm{HS}} = \|Y\|^2_{\mathrm{HS}} - \|X - Y\|^2_{\mathrm{HS}} \leq \|Y\|^2_{\mathrm{HS}},$$

with the matrices in (5.41). Apply its Euclidean vector version to $A^{1/2}w$ and $A^{-1/2}z$ and add. The three identities in (A2) give exactly (A3).

For (A4), the symmetry $J_{ijk} = J_{ikj}$ gives

$$|\operatorname{Sym} J|^2 = \frac{|J|^2 + 2\langle J, J^{(12)}\rangle}{3}, \qquad \langle S, \operatorname{Sym} J\rangle = \langle S, J\rangle,$$

where the second identity also uses full symmetry of S. Moreover,

$$|J - J^{(12)}|^2 = 2(|J|^2 - \langle J, J^{(12)}\rangle).$$

Expanding the squares on the right of (A4) proves the identity.

For (A5), set $a_0 = 0$, and let all sums below run from $i = 0$ to m. For an arbitrary $e \in \mathbb{R}^n$, choose

$$z = \left(\sum_i A_i\right)^{-1} \left(\sum_i a_i A_i\right) e.$$

Since every $A_i \geq 0$,

$$\begin{aligned}
0 &\leq \sum_i \langle A_i(a_i e - z), a_i e - z\rangle \\
&= \sum_i a_i^2 \langle A_i e, e\rangle - 2\langle \left(\sum_i a_i A_i\right) e, z\rangle \\
&\qquad + \langle \left(\sum_i A_i\right) z, z\rangle \\
&= \sum_i a_i^2 \langle A_i e, e\rangle - \langle \left(\sum_i a_i A_i\right) e, z\rangle \\
&= \sum_i a_i^2 \langle A_i e, e\rangle \\
&\qquad - \langle \left(\sum_i a_i A_i\right) e, \left(\sum_i A_i\right)^{-1} \left(\sum_i a_i A_i\right) e\rangle.
\end{aligned}$$

Since e is arbitrary, this proves (A5). □

5.5 Combining integration by parts with linear algebra into a crucial estimate

We now combine the integration-by-parts identities from Subsection 5.3 and the matrix inequalities from Subsection 5.4 and explain how the moment equation and the tensor square completion cooperate. In fact, the Proposition below uses the same differentiated Monge–Ampère identity and the same third-order tensor symmetry as was used in the work of Chen and Klartag [35, Lemmas 3.1 and 3.5], but incorporates an arbitrary test function u. The twice differentiated equation is the identity in their Lemma 3.1; their Lemma 3.5 also uses the full symmetry of the third derivatives, although the square completion below additionally

involves derivatives of u. The resulting coercive estimate corresponds to [131, Lemma 6.1].

In fact, in the following section we will show that this inequality expresses convexity of entropy along the appropriate flow, which is stated in Theorem D.

Proposition 5.17 (Coercivity for the iterated moment operator)**.** *Let $\mu = Z^{-1}e^{-V}dx$ be a smooth moment measure with convex V, and let $u \in C_c^\infty(\mathbb{R}^n)$. With $B = (D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}$ and $w = \delta_\mu B$ as in (5.29), we have*

$$\begin{aligned}\int (Pu)^2\,d\mu \geq \int \langle D^2Vw, w\rangle\,d\mu \\ + \int \mathrm{Tr}\Big(D^2\varphi^* B D^2V B\Big)\,d\mu.\end{aligned} \tag{5.43}$$

More precisely, the derivative term in the compact Bochner formula satisfies

$$\int \mathrm{Tr}((D_x w)^2)\,d\mu \geq \int \mathrm{Tr}\Big(D^2\varphi^* B D^2V B\Big)\,d\mu. \tag{5.44}$$

Proof. Recall (5.39), applied to $w = \delta_\mu B$ with $\delta_\mu w = Pu$:

$$\int (Pu)^2\,d\mu = \int \langle D^2Vw, w\rangle\,d\mu + \int \mathrm{Tr}((D_x w)^2)\,d\mu.$$

Thus (5.44) implies (5.43). We expand its two sides and identify their difference as a sum of squares. In the source calculations we retain $\widetilde{w} = w\circ\nabla\varphi$, $\widetilde{B} = B\circ\nabla\varphi$, and $U = (D_x^2u)\circ\nabla\varphi$ from Subsection 5.3. All ∂_k and subscripts on φ in these calculations are ordinary y-derivatives.

The derivative term. The representation $\widetilde{w}_a = \partial_k^*((D_y^2\varphi\, U)_{ak})$ in (5.38), together with $[\partial_l, \partial_k^*] = \varphi_{kl}$, gives

$$(D_y\widetilde{w} - \widetilde{B})_{al} = \partial_k^*\partial_l((D_y^2\varphi\, U)_{ak}).$$

Define the target-coordinate remainder by

$$\boxed{\begin{aligned}R = (D^2\varphi^*)^{1/2}D_x w(D^2\varphi^*)^{-1/2} \\ - (D^2\varphi^*)^{1/2}B(D^2\varphi^*)^{1/2}.\end{aligned}} \tag{5.45}$$

Its source expression is

$$R\circ\nabla\varphi = (D_y^2\varphi)^{-1/2}(D_y\widetilde{w} - \widetilde{B})(D_y^2\varphi)^{-1/2}.$$

Both $D_y\widetilde{w}$ and $\widetilde{B}$ are symmetric. Expanding the square, the cross term before integration by parts is $2U_{al}(D_y\widetilde{w} - \widetilde{B})_{al}$, since $(D_y^2\varphi)^{-1}\widetilde{B}(D_y^2\varphi)^{-1} = U$. Apply (5.32),

$$\int a\,\partial_k^* b\,d\nu = \int (\partial_k a)b\,d\nu,$$

with $a = U_{al}$ and $b = \partial_l((D_y^2\varphi\, U)_{ak})$, and sum over a, l, k. Changing variables in the terms that contain only target fields gives

$$\int \operatorname{Tr}((D_x w)^2)\, d\mu = \int \operatorname{Tr}(D^2\varphi^* B D^2\varphi^* B)\, d\mu + 2\int (\partial_k U_{al})\partial_l((D_y^2\varphi\, U)_{ak})\, d\nu + \int \|R\|_{\mathrm{HS}}^2\, d\mu. \tag{5.46}$$

The curvature term. The moment equation (5.2) is

$$\log\det D_y^2\varphi = V(\nabla_y\varphi) + \log Z - \varphi.$$

Its first derivative and the derivative of the inverse Hessian give

$$\begin{aligned} \varphi_{ik}(V_k \circ \nabla\varphi) &= \varphi_i + \varphi^{ab}\varphi_{abi}, \\ \partial_d\varphi^{dk} - \varphi_d\varphi^{dk} &= -V_k \circ \nabla\varphi. \end{aligned}$$

Here $V_k = \partial_{x_k} V$. Differentiating the first equality in y_j and using the second to collect the divergence terms yields

$$\begin{aligned} &\Big(D_y^2\varphi\,(D_x^2 V)(\nabla\varphi)D_y^2\varphi - D_y^2\varphi\Big)_{ij} \\ &\quad = (\partial_d - \varphi_d)(\varphi^{dk}\varphi_{ijk}) - \operatorname{Tr}\Big((D_y^2\varphi)^{-1}D_y^2\varphi_i(D_y^2\varphi)^{-1}D_y^2\varphi_j\Big). \end{aligned} \tag{5.47}$$

The derivative of $\log\det D_y^2\varphi$ contributes $\varphi^{ab}\varphi_{abij}$ minus the trace in this formula; the term $(V_k \circ \nabla\varphi)\varphi_{ijk}$ is absorbed into the source divergence.

Contract (5.47) with $(UD_y^2\varphi\, U)_{ij}$ and integrate against ν. In the divergence term apply

$$\int a(\partial_d - \varphi_d)b\, d\nu = -\int (\partial_d a) b\, d\nu$$

with $a = (UD_y^2\varphi\, U)_{ij}$ and $b = \varphi^{dk}\varphi_{ijk}$. Cyclicity of trace, $\widetilde{B} = (D_y^2\varphi)U(D_y^2\varphi)$, and the pushforward relation give

$$\begin{aligned} &\int \operatorname{Tr}(D^2\varphi^* B D^2 V B)\, d\mu \\ &\quad = \int \operatorname{Tr}(D^2\varphi^* B D^2\varphi^* B)\, d\mu \\ &\qquad - \int \partial_d((UD_y^2\varphi\, U)_{ij})\varphi^{dk}\varphi_{ijk}\, d\nu \\ &\qquad - \int (UD_y^2\varphi\, U)_{ij} \operatorname{Tr}\Big((D_y^2\varphi)^{-1}D_y^2\varphi_i(D_y^2\varphi)^{-1}D_y^2\varphi_j\Big)\, d\nu. \end{aligned} \tag{5.48}$$

The first term is the same as in (5.46) and cancels on subtraction.

The remaining algebra. The remaining contractions are most transparent in coordinates where the source Hessian is the identity at the point under consideration. Fix y_0 and use the constant dual linear change

$$\boxed{y = Ty', \qquad x' = T^T x, \qquad T = (D_y^2\varphi(y_0))^{-1/2}.} \tag{5.49}$$

The transformed potential is $\varphi(Ty') - \log|\det T|$ and the transformed test is $u(T^{-T}x')$. The moment equation is preserved, and the transformed source Hessian at $T^{-1}y_0$ is I.

In the normalized coordinates, with primes suppressed, let $x_0 = \nabla\varphi(y_0)$ and put

$$\boxed{\begin{aligned} &S_{ijk} = u_{ijk}(x_0), \qquad J_{ijk} = \sum_a U_{ia}(y_0)\varphi_{ajk}(y_0), \\ &J^{(12)}_{ijk} = J_{jik}, \qquad (\mathrm{Sym}\, J)_{ijk} = \frac{J_{ijk} + J_{jik} + J_{kij}}{3}. \end{aligned}} \tag{5.50}$$

Here S is fully symmetric, and J is symmetric in its last two indices. At y_0, the chain rule gives $\partial_k U_{al} = u_{alk}(x_0)$. Expanding the other derivatives at the same point gives

$$\begin{aligned} 2(\partial_k U_{al})\partial_l((D_y^2\varphi\, U)_{ak}) &= 2|S|^2 + 2\langle S, J\rangle, \\ \partial_d((UD_y^2\varphi\, U)_{ij})\varphi_{ijd} &= 2\langle S, J\rangle + \langle J, J^{(12)}\rangle, \\ (UD_y^2\varphi\, U)_{ij}\,\mathrm{Tr}(D_y^2\varphi_i D_y^2\varphi_j) &= |J|^2. \end{aligned}$$

In the middle line, differentiating the two copies of U gives the two copies of $\langle S, J\rangle$; differentiating the middle Hessian gives $\langle J, J^{(12)}\rangle$. These are precisely the terms arising when (5.48) is subtracted from (5.46).

Apply (A4) of Lemma 5.16 to the arrays in (5.50):

$$2|S|^2 + 4\langle S, J\rangle + |J|^2 + \langle J, J^{(12)}\rangle = 2|S + \mathrm{Sym}\, J|^2 + \frac{1}{6}|J - J^{(12)}|^2 \geq 0.$$

Constructing these scalar squares at each source point by the same normalization gives the exact remainder formula

$$\begin{aligned} &\int \mathrm{Tr}((D_x w)^2)\, d\mu - \int \mathrm{Tr}(D^2\varphi^* B D^2 V B)\, d\mu \\ &\quad = \int \|R\|_{\mathrm{HS}}^2\, d\mu + \int \left(2|S + \mathrm{Sym}\, J|^2 + \frac{1}{6}|J - J^{(12)}|^2\right) d\nu \geq 0. \end{aligned} \tag{5.51}$$

The scalar remainder is independent of the orthogonal freedom in the normalized coordinates. This proves (5.44); substitution into the vector Bochner identity at the start of the proof proves (5.43). □

5.6 The variance theorem modulo affine functions

We are ready to present the second-order analogue of the Brascamp–Lieb argument in the moment setting.

Proof of Theorem 5.1. We assume that V and φ are smooth and that $(D^2\varphi^*)^{-1} \le CI$. Proposition A.1 in Appendix A removes this bound and reduces the regularity to that stated in the theorem.

Let ℓ_f be the orthogonal projection of f onto the finite-dimensional space Aff, and set $f_0 = f - \ell_f$. Lemma 5.14 (approximation modulo affine functions) says that f_0 can be approximated in $L^2(\mu)$ by Pu with $u \in C_c^\infty(\mathbb{R}^n)$. It also says that every such Pu is orthogonal to Aff. Consequently,

$$\sup_{u \in C_c^\infty(\mathbb{R}^n)} \left(2 \int f Pu \, d\mu - \int (Pu)^2 \, d\mu \right) = \inf_{a \in \mathbb{R},\, b \in \mathbb{R}^n} \int (f - a - \langle b, x \rangle)^2 \, d\mu.$$

Thus our goal is to estimate from above

$$2 \int f Pu \, d\mu - \int (Pu)^2 \, d\mu$$

for an arbitrary compact smooth u.

Recall the notation of (5.29):

$$B = (D^2\varphi^*)^{-1} D^2 u (D^2\varphi^*)^{-1}, \qquad w = \delta_\mu B, \qquad Pu = \delta_\mu w.$$

The pairing identity (5.36) (integration by parts for P) reads

$$\int qPu \, d\mu = \int \langle w, \nabla q \rangle \, d\mu = \int \operatorname{Tr}(BD^2 q) \, d\mu.$$

Apply its Hessian form with $q = g$, where g is our auxiliary function, and its gradient form with $q = f - g$. Adding these gives

$$\int f Pu \, d\mu = \int \operatorname{Tr}(BD^2 g) \, d\mu + \int \langle w, \nabla (f - g) \rangle \, d\mu. \tag{5.52}$$

Proposition 5.17 (our key estimate from the previous subsection), applied to this same u, gives

$$\int (Pu)^2 \, d\mu \ge \int \operatorname{Tr}\left(D^2\varphi^* B D^2 V B \right) d\mu + \int \langle D^2 V w, w \rangle \, d\mu.$$

Combining this bound with (5.52), we obtain

$$\begin{aligned} 2 \int f Pu \, d\mu - \int (Pu)^2 \, d\mu \le & \int \left\{ 2 \operatorname{Tr}(BD^2 g) - \operatorname{Tr}\left(D^2\varphi^* B D^2 V B \right) \right\} d\mu \\ & + \int \left\{ 2 \langle w, \nabla (f - g) \rangle - \langle D^2 V w, w \rangle \right\} d\mu. \end{aligned}$$

Both integrands have the form $2\langle X, Y\rangle - |X|^2$. To see this explicitly for the matrices, use (A2) in Lemma 5.16 (the weighted matrix pairing), with

$$A = D^2V, \qquad H = (D^2\varphi^*)^{-1}, \qquad C = D^2g$$

and with the same B as above. Thus the matrices and vectors to which we apply the pointwise square completion are

$$\boxed{\begin{aligned} X &= (D^2V)^{1/2} B (D^2\varphi^*)^{1/2}, \\ Y &= (D^2V)^{-1/2} D^2g (D^2\varphi^*)^{-1/2}, \\ \xi &= (D^2V)^{1/2} w, \qquad \zeta = (D^2V)^{-1/2} \nabla(f-g). \end{aligned}} \tag{5.53}$$

The identities in (A2), for these choices, are

$$\begin{aligned} \langle X, Y\rangle_{\mathrm{HS}} &= \mathrm{Tr}(BD^2g), \\ \|X\|_{\mathrm{HS}}^2 &= \mathrm{Tr}\left(D^2\varphi^* B D^2V B\right), \\ \|Y\|_{\mathrm{HS}}^2 &= \mathrm{Tr}\left((D^2\varphi^*)^{-1} D^2g (D^2V)^{-1} D^2g\right). \end{aligned}$$

The vector identities are

$$\begin{aligned} \langle \xi, \zeta\rangle &= \langle w, \nabla(f-g)\rangle, \qquad |\xi|^2 = \langle D^2V w, w\rangle, \\ |\zeta|^2 &= \langle (D^2V)^{-1}\nabla(f-g), \nabla(f-g)\rangle. \end{aligned}$$

Now use (A3) (pointwise completion of squares), namely

$$2\langle X, Y\rangle_{\mathrm{HS}} - \|X\|_{\mathrm{HS}}^2 = \|Y\|_{\mathrm{HS}}^2 - \|X - Y\|_{\mathrm{HS}}^2 \le \|Y\|_{\mathrm{HS}}^2,$$

and the identical formula for ξ, ζ. Integrating these pointwise inequalities proves

$$2\int f Pu \, d\mu - \int (Pu)^2 \, d\mu \le Q(f, g), \qquad u \in C_c^\infty(\mathbb{R}^n), \tag{5.54}$$

where, for later reference,

$$\boxed{\begin{aligned} Q(f,g) = &\int \mathrm{Tr}\left((D^2\varphi^*)^{-1} D^2g (D^2V)^{-1} D^2g\right) d\mu \\ &+ \int \langle (D^2V)^{-1}\nabla(f-g), \nabla(f-g)\rangle \, d\mu. \end{aligned}} \tag{5.55}$$

This proves the required estimate for every compact smooth test u.

Taking the supremum over u completes the proof in the smooth bounded-Hessian case, and the approximation argument is presented in Appendix A. $\square$

5.7 Proof of Theorem C

Proof of Theorem C. By approximation we may assume that $D^2V_0 \geq \varepsilon I$ for some $\varepsilon > 0$. Indeed, replace V_0 by $V_0 + \varepsilon|x|^2/2$ and let $\varepsilon \downarrow 0$ at the end. The unregularized probability measure has an exponential tail, since its total potential is convex and its normalizing integral is finite. The growth assumptions therefore make every term in the claimed inequality integrable. Dominated convergence passes the normalizing constants, both moments defining the variance, and the right-hand side to their limits. We henceforth work with the regularized background and suppress the extra notation.

Fix the coefficients a_i, and put

$$\boxed{V = V_0 + \sum_{i=1}^{m} V_i, \qquad F = \sum_{i=1}^{m} a_i V_i, \qquad d\mu = Z^{-1} e^{-V} dx.} \tag{5.56}$$

With these choices, all the integrals in (5.4) are well-defined and finite, thanks to Theorem 5.6 (smooth moment coordinates and Klartag's transport estimate), Lemma 5.14 (approximation modulo affine functions) and the uniform convexity of V_0.

We apply the matrix inequality (A5) in Lemma 5.16:

$$\left(\sum_{i=1}^{m} a_i A_i\right) A^{-1} \left(\sum_{i=1}^{m} a_i A_i\right) \leq \sum_{i=1}^{m} a_i^2 A_i, \qquad A = A_0 + \sum_{i=1}^{m} A_i > 0,$$

with

$$\boxed{\begin{aligned} A_0 = D^2V_0, \qquad A_i = D^2V_i \quad (1 \leq i \leq m), \\ A = D^2V, \qquad \sum_i a_i A_i = D^2F. \end{aligned}} \tag{5.57}$$

Convexity makes every A_i positive semidefinite, while $A_0 \geq \varepsilon I$ makes A invertible. Hence

$$D^2F(D^2V)^{-1}D^2F \leq \sum_i a_i^2 D^2V_i. \tag{5.58}$$

Next use (A1) (monotonicity of the trace against a positive matrix):

$$A_- \leq A_+,\ C \geq 0 \quad \Longrightarrow \quad \mathrm{Tr}(CA_-) \leq \mathrm{Tr}(CA_+).$$

Apply it with

$$C = (D^2\varphi^*)^{-1} + xx^T, \qquad A_- = D^2F(D^2V)^{-1}D^2F, \qquad A_+ = \sum_i a_i^2 D^2V_i.$$

Inserting the result into Corollary 5.3 (the variance bound for a radial derivative) yields

$$\begin{aligned}\operatorname{Var}_\mu\left(\sum_i a_i\langle x,\nabla V_i\rangle\right) &\le \sum_i a_i^2\int \operatorname{Tr}\left(((D^2\varphi^*)^{-1}+xx^T)D^2V_i\right)d\mu\\ &= \sum_i a_i^2\int\left\{\operatorname{Tr}((D^2\varphi^*)^{-1}D^2V_i)+\langle D^2V_i x,x\rangle\right\}d\mu.\end{aligned}$$

Finally, recall (5.13) in Lemma 5.7 (the Stein identity and its Hessian-trace consequence). For a fixed i, apply the Hessian formula with $F = V_i$ to obtain

$$\int \operatorname{Tr}((D^2\varphi^*)^{-1}D^2V_i)\,d\mu = \int\langle x,\nabla V_i\rangle\,d\mu.$$

For the possibly noncompact tests $\partial_{x_j}V_i$, first multiply by a cutoff. The bounded moment Hessian and the polynomial growth assumptions make the annular error tend to zero.

Substituting (5.13) into the preceding variance bound proves

$$\operatorname{Var}_\mu\left(\sum_i a_i\langle x,\nabla V_i\rangle\right) \le \sum_i a_i^2\int\left(\langle x,\nabla V_i\rangle+\langle D^2V_i x,x\rangle\right)d\mu,$$

which is Theorem C. □

6 Entropy, convex potentials, and interpolation

We now prove Theorem D. The convex function in its statement is selected by a variational problem, and the coercivity proposition becomes exactly a nonnegative entropy second variation. We then formulate the finite interpolation property equivalent to Theorem B, and construct a simpler interpolation when all potentials are unconditional.

6.1 The moment map as the optimizer

Fix a centered probability measure μ with finite first moment whose support is not contained in a hyperplane. Consider

$$\inf_{\varphi\ \text{convex}}\left\{\int\varphi^*\,d\mu-\log\int e^{-\varphi(y)}\,dy\right\}, \tag{6.1}$$

over proper lower semicontinuous convex functions for which the terms are finite. Adding a constant to φ leaves the functional unchanged; source translations

also leave it unchanged because μ is centered. We normalize $\int e^{-\varphi} = 1$. Turns out that the optimizer is precisely the *moment potential* of μ and the existence theorem of Cordero-Erausquin and Klartag [42, Proposition 12 and Theorem 2] gives an essentially continuous optimizer, unique up to source translation. In the uniformly convex smooth setting of Theorem D, Theorem 5.6 supplies smooth global coordinates. For even μ, the optimizer can be chosen even.

The optimality equation explains why the moment map appears here. For a smooth compactly supported g, make the Legendre variation $\varphi_\epsilon = (\varphi^* + \epsilon g)^*$; it is admissible for small $|\epsilon|$ in the smooth strictly convex setting. The Legendre derivative is

$$\frac{d}{d\epsilon}\varphi_\epsilon(y)\bigg|_{\epsilon=0} = -g(\nabla\varphi(y)).$$

Stationarity of (6.1) gives

$$0 = \int g\, d\mu - \int g(\nabla\varphi(y))e^{-\varphi(y)}\, dy.$$

Thus the optimizer satisfies

$$(\nabla\varphi)_\#(e^{-\varphi(y)}dy) = \mu. \tag{6.2}$$

This is the moment-map equation. Santambrogio's entropy-and-transport minimization gives the same optimizer; see [142] and (8.8) below. For fixed μ, this optimization selects the convex potential rather than leaving an additional free choice of its Hessian.

It also explains the normalization in Theorem D: Stein's identity (5.12) is precisely (1.25). Conversely, the latter normalization identifies the same potential in the smooth setting. Indeed, symmetry of the third derivatives of φ^* gives

$$\operatorname{div}\Big((D^2\varphi^*)^{-1}\Big) = -(D^2\varphi^*)^{-1}\nabla \log\det D^2\varphi^*.$$

Consequently (1.25) implies

$$\nabla\Big(V + \log\det D^2\varphi^*\Big) = D^2\varphi^* x = \nabla\Big(\langle x, \nabla\varphi^*\rangle - \varphi^*\Big).$$

After an additive normalization, changing variables $y = \nabla\varphi^*(x)$ recovers (6.2). Thus the existential statement in the introduction uses exactly the potential selected by (6.1).

6.2 Entropy second variation and the coercivity estimate

Take the optimizer just constructed and use the interpolation (1.26). Its matrix field is symmetric positive definite: explicitly,

$$\begin{aligned}&\exp\Big(t(D^2\varphi^*)^{-1}D^2u\Big)\,(D^2\varphi^*)^{-1}\\&\quad=(D^2\varphi^*)^{-1/2}\exp\Big(t(D^2\varphi^*)^{-1/2}D^2u(D^2\varphi^*)^{-1/2}\Big)\,(D^2\varphi^*)^{-1/2}.\end{aligned}$$

Since u is compactly supported, $T_t-\mathrm{Id}$ is compactly supported and converges to zero in C^1 as $t\to 0$. For sufficiently small $|t|$, its Lipschitz constant is less than one. The contraction mapping theorem applied to $x\mapsto z-(T_t(x)-x)$, for each z, gives a global inverse. The Jacobian is then positive by continuity from $D_xT_0=I$. This justifies the local changes of variables below.

The relation to source coordinates can be written without an evaluation convention. If $S_t(y)=T_t(\nabla\varphi(y))$, then

$$D_yS_t(y)=D_xT_t(\nabla\varphi(y))D_y^2\varphi(y),$$

and hence

$$\begin{aligned}\mathrm{Ent}((T_t)_{\#}\mu)-\mathrm{Ent}(\mu)&=-\int\log\det D_xT_t\,d\mu\\&=-\int\log\frac{\det D_yS_t(y)}{\det D_y^2\varphi(y)}e^{-\varphi(y)}\,dy.\end{aligned}\tag{6.3}$$

Thus $T_0=\mathrm{Id}$ corresponds to $S_0=\nabla\varphi$.

Proposition 6.1 (Exact entropy interpretation of coercivity)**.** *For a smooth pair satisfying* (6.2)*, the inequality* (1.27) *for every compactly supported smooth u is equivalent to Proposition 5.17 for the same pair.*

Proof. Retain the vector field w from (5.29). Differentiating the displayed interpolation gives

$$\begin{aligned}\dot T_0&=\delta_\mu\Big[(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}\Big]=w,\\\ddot T_0&=\delta_\mu\Big[(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}\Big].\end{aligned}\tag{6.4}$$

Lemma 5.15 gives

$$w=-(D^2\varphi^*)^{-1}\nabla(L_\mu^m+I)u,\qquad \delta_\mu w=Pu.$$

Equivalently, the first variation of the transported density is

$$\frac{d}{dt}\frac{d((T_t)_{\#}\mu)}{d\mu}\bigg|_{t=0}=Pu.$$

Differentiate (6.3) twice. The second derivative of the logarithmic determinant at the identity yields

$$\frac{d^2}{dt^2}\operatorname{Ent}((T_t)_{\#}\mu)\Big|_{t=0} = \int \operatorname{Tr}((D_x w)^2)\,d\mu - \int \operatorname{div}\ddot{T}_0\,d\mu.$$

Two integrations by parts in the acceleration term give

$$\begin{aligned}\int \operatorname{div}\ddot{T}_0\,d\mu &= \int \langle \nabla V, \ddot{T}_0\rangle\,d\mu \\ &= \int \operatorname{Tr}\Big[D^2V(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}\Big]\,d\mu.\end{aligned}$$

Finally, use the vector Bochner identity (5.39) and $\delta_\mu w = Pu$. We obtain the exact equality

$$\boxed{\begin{aligned}&\frac{d^2}{dt^2}\operatorname{Ent}((T_t)_{\#}\mu)\Big|_{t=0} \\ &\quad= \int (Pu)^2\,d\mu - \int \langle D^2V w, w\rangle\,d\mu \\ &\qquad- \int \operatorname{Tr}\Big[D^2V(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}D^2u(D^2\varphi^*)^{-1}\Big]\,d\mu.\end{aligned}} \tag{6.5}$$

By cyclicity of the trace, the last integral is exactly $\int \operatorname{Tr}(D^2\varphi^* B D^2 V B)\,d\mu$ with the already defined B in (5.29). Thus nonnegativity of the left-hand side is precisely (5.43), in both directions. □

In view of the preceeding proposition, we see that the coercivity estimate of Proposition 5.17 is precisely equivalent to the statement of Theorem D.

Proof of Theorem D. As explained in Subsection 6.1 we take φ to be the moment potential of μ which guarantees (1.25). The preceding local invertibility argument justifies the entropy identity. Proposition 5.17, inserted into (6.5), proves (1.27). □

In one dimension the full entropy convexity does hold on any interval where $T_t' > 0$ and the change of variables is admissible. The formula becomes

$$T_t(x) = \exp\left(t\frac{u''}{\varphi^{*\prime\prime}}\right)\left[x - \frac{t}{\varphi^{*\prime\prime}}\left(\frac{u''}{\varphi^{*\prime\prime}}\right)'\right],$$

and its derivative is

$$T_t'(x) = \exp\left(t\frac{u''}{\varphi^{*\prime\prime}}\right)\left[1 + t\left\{x\left(\frac{u''}{\varphi^{*\prime\prime}}\right)' - \left(\frac{1}{\varphi^{*\prime\prime}}\left(\frac{u''}{\varphi^{*\prime\prime}}\right)'\right)'\right\} - \frac{t^2}{\varphi^{*\prime\prime}}\left(\left(\frac{u''}{\varphi^{*\prime\prime}}\right)'\right)^2\right].$$

The bracket is a concave quadratic in t. On its positivity interval its negative logarithm is convex; the logarithm of the preceding exponential is affine. Formula (6.3) then proves the assertion after integration. This calculation does not require evenness.

6.3 Finite entropy interpolation and Theorem B

The finite statement needed for the Gibbs variational argument includes potential energy as well as entropy. We give its exact relation to Theorem B. This also identifies what an optimization over convex potentials can provide.

For the notation $Z(s)$ and μ_s of (1.20), the Gibbs variational identity is

$$\operatorname{Ent}(\eta) + \int \left[V_0(x) + \sum_i V_i(e^{s_i}x)\right] d\eta(x) = -\log Z(s) + D(\eta\|\mu_s). \tag{6.6}$$

Here D is relative entropy, which is nonnegative. The identity holds for probability densities whenever the terms are finite.

Proposition 6.2 (An equivalent finite interpolation property)**.** *In the smooth finite coercive even setting, Theorem B is equivalent to the following assertion. For every $s, t \in \mathbb{R}^m$ and $0 < \lambda < 1$, there exists an even log-concave probability measure η such that*

$$\operatorname{Ent}(\eta) \le (1-\lambda)\operatorname{Ent}(\mu_s) + \lambda \operatorname{Ent}(\mu_t) \tag{6.7}$$

and

$$\begin{aligned}\int \left[V_0(x) + \sum_i V_i(e^{(1-\lambda)s_i + \lambda t_i}x)\right] d\eta(x) \\ \le (1-\lambda)\int \left[V_0(x) + \sum_i V_i(e^{s_i}x)\right] d\mu_s(x) \\ + \lambda \int \left[V_0(x) + \sum_i V_i(e^{t_i}x)\right] d\mu_t(x).\end{aligned} \tag{6.8}$$

The resulting integral inequality extends to the general potentials of Theorem B by the approximation in Appendix B.

Proof. Adding (6.7) and (6.8), and applying (6.6) at the interpolated parameter, proves

$$-\log Z((1-\lambda)s+\lambda t) \le -(1-\lambda)\log Z(s) - \lambda \log Z(t).$$

This is Theorem B.

For the converse, subtract the minima of the potentials, so that $V_i(0) = 0$. Write $r = (1-\lambda)s + \lambda t$, and let H and C denote the right-hand sides of (6.7) and (6.8), respectively. These are the two scalar bounds to be met. Minimize $\mathrm{Ent}(\eta)$ over all probability densities satisfying

$$\int \left[V_0(x) + \sum_i V_i(e^{r_i} x) \right] d\eta(x) \le C.$$

For $0 < C < \infty$, the unique optimizer is

$$d\eta_\beta(x) = \frac{\exp(-\beta[V_0(x) + \sum_i V_i(e^{r_i}x)])}{\int \exp(-\beta[V_0 + \sum_i V_i(e^{r_i}\cdot)])}\, dx, \tag{6.9}$$

where $\beta > 0$ is chosen to make its energy equal to C. Such a β exists uniquely. The energy is continuous and strictly decreasing in β, since increasing β reweights by a decreasing function of the total potential. As $\beta \downarrow 0$, the normalizing integral tends to infinity while every bounded sublevel set has finite volume, so the energy tends to infinity. As $\beta \to \infty$, concentration near the minimum makes the energy tend to zero.

For any competitor with energy at most C, nonnegativity of $D(\eta \| \eta_\beta)$ gives

$$\begin{aligned} \mathrm{Ent}(\eta) &\ge -\beta C - \log \int e^{-\beta[V_0(x) + \sum_i V_i(e^{r_i}x)]}\, dx \\ &= \mathrm{Ent}(\eta_\beta). \end{aligned}$$

Thus the optimizer exists and is even and log-concave. In particular, the moment-potential existence theorem realizes it as $(\nabla\varphi)_\#(e^{-\varphi}dy)$ for an essentially continuous convex φ. Optimizing over those convex potentials therefore has the same attained value.

Now apply Theorem B to the potentials βV_i. The Gibbs inequality at each endpoint, evaluated using the original measures μ_s and μ_t, gives

$$\begin{aligned} &-\log \int e^{-\beta[V_0(x) + \sum_i V_i(e^{r_i}x)]}\, dx \\ &\quad \le -(1-\lambda)\log \int e^{-\beta[V_0(x) + \sum_i V_i(e^{s_i}x)]}\, dx \\ &\qquad - \lambda \log \int e^{-\beta[V_0(x) + \sum_i V_i(e^{t_i}x)]}\, dx \\ &\quad \le H + \beta C. \end{aligned}$$

Together with the optimizer's energy equality, this yields $\mathrm{Ent}(\eta_\beta) \le H$. Hence η_β satisfies both required bounds. The smooth coercive case has $C > 0$; limiting cases are handled at the level of the integral inequality by approximation. □

The optimization proves existence of a best candidate midpoint. Its entropy bound is supplied by Theorem B in the converse argument; existence alone does not give that bound. To obtain a finite proof from convexity along (1.26), one would need a curve joining μ_s and μ_t which also satisfies (6.8). This is the additional interpolation problem in Question 1.

6.4 A simpler interpolation in the unconditional case

Suppose now that $V_0, \dots, V_m$ are unconditional: they are invariant under changing the sign of any coordinate. On the positive orthant, each is coordinatewise nondecreasing. Indeed, an even convex function of one coordinate is nondecreasing on the positive half-line, with the other coordinates fixed. Consequently

$$z \longmapsto V_i(e^{z_1}, \dots, e^{z_n})$$

is convex: apply the coordinatewise convexity of the exponential, then monotonicity, and finally convexity of V_i.

There is a direct entropy interpolation in these coordinates. Let μ_0, μ_1 be unconditional probability densities with finite entropy and finite logarithmic second moments, and let μ_0^+, μ_1^+ be their normalized restrictions to the positive orthant. Take the quadratic optimal transport $\nabla\psi$ between $(\log)_\# \mu_0^+$ and $(\log)_\# \mu_1^+$, where logarithms and exponentials are coordinatewise. In the smooth setting with finite logarithmic second moments, Brenier's theorem supplies the convex function ψ [31]. Define on the positive orthant

$$\boxed{T_\lambda(x) = \exp\Big((1-\lambda)\log x + \lambda \nabla\psi(\log x)\Big), \qquad 0 \le \lambda \le 1,} \tag{6.10}$$

and extend the map to the other orthants by the same coordinate signs. Then $T_0 = \mathrm{Id}$ and $(T_1)_\# \mu_0 = \mu_1$. This interpolation is constructed in logarithmic coordinates; it is not an identification of (1.26) with a coordinatewise formula for general unconditional data.

Proposition 6.3 (Entropy convexity in logarithmic coordinates)**.** *Along* (6.10)*, the function* $\lambda \mapsto \mathrm{Ent}((T_\lambda)_\# \mu_0)$ *is convex. When the endpoint measures are the Gibbs measures* μ_s, μ_t *for unconditional potentials, the same interpolation satisfies* (6.8)*. It therefore proves Theorem B in the unconditional case without taking parameter derivatives.*

Proof. Put $\mu_\lambda = (T_\lambda)_\# \mu_0$ and $\nu_\lambda = (\log)_\# \mu_\lambda^+$. Then ν_λ is ordinary displacement interpolation between ν_0 and ν_1. The Jacobian of $x = \exp z$ gives

$$\mathrm{Ent}(\mu_\lambda) = \mathrm{Ent}(\nu_\lambda) - \int \sum_{j=1}^{n} z_j \, d\nu_\lambda(z) - n \log 2. \tag{6.11}$$

McCann's theorem makes the first term convex in λ; the integral of $\sum_j z_j$ is affine because the particles in z-coordinates move along straight lines. This proves the entropy assertion, and in particular (6.7).

For the Gibbs endpoints, convexity of $z \mapsto V_i(e^z)$ gives pointwise on the positive orthant

$$\begin{aligned} &V_i\Big(e^{(1-\lambda)s_i + \lambda t_i} T_\lambda(x)\Big) \\ &\quad \le (1-\lambda) V_i(e^{s_i} x) + \lambda V_i(e^{t_i} T_1(x)). \end{aligned}$$

For V_0, use the same assertion with both parameters equal to zero. Sum these inequalities and integrate against μ_s, using $(T_1)_\# \mu_s = \mu_t$. Unconditionality extends the calculation from one orthant to all of $\mathbb{R}^n$, proving (6.8). Adding the entropy and energy bounds and using (6.6) gives the functional dilation inequality. Standard smoothing and convex approximation yield the extended-valued case, as in Appendix B; this is the logarithmic-coordinate Prékopa mechanism behind the unconditional results [46, 143]. □

7 Hypercontractivity and a semigroup proof of Theorem B

The operator $P = L_\mu^m (L_\mu^m + I)$ defines an evolution which removes the part of a function orthogonal to affine functions. We describe its Hessian energy, prove hypercontractivity by the classical Gross argument for the moment diffusion followed by an L^2 comparison, and apply the evolution to the functional dilation inequality. The sharp estimate from Section 5 remains the main analytic input to that last step.

Throughout, $d\mu = Z^{-1} e^{-V} dx$ is a smooth even log-concave probability measure, and φ is its smooth even moment potential, with positive Hessian and a global gradient diffeomorphism. All derivatives in target formulas are ordinary x-derivatives; the diffusion matrix is $(D^2 \varphi^*)^{-1}$, by (2.17). Norms and pairings refer to μ. The operator domains and approximation arguments are justified in Appendices A and C.

7.1 The evolution and its affine limit

Lemma 7.1 (Semigroup, affine projection, and dissipation)**.** *The closed realization of* $P = L_\mu^m(L_\mu^m + I)$ *is nonnegative and self-adjoint, with* $\ker P = \mathrm{Aff}$*. Its spectral semigroup*

$$S_t = e^{-tP}, \qquad f_t = S_t f, \qquad \partial_t f_t = -P f_t \tag{7.1}$$

is a strongly continuous self-adjoint contraction semigroup on $L^2(\mu)$*; the derivative holds for* $t > 0$*. For* $f \in L^2(\mu)$*, its orthogonal projection onto* Aff *is*

$$\Sigma = \int x x^T \, d\mu, \qquad \ell_f(x) = \int f \, d\mu + \left\langle \Sigma^{-1} \int x f \, d\mu, x \right\rangle. \tag{7.2}$$

Then $\ell_{S_t f} = \ell_f$ *and* $S_t f \to \ell_f$ *in* L^2 *as* $t \to \infty$*. For even* f*,* $\ell_f = \int f \, d\mu$*. Moreover, for* $t > 0$*,*

$$\frac{1}{2}\frac{d}{dt}\|S_t f - \ell_f\|_2^2 = -\int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 S_t f (D^2\varphi^*)^{-1} D^2 S_t f) \, d\mu. \tag{7.3}$$

Proof. The closed Hessian pairing in (C.1) identifies the quadratic form of P with the nonnegative Hessian integral in (7.3). Its zero-energy functions have $D^2 f = 0$, hence are affine. This gives the stated realization and kernel, as justified in Appendix C. Spectral calculus then defines S_t. On a spectral value λ of $-L_\mu^m$, its multiplier is $e^{-t\lambda(\lambda-1)}$; these multipliers lie in $[0, 1]$ and tend to 1 as $t \downarrow 0$. They also show $S_t f \in \mathcal{D}(P)$ and $\partial_t S_t f = -P S_t f$ for $t > 0$.

Since μ is centered, the formula for ℓ_f gives

$$\int (f - \ell_f) \, d\mu = 0, \qquad \int x(f - \ell_f) \, d\mu = 0.$$

Here $\Sigma > 0$ because μ has a positive density; all these integrals are finite by Cauchy–Schwarz. Thus $f - \ell_f \perp \mathrm{Aff}$. For every $\ell \in \mathrm{Aff}$, self-adjointness gives

$$\int S_t f \, \ell \, d\mu = \int f S_t \ell \, d\mu = \int f \ell \, d\mu,$$

so the projection is conserved. As $t \to \infty$, the spectral multiplier tends to the indicator of $\ker P$, proving the L^2 limit by dominated convergence. For even f, xf is odd, so the linear coefficient vanishes. Finally,

$$\frac{1}{2}\frac{d}{dt}\|S_t f - \ell_f\|_2^2 = -\int (S_t f - \ell_f) P S_t f \, d\mu = -\int S_t f \, P S_t f \, d\mu;$$

the Hessian pairing (5.36), extended by closure, gives (7.3). □

Lemma 7.2 (Affine spectral gap and diffusion comparison)**.** *For $h \in \mathcal{D}(L_\mu^m)$,*

$$\|h - \ell_h\|_2^2 \le 4 \int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 h (D^2\varphi^*)^{-1} D^2 h)\, d\mu. \tag{7.4}$$

Consequently $P \ge \frac{1}{4} I$ on $\mathrm{Aff}^\perp$. With $\delta = (\sqrt{2}-1)/2$,

$$\begin{aligned} P &\ge -\delta L_\mu^m \quad \text{on } \mathrm{Aff}^\perp, \\ \|S_t f - \ell_f\|_2 &\le e^{-t/4} \|f - \ell_f\|_2, \qquad t \ge 0. \end{aligned} \tag{7.5}$$

Proof. For $u \in C_c^\infty$, retain B, w, R from (5.29) and (5.45). Convexity of V and the remainder identity (5.51) give

$$\int \|R\|_{\mathrm{HS}}^2 \, d\mu \le \int \mathrm{Tr}((D_x w)^2)\, d\mu.$$

The definition of R expresses $(D^2\varphi^*)^{1/2} B (D^2\varphi^*)^{1/2}$ as the difference of $(D^2\varphi^*)^{1/2} D_x w (D^2\varphi^*)^{-1/2}$ and R. Applying $\|U - V\|_{\mathrm{HS}}^2 \le 2\|U\|_{\mathrm{HS}}^2 + 2\|V\|_{\mathrm{HS}}^2$ and then (5.37) yields

$$\begin{aligned} \int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 u (D^2\varphi^*)^{-1} D^2 u)\, d\mu &\le 2 \int \mathrm{Tr}((D_x w)^2)\, d\mu + 2 \int \|R\|_{\mathrm{HS}}^2 \, d\mu \\ &\le 4 \int \mathrm{Tr}((D_x w)^2)\, d\mu \le 4 \|Pu\|_2^2. \end{aligned} \tag{7.6}$$

For compact smooth h, Cauchy–Schwarz in the Hessian pairing (5.36) therefore gives

$$\left| \int h P u \, d\mu \right| \le 2 \left(\int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 h (D^2\varphi^*)^{-1} D^2 h)\, d\mu \right)^{1/2} \|Pu\|_2.$$

Lemma 5.14, with the extension in Appendix A, makes the compact-test range dense in $\mathrm{Aff}^\perp$. Hence

$$\|h - \ell_h\|_2 = \sup_{\substack{u \in C_c^\infty \\ \|Pu\|_2 \le 1}} \left| \int h P u \, d\mu \right|.$$

Taking this supremum and squaring proves (7.4); closing the form extends it to $\mathcal{D}(L_\mu^m)$ and proves $P \ge \frac{1}{4} I$ on $\mathrm{Aff}^\perp$.

Moment Brascamp–Lieb (5.22) puts the nonzero spectrum of $-L_\mu^m$ in $[1, \infty)$. On $\mathrm{Aff}^\perp$, the gap therefore gives the scalar implications

$$\lambda(\lambda - 1) \ge \tfrac{1}{4} \implies \lambda \ge \frac{1 + \sqrt{2}}{2} = 1 + \delta \implies \lambda(\lambda - 1) \ge \delta \lambda.$$

The last inequality is the operator comparison in (7.5). The decay estimate follows from $e^{-t\lambda(\lambda-1)} \le e^{-t/4}$ on the same spectral subspace and $S_t f - \ell_f = S_t(f - \ell_f)$. □

7.2 The moment diffusion and Gross's argument

Put $T_s = e^{sL_\mu^m}$. The two evolutions have the spectral rules

$$-L_\mu^m v = \lambda v \quad \Longrightarrow \quad T_s v = e^{-s\lambda} v, \qquad S_t v = e^{-t\lambda(\lambda-1)} v.$$

Thus T_s fixes constants and contracts linear functions by e^{-s}, while S_t fixes both. Gross's classical argument converts a logarithmic Sobolev inequality into growth of the integrability exponent for T_s. We then place this diffusion on either side of an L^2 spectral comparison for S_t.

The moment metric $D^2\varphi^*$ has weighted Ricci curvature at least one half of the metric [86, Theorem 4.3]; see also [93, Theorem 2.4]. With completeness as in Appendix A, the Bakry–Émery criterion [6, 7] gives

$$\operatorname{Ent}_\mu(v^2) \le 4 \int \langle (D^2\varphi^*)^{-1} \nabla v, \nabla v \rangle \, d\mu. \tag{7.7}$$

Take $h = \varepsilon + T_\eta k$, where $k \ge 0$ is bounded and $\varepsilon, \eta > 0$, and set $h_s = T_s h$. For a differentiable exponent $q = q(s) > 1$, differentiation gives

$$\frac{d}{ds} \int h_s^q \, d\mu = q' \int h_s^q \log h_s \, d\mu + q \int h_s^{q-1} L_\mu^m h_s \, d\mu.$$

By (5.17), the second integral equals $-(q-1) \int h_s^{q-2} \langle (D^2\varphi^*)^{-1} \nabla h_s, \nabla h_s \rangle \, d\mu$. Differentiating $q^{-1} \log \int h_s^q \, d\mu$ therefore yields

$$\begin{aligned} \frac{d}{ds} \log \|h_s\|_q = {} & \frac{q'}{q^2 \int h_s^q \, d\mu} \operatorname{Ent}_\mu(h_s^q) \\ & - \frac{q-1}{\int h_s^q \, d\mu} \int h_s^{q-2} \langle (D^2\varphi^*)^{-1} \nabla h_s, \nabla h_s \rangle \, d\mu. \end{aligned} \tag{7.8}$$

Since $\nabla(h_s^{q/2}) = (q/2) h_s^{q/2-1} \nabla h_s$, (7.7) gives

$$\operatorname{Ent}_\mu(h_s^q) \le q^2 \int h_s^{q-2} \langle (D^2\varphi^*)^{-1} \nabla h_s, \nabla h_s \rangle \, d\mu.$$

For $q' \ge 0$, substitution in (7.8) gives

$$\frac{d}{ds} \log \|h_s\|_q \le \frac{q' - (q-1)}{\int h_s^q \, d\mu} \int h_s^{q-2} \langle (D^2\varphi^*)^{-1} \nabla h_s, \nabla h_s \rangle \, d\mu.$$

Choose $q' = q - 1$ and $q(0) = p$, so $q(s) = 1 + (p-1)e^s$. The derivative is nonpositive, and integration gives Gross's estimate [71]:

$$\|T_s h\|_{1+(p-1)e^s} \le \|h\|_p, \qquad s \ge 0, \quad p > 1. \tag{7.9}$$

Letting $\varepsilon, \eta \downarrow 0$ and then truncating gives nonnegative L^p data; $|T_s h| \le T_s |h|$ gives signed data. The closed-form chain rule and these passages are justified in Appendix C.

7.3 Hypercontractivity modulo affine functions

The following estimate measures convergence to ℓ_f and is unchanged when an affine function is added to f. Its proof combines Gross's change of exponent with the spectral comparison above.

Theorem 7.3 (Affine hypercontractivity)**.** *Let $\delta = (\sqrt{2}-1)/2$. For $1 < p \le 2 \le q < \infty$, put*

$$\tau = \delta^{-1} \log \frac{q-1}{p-1}.$$

Then, for $f \in L^2(\mu)$ and $t \ge \tau$,

$$\|S_t f - \ell_f\|_q \le e^{-(t-\tau)/4} \inf_{\ell \in \mathrm{Aff}} \|f - \ell\|_p. \tag{7.10}$$

The estimate extends by density to $f \in L^p(\mu)$.

Proof. Passage from L^p to L^2. Choose

$$a = \log \frac{1}{p-1}, \qquad b = \log(q-1), \qquad a + b = \delta\tau.$$

Thus $(p-1)e^a = 1$ and $1 + e^b = q$, so (7.9) gives

$$\|T_a g\|_2 \le \|g\|_p, \qquad \|T_b g\|_q \le \|g\|_2.$$

Since $T_a 1 = 1$, $T_a x_i = e^{-a} x_i$, and T_a is self-adjoint, (7.2) gives

$$\ell_{T_a f} = \int f \, d\mu + e^{-a} \left(\ell_f - \int f \, d\mu \right) = T_a \ell_f.$$

Set $h = T_a f - \ell_{T_a f} = T_a(f - \ell_f) \in \mathrm{Aff}^{\perp}$. For every $\ell \in \mathrm{Aff}$, subtracting the affine function $T_a \ell$ does not change this orthogonal component. Consequently,

$$\begin{aligned} \|h\|_2^2 &= \|T_a(f-\ell)\|_2^2 - \|\ell_{T_a(f-\ell)}\|_2^2 \\ &\le \|T_a(f-\ell)\|_2^2 \le \|f-\ell\|_p^2. \end{aligned}$$

The remaining evolution in L^2. All operators below are spectral functions of L_μ^m. Commutation, $a + b = \delta\tau$, and $S_t \ell_f = \ell_f$ give the exact identity

$$\begin{aligned} T_b e^{-(t-\tau)P} e^{-\tau(P + \delta L_\mu^m)} h &= e^{-tP} e^{(a+b-\delta\tau) L_\mu^m} (f - \ell_f) \\ &= S_t f - \ell_f. \end{aligned} \tag{7.11}$$

On $\mathrm{Aff}^{\perp}$, Lemma 7.2 gives $\lambda \geq 1+\delta$ and $\lambda(\lambda-1) \geq 1/4$ for the spectral values of $-L_{\mu}^{m}$. Thus

$$e^{-\tau[\lambda(\lambda-1)-\delta\lambda]} \leq 1,$$
$$e^{-(t-\tau)\lambda(\lambda-1)} \leq e^{-(t-\tau)/4}.$$

The spectral theorem applies these scalar bounds to h, giving

$$\left\| e^{-(t-\tau)P} e^{-\tau(P+\delta L_{\mu}^{m})} h \right\|_2 \leq e^{-(t-\tau)/4} \|h\|_2.$$

Passage from L^2 to L^q. Apply the estimate for T_b to (7.11). For every $\ell \in \mathrm{Aff}$,

$$\begin{aligned} \|S_t f - \ell_f\|_q &\leq \left\| e^{-(t-\tau)P} e^{-\tau(P+\delta L_{\mu}^{m})} h \right\|_2 \\ &\leq e^{-(t-\tau)/4} \|h\|_2 \leq e^{-(t-\tau)/4} \|f-\ell\|_p. \end{aligned}$$

Taking the infimum proves the assertion. Gross's diffusion estimate supplies both changes of exponent; the fourth-order spectral gap supplies the decay factor.

Finally, μ has all finite moments, so Hölder's inequality in (7.2) makes $f \mapsto \ell_f$ continuous from L^p into every finite L^q. For bounded truncations $f_j \to f$ in L^p, the estimate just proved gives

$$\|S_t(f_j - f_k)\|_q \leq e^{-(t-\tau)/4} \|f_j - f_k\|_p + \|\ell_{f_j - f_k}\|_q \longrightarrow 0.$$

This defines the extension, and passage to the limit in the bound for each fixed $\ell \in \mathrm{Aff}$ proves the same inequality for f. □

Remark 7.4 (Semigroup derivation of Theorem B). *Work first under the smooth growth and uniform-convexity assumptions used in Section 5.7. Fix a dilation parameter z and a direction a, and put $W_i(x) = V_i(e^{z_i}x)$. Use V, F, μ from (5.56), with V_i replaced by W_i, and set $f = \langle x, \nabla F\rangle - \int \langle x, \nabla F\rangle \, d\mu$. Thus f is centered and even. Integrating the evolution gives an approximate inverse. The compact testing inequality (5.54) with $g = F$, the matrix bound (5.58), and Stein's identity (5.13) imply*

$$u_t = \int_0^t S_r f \, dr, \qquad Pu_t = f - S_t f,$$
$$\begin{aligned} \|f\|_2^2 - \|S_t f\|_2^2 &= 2\int f P u_t \, d\mu - \|Pu_t\|_2^2 \\ &\leq \sum_i a_i^2 \int \Big(\langle x, \nabla W_i\rangle + \langle D^2 W_i x, x\rangle \Big) \, d\mu. \end{aligned}$$

Here spectral calculus justifies the identity for Pu_t; Lemma 5.14 allows the compact testing inequality to pass to Pu_t by L^2 approximation. Affine hypercontractivity, with $p = 2$ and $q = 1 + e^{\delta t/2}$, gives

$$\|S_t f\|_2^2 \leq \|S_t f\|_{1+e^{\delta t/2}}^2 \leq e^{-t/4} \|f\|_2^2 \longrightarrow 0.$$

Letting $t \to \infty$ and using (1.21) proves concavity of $z \mapsto \log \int e^{-V_0 - \sum_i V_i(e^{z_i} x)} dx$. Exponentiation proves Theorem B; Appendix B passes to general convex potentials.

Remark 7.5 (An evolution between two symmetric convex bodies). *Let K, L be origin-symmetric convex bodies. On $Q = [-1,1]^n$ with uniform probability measure μ, use the tensor Legendre realization of $L_\mu^m = \frac{1}{2}\sum_i (1 - x_i^2)\partial_{ii} - x \cdot \nabla$, as in (5.7). For $J = K, L$, put $F_J(0) = 0$ and*

$$F_J(x) = \frac{\|x\|_\infty}{\|x\|_J}\, x \quad (x \neq 0), \qquad A_J = 3\int_Q F_J(x) x^T \, d\mu(x) > 0.$$

Thus $F_J(Q) = J$ and $\ell_{F_J}(x) = A_J x$, with projections understood componentwise. Positive definiteness follows directly from the radial formula. Apply both semigroups componentwise and set

$$F_t = F_L + T_t S_t (F_K - F_L), \qquad K_t = \operatorname{conv} F_t(Q),$$
$$T_t S_t = e^{t L_\mu^m - t L_\mu^m (L_\mu^m + I)} = e^{-t (L_\mu^m)^2}.$$

Equivalently, $\partial_t F_t = -(L_\mu^m)^2 (F_t - F_L)$ for $t > 0$, with $F_0 = F_K$. The constant mode of $F_K - F_L$ vanishes by oddness; every other Legendre mode decays, with multiplier $e^{-t\lambda^2}$ for an eigenvalue $\lambda = \sum_i k_i(k_i+1)/2 > 0$ of $-L_\mu^m$. These multipliers dominate the polynomial growth of the normalized Legendre polynomials, giving uniform convergence $F_t \to F_L$. Continuity in $C(Q)$ at zero follows by lifting $(L_\mu^m)^2$ to $\frac{1}{4}\Delta^2$ on $(\mathbb{S}^2)^n$.[1] *Moreover,*

$$\ell_{F_t}(x) = \left(e^{-t} A_K + (1 - e^{-t}) A_L\right) x,$$
$$K_0 = K, \qquad d_H(K_t, L) \le \sup_{x \in Q} \|F_t(x) - F_L(x)\| \longrightarrow 0.$$

The positive definite affine coefficient prevents $F_t(Q)$ from lying in a hyperplane. Hence K_t is a continuous family of origin-symmetric convex bodies joining K to L.

8 Rigidity and optimality of moment coordinates

As we have already discussed, the fact that the first nonconstant eigenspace of the moment operator L_μ^m consists of linear functions makes it an excellent

[1] See W. He and Y. Zeng, *The Calabi flow with rough initial data*, Section 3, arXiv:1701.06943.

fit for understanding convexity. But there are many operators with linear eigenfunctions: every operator associated with a Stein kernel satisfies the equation $L_M x_i = -x_i$.

Why does the moment kernel work particularly well in the argument above? Firstly, it attains the largest spectral gap allowed by the Stein normalization, and secondly its weighted coordinate derivatives commute. The latter explains the cancellations in the iterated operator.

Throughout this section, $d\mu = Z^{-1}e^{-V}dx$ is a centered probability measure with smooth positive density on $\mathbb{R}^n$ and finite second moment. For a smooth symmetric positive definite Stein kernel M, use the weighted divergence from (5.15) and set

$$\begin{gathered} \delta_\mu M = x, \\ L_M u = e^V \operatorname{div}(e^{-V} M \nabla u) = \operatorname{Tr}(M D^2 u) - \langle x, \nabla u \rangle, \\ P_M = L_M(L_M + I). \end{gathered} \tag{8.1}$$

We use the closed-form realization of L_M from Section 2 and define P_M by spectral calculus. Assume that the Stein identity extends to linear tests and that linear functions belong to the operator domain. Then

$$\int M \, d\mu = \int x x^T \, d\mu = \operatorname{Cov}(\mu), \qquad L_M x_i = -x_i. \tag{8.2}$$

Thus P_M annihilates affine functions.

8.1 An optimal spectral gap, with a fixed normalization

Define

$$\lambda(M) = \inf_{\substack{f \text{ in the form domain} \\ \operatorname{Var}_\mu f > 0}} \frac{\int \langle M \nabla f, \nabla f \rangle \, d\mu}{\operatorname{Var}_\mu f}. \tag{8.3}$$

Proposition 8.1 (Spectral optimality among normalized Stein kernels)**.** *Every such kernel satisfies $\lambda(M) \le 1$. For the moment kernel, under the assumptions of Corollary 5.11,*

$$\lambda((D^2\varphi^*)^{-1}) = 1.$$

Consequently the largest possible spectral gap in this class is one, and the moment kernel attains it.

Proof. For $f(x) = \langle b, x \rangle$, $b \neq 0$, (8.2) identifies its energy with its variance, so its Rayleigh quotient is one. For the moment kernel, Corollary 5.11 gives the reverse bound $\lambda((D^2\varphi^*)^{-1}) \ge 1$. □

This is the normalization of the metric optimization studied by Cui, Tong, and Zahm [50, equation (8) and Theorem 2.1]. They allow nonnegative matrix fields under the weaker scalar constraint

$$\int \operatorname{Tr} M \, d\mu = \operatorname{Tr} \operatorname{Cov}(\mu),$$

and show that the moment kernel attains the optimal Poincaré constant one. Their Theorem 3.2 shows that a matrix field with this normalization and Poincaré constant one must be a Stein kernel. The underlying weighted Poincaré inequality is the moment form of Brascamp–Lieb; see Klartag [83, Section 6] and Fathi [58, Section 4].

Remark 8.2. *The same elementary normalization explains the Gaussian measure's spectral optimality among isotropic probability measures: linear tests bound the Euclidean spectral gap by one, and the standard Gaussian attains one. The upper bound does not require log-concavity.*

Remark 8.3 (The optimal gap does not determine the kernel). *Even the standard Gaussian admits other positive Stein kernels with gap one. In dimension two, choose a nonzero even* $a \in C_c^\infty(\mathbb{R}^2)$ *and put*

$$K(x) = e^{|x|^2/2} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix}, \qquad a_{ij} = \partial_{ij} a. \tag{8.4}$$

Its Gaussian weighted row divergences vanish by equality of mixed partial derivatives. Hence $M = I + \varepsilon K$ *is a Stein kernel, and is positive definite for any nonzero* ε *with* $|\varepsilon| \|K\|_{\mathrm{op},\infty} < 1/2$. *For the orthogonal decomposition* $f = a_0 + \langle b, x \rangle + g$ *with* $g \perp \mathrm{Aff}$, *the cross energy vanishes by the Stein identity. The Gaussian gap above the affine modes is two, so*

$$\begin{aligned} \int \langle M \nabla f, \nabla f \rangle \, d\gamma &\geq |b|^2 + 2\Big(1 - |\varepsilon| \|K\|_{\mathrm{op},\infty}\Big) \|g\|_{L^2(\gamma)}^2 \\ &\geq \operatorname{Var}_\gamma f. \end{aligned}$$

These kernels have gap one, although the Gaussian moment kernel is I.

8.2 Compatibility with Hessian coordinates

Define the weighted coordinate fields and their dual one-forms by

$$X_i = \sum_j M_{ij} \partial_{x_j}, \qquad \alpha_i = \sum_j (M^{-1})_{ij} \, dx_j. \tag{8.5}$$

Writing $[X_i, X_b]^a$ for the a-component of the commutator,

$$\Big\{\delta_\mu(MD^2uM) + M\nabla(L_M + I)u\Big\}_i = \sum_{a,b}[X_i, X_b]^a u_{ab}. \tag{8.6}$$

Indeed, the Stein equation gives, with repeated indices summed,

$$\Big\{\delta_\mu(MD^2uM)\Big\}_i = M_{ia}u_{ab}x_b - M_{bj}(\partial_j M_{ia})u_{ab} - M_{ia}M_{bj}u_{abj}.$$

Adding $M_{ic}\partial_c(L_M + I)u$ cancels the drift and third derivatives, leaving $(M_{ic}\partial_c M_{ab} - M_{bc}\partial_c M_{ia})u_{ab}$. For the moment kernel, X_i is the source derivative ∂_{y_i}, so the commutators vanish.

The following proposition records why these commuting derivatives lead to moment coordinates. Its local part is Hessian integrability and Legendre duality; see, for example, [147]. The density calculation reverses Fathi's construction [58, Theorem 2.3], and global uniqueness follows from [42, Theorem 2].

Proposition 8.4 (Hessian-coordinate rigidity)**.** *For a smooth symmetric positive definite M on an open set, the following conditions are locally equivalent:*

1. *The fields X_i commute pairwise.*
2. *The one-forms α_i are closed.*
3. *There is a strictly convex ψ with $D_x^2\psi = M^{-1}$.*
4. *There are source coordinates y and a strictly convex φ such that $x = \nabla\varphi(y)$ and $M(x) = D_y^2\varphi(y)$.*

If also $\delta_\mu M = x$, the pullback density is a constant multiple of $e^{-\varphi}$. If the coordinates extend to a global diffeomorphism $y : \mathbb{R}^n \to \mathbb{R}^n$, an additive normalization gives

$$(\nabla\varphi)_\#(e^{-\varphi(y)}dy) = \mu.$$

Within this global class, M is uniquely determined by μ.

Proof. Since $\alpha_i(X_j) = \delta_{ij}$,

$$d\alpha_i(X_j, X_k) = -\alpha_i([X_j, X_k]),$$

which proves (1)$\Leftrightarrow$(2). Locally, closedness gives $dy_i = \alpha_i$. Symmetry of M^{-1} then makes $\sum_i y_i\,dx_i$ closed, so $y = \nabla\psi$ and $D^2\psi = M^{-1} > 0$. The converse follows by equality of mixed derivatives. Inverting $y = \nabla\psi$ and using Legendre duality gives (4), with $\varphi = \psi^*$, and proves its converse.

Write the pullback of μ as $\rho(y)dy$. For compact tests in a coordinate neighborhood the Stein identity becomes

$$\int \varphi_i q\rho \, dy = \int \partial_{y_i} q\rho \, dy.$$

Thus $\partial_{y_i} \log \rho = -\varphi_i$, so $\rho = ce^{-\varphi}$. A global coordinate map allows the constant to be absorbed into φ. The resulting smooth global moment potentials are unique up to source translation by [42, Theorem 2]. Source translation changes φ^* by a linear function, so $(D_x^2\varphi^*)^{-1}$ is unchanged. This proves uniqueness of M. □

8.3 The variance estimate for a general Stein kernel

We now retain the two errors which disappear in moment coordinates. Assume in addition that $D^2V > 0$. The displayed integrals are assumed finite and the integrations by parts justified.

Proposition 8.5 (Variance estimate with Stein-kernel errors)**.** *Let ℓ_f be the orthogonal projection of $f \in C^1(\mathbb{R}^n) \cap L^2(\mu)$ onto* Aff*. Suppose that $P_M u = f - \ell_f$ has a smooth solution in $\mathcal{D}(P_M)$ for which the integrations by parts below are justified. For a smooth auxiliary function g, put*

$$B = MD^2uM, \qquad w = -M\nabla(L_M + I)u.$$

Then

$$\begin{aligned} \|f - \ell_f\|_2^2 \le & \int \mathrm{Tr}\Big(MD^2g(D^2V)^{-1}D^2g\Big)\, d\mu \\ & + \int \langle (D^2V)^{-1}\nabla(f-g), \nabla(f-g)\rangle \, d\mu \\ & + 2\int \langle w - \delta_\mu B, \nabla g\rangle \, d\mu \\ & + \left(\int \Big\{\mathrm{Tr}(M^{-1}BD^2VB) - \mathrm{Tr}((Dw)^2)\Big\}\, d\mu\right)_+ . \end{aligned} \tag{8.7}$$

Here $t_+ = \max\{t, 0\}$, taken after integration, and Dw is the Euclidean Jacobian.

Proof. Since $\delta_\mu w = P_M u$, integration by parts gives

$$\begin{aligned} \int f P_M u \, d\mu = & \int \mathrm{Tr}(BD^2g)\, d\mu + \int \langle w, \nabla(f-g)\rangle \, d\mu \\ & + \int \langle w - \delta_\mu B, \nabla g\rangle \, d\mu. \end{aligned}$$

The vector Bochner formula (5.39) gives

$$\|P_M u\|_2^2 = \int \langle D^2 V w, w\rangle \, d\mu + \int \mathrm{Tr}((Dw)^2)\, d\mu.$$

Apply the matrix and vector square completions (A3), with $A = D^2V$, $H = M$, $C = D^2 g$ and $z = \nabla(f - g)$, to the first two pairings. Subtracting the Bochner identity bounds $2\int f P_M u \, d\mu - \|P_M u\|_2^2$ by the right side of (8.7), even before replacing the last signed integral by its positive part. Since $P_M u = f - \ell_f$, the left side is $\|f - \ell_f\|_2^2$. □

Why the errors disappear for the moment kernel. First, (8.6) and the commuting source derivatives give $\delta_\mu B = w$. Second, for compact smooth tests the compensated estimate (5.44) gives

$$\int \mathrm{Tr}((Dw)^2)\, d\mu \geq \int \mathrm{Tr}(M^{-1} B D^2 V B)\, d\mu.$$

Thus both corrections in (8.7) vanish. The signed difference in the last line may be strictly negative; it is its positive part that vanishes. The moment range-density result supplies the approximate solutions modulo Aff, and we recover Theorem 5.1. The two special ingredients are therefore the exact Hessian pairing and the compensated Bochner estimate.

Remark 8.6 (A global variational characterization). *There is also an entropy-and-transport characterization of the moment source. For a probability density ρ, set*

$$\begin{aligned} \mathcal{H}(\rho) &= \int \rho \log \rho \, dy, \\ \mathcal{T}(\rho, \mu) &= \sup_{\pi \in \Pi(\rho(y)dy, \mu)} \int \langle y, x\rangle \, d\pi(y, x), \end{aligned} \tag{8.8}$$

where Π is the set of couplings. Santambrogio [142, Theorems 4.1–4.2 and Proposition 5.1] shows that, for centered μ with finite first moment not supported on a hyperplane,

$$\inf_{\rho(y)dy \in \mathcal{P}_1(\mathbb{R}^n)} \{\mathcal{H}(\rho) + \mathcal{T}(\rho, \mu)\}$$

is attained precisely by its normalized essentially continuous moment source, up to translation. Singular competitors have infinite entropy; the functionals are understood in their extended sense. This gives a global transport formulation of the existence and uniqueness theory of [42], complementing the coordinate characterization above.

9 Moment geometry and E. Milman's centroaffine geometry

We outline the connection between the moment map geometry and E. Milman's centroaffine geometry. Consider a source $d\nu = e^{-\varphi}dy$ whose potential has level sets that are dilates of the same convex body. Its moment map separates into a radial factor and a map of directions, while its Hessian metric separates into radial and angular parts. The angular metric is precisely the centroaffine metric studied by E. Milman [121]; see also [89, Remark 1.3 and Section 4.3]. Kolesnikov [87] developed a related transport framework on the sphere, with an associated Riemannian metric and diffusion operator, and compared the logarithmic Minkowski problem with the moment-measure problem.

9.1 The angular part of a moment map

Let $n \geq 2$ and let K be a smooth convex body containing the origin in its interior, with positive curvature. Write $r(y) = \|y\|_K$ and $y = R\xi$, where $R > 0$ and $\xi \in \partial K$. A direction is thus represented by its intersection with ∂K. Define

$$\begin{aligned} F_K(\xi) = \frac{N_K(\xi)}{\langle \xi, N_K(\xi)\rangle} \in \partial K^\circ, \qquad g_K = \frac{\mathrm{II}_K}{\langle \xi, N_K(\xi)\rangle}, \\ d\sigma_K(\xi) = \frac{\langle \xi, N_K(\xi)\rangle}{n|K|}\, dS(\xi). \end{aligned} \tag{9.1}$$

Here N_K is the outer unit normal, II_K is the positive second fundamental form, and σ_K is normalized cone measure. The conormal F_K records the supporting hyperplane in the normalization $\langle \xi, F_K(\xi)\rangle = 1$.

For the source potential $\varphi_p = r^p/p$, $p > 1$, direct differentiation gives

$$\nabla\varphi_p(R\xi) = R^{p-1}F_K(\xi), \qquad \frac{\nabla\varphi_p(R\xi)}{|\nabla\varphi_p(R\xi)|} = N_K(\xi). \tag{9.2}$$

Thus its angular part is the conormal map; retaining only the Euclidean unit direction gives the Gauss map. On tangent vectors, $D^2 r(\xi) = g_K$, since $r = 1$ on ∂K and $\langle \xi, \nabla r(\xi)\rangle = 1$. In the coordinates (R, ξ) this yields

$$D_y^2\varphi_p = (p-1)R^{p-2}\, dR^2 + R^p g_K. \tag{9.3}$$

For $p = 2$ this is the metric cone $dR^2 + R^2 g_K$. As $p \downarrow 1$, the radial metric degenerates and the gradient becomes constant on rays: $\nabla r(R\xi) = F_K(\xi)$.

The dual metric at $p = 2$ explains the weight in the odd inequality of Section 3.3. Set $\Phi = r^2/2$. At $\xi \in \partial K$, (9.3) gives

$$\langle (D^2\Phi(\xi))^{-1}\nabla f(\xi), \nabla f(\xi)\rangle = \langle \xi, \nabla f(\xi)\rangle^2 + |\nabla_{g_K}(f|_{\partial K})(\xi)|^2_{g_K}.$$

This radial and tangential decomposition is used in [155, Lemma 4.1] and [156, Lemma 8.1].

On the other hand, polar integration with respect to K gives

$$\frac{e^{-r(y)}}{n!|K|}\,dy = \frac{e^{-R}R^{n-1}}{\Gamma(n)}\,dR\,d\sigma_K(\xi). \tag{9.4}$$

The corresponding moment measure is therefore

$$\mu_K = (F_K)_\# \sigma_K. \tag{9.5}$$

It is supported on ∂K°.

Surface area and cone volume. Let $S_K = (N_K)_\# dS$ be the surface-area measure on the sphere and let $dV_K = \frac{1}{n} h_K\, dS_K$ be the cone-volume measure. For $\pi(x) = x/|x|$, the identities $\pi \circ F_K = N_K$ and $|F_K(\xi)| = 1/h_K(N_K(\xi))$ give

$$\pi_\# \mu_K = \frac{V_K}{|K|}, \qquad \pi_\#\Big(|x|\, d\mu_K(x)\Big) = \frac{S_K}{n|K|}. \tag{9.6}$$

Thus retaining the direction of the moment slope gives normalized cone volume, while weighting by its length recovers surface area. The first identity also holds for the moment measure of a normalized source proportional to $e^{-r^p/p}dy$, since its boundary direction ξ has law σ_K and (9.2) gives the same Gauss map.

The terminology "functional surface area" comes from the first-variation identity (5.6): the unnormalized moment measure pairs with additive variations of the Legendre transform [38, 139]. For a smooth variation K_t with $K_0 = K$ and support functions h_t, the corresponding geometric identities are

$$\frac{d}{dt}|K_t|\bigg|_{t=0} = \int_{\mathbb{S}^{n-1}} \dot{h}_0\, dS_K = n\int_{\mathbb{S}^{n-1}} \frac{d}{dt}\log h_t\bigg|_{t=0} dV_K.$$

Surface area therefore pairs with additive support variations, and cone volume with logarithmic support variations.

The Poincaré inequality also passes to the angular part, as was observed in [89, Remark 2.3]. Under $d\nu_p \propto e^{-r^p/p}dy$, the radius and direction are independent, and $\mathbb{E}_{\nu_p} R^{-p} = 1/(n-p)$ for $1 < p < n$. Apply source Brascamp–Lieb to the

degree-zero function $u(R\xi) = u(\xi)$ and use (9.3). Convex approximation and radial cutoffs justify the calculation at zero and infinity. Letting $p \downarrow 1$ gives

$$\operatorname{Var}_{\sigma_K} u \le \frac{1}{n-1} \int |\nabla_{g_K} u|_{g_K}^2 \, d\sigma_K =: \mathcal{E}_K(u,u). \tag{9.7}$$

This is the centroaffine Poincaré inequality, or local classical Brunn–Minkowski inequality [121, Theorems 5.3–5.4].

The weighted boundary operator. For $d\mu = e^{-V} dx$ with V even, C^1, and strictly convex, set

$$dV_{K,\mu}(\xi) = \frac{\langle \xi, N_K(\xi)\rangle e^{-V(\xi)}}{n\mu(K)} \, dS(\xi), \qquad b(\xi) = \langle \xi, \nabla V(\xi)\rangle.$$

Write $\Delta_{K,\mu}$ for the nonpositive weighted Laplacian of $(\partial K, g_K, V_{K,\mu})$. The operator of Armeniakos, Kolesnikov, and Ulivelli is $f \mapsto -\Delta_{K,\mu} f + bf$.

For $0 \le p \le 1$, the weighted L_p-Brunn–Minkowski inequality uses the p/n power of μ, with the logarithmic interpretation at $p = 0$. By [155, Theorem 1.1], it holds for all symmetric bodies if and only if every smooth symmetric K of positive curvature and every smooth even f on ∂K satisfy

$$\begin{aligned} &\int_{\partial K} \left(|\nabla_{g_K} f|_{g_K}^2 + bf^2 \right) dV_{K,\mu} \\ &\qquad \ge (n-p) \left\{ \int_{\partial K} f^2 \, dV_{K,\mu} - \left(\int_{\partial K} f \, dV_{K,\mu} \right)^2 \right\}. \end{aligned}$$

The normalization matters even for $f = 1$. By the divergence theorem, $V_{K,\mu}(\partial K) = 1 - (n\mu(K))^{-1} \int_K \langle x, \nabla V(x)\rangle \, d\mu(x)$. At $p = 0$, testing $f = 1$ for every K characterizes the B-property. Testing all even f gives weighted LogBM. At $p = 1$ the same criterion gives dimensional Brunn–Minkowski.

9.2 A short glossary

Define the symmetric boundary operator L_K by Green's identity

$$-\int u L_K v \, d\sigma_K = \frac{1}{n-1} \int \langle \nabla_{g_K} u, \nabla_{g_K} v \rangle_{g_K} \, d\sigma_K = \mathcal{E}_K(u,v).$$

Up to normalization, this is Milman's centroaffine Laplacian Δ_K, or the Hilbert–Brunn–Minkowski operator [121, Theorem 4.8]. We use

$$L_K = \frac{\Delta_K}{n-1}, \qquad P_K = L_K(L_K + I), \qquad \ell_a(\xi) = \langle a, F_K(\xi)\rangle. \tag{9.8}$$

The first nonconstant eigenfunctions are precisely the ℓ_a, and $L_K \ell_a = -\ell_a$ [121, Proposition 4.9 and Theorem 5.4].

Moment geometry	**Centroaffine geometry**
Map. The gradient $\nabla\varphi$.	**Angular map.** The conormal $F_K = \nabla r\|_{\partial K}$; its unit direction is the Gauss map.
Metric. Source Hessian $D^2_y\varphi$.	**Angular metric.** $g_K = \mathrm{II}_K/\langle \xi, N_K\rangle$; see (9.3).
Probability measure. $(\nabla\varphi)_\#(e^{-\varphi}dy)$ for a normalized source.	**Homogeneous passage.** Cone measure σ_K gives $\mu_K = (F_K)_\#\sigma_K$.
First variation. The unnormalized moment measure describes variation of $\int e^{-\varphi}$ in Legendre coordinates.	**First variation.** Surface-area measure S_K describes variation of $\|K\|$ in support-function coordinates [38, 139].
Operator and first gap. L^m_μ has gap one; linear functions $\langle a, x\rangle$ are its -1 modes.	**Operator and first gap.** L_K has gap one; conormal coordinates ℓ_a are its -1 modes.
Poincaré inequality. $\mathrm{Var}_\mu f \le \int \langle (D^2\varphi^*)^{-1}\nabla f, \nabla f\rangle\, d\mu.$	**Poincaré inequality.** $\mathrm{Var}_{\sigma_K} u \le \mathcal{E}_K(u,u) = \frac{1}{n-1}\int \lvert\nabla_{g_K} u\rvert^2_{g_K}\, d\sigma_K.$

10 Concluding remarks and open questions

In this section I discuss some open problems and potential directions, which neither Astra nor I know how to close off-hand, neither affirmatively nor negatively.

10.1 A cost inequality and its radial specialization

The intersection inequality also has a Blaschke–Santaló formulation for pairs of functions. Here the functions need not be even or convex. Let ρ be a measure with a nonzero even log-concave density, let $K_1, \dots, K_m$ be symmetric convex bodies, and let $p \ge 1$. Put

$$R_i(x) = \|x\|_{K_i}, \qquad K = \bigcap_i K_i, \qquad U(x) = \max_i R_i(x)^p = \|x\|_K^p, \tag{10.1}$$

and define the cost

$$c_{\mathcal{K},p}(x,y) = 2\max_i \Big(R_i(x)R_i(y)\Big)^{p/2}. \tag{10.2}$$

Theorem 10.1 (A cost inequality for a finite family of gauges)**.** *If Borel functions $A, B : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ have positive finite exponential integrals and satisfy*

$$A(x) + B(y) \geq c_{\mathcal{K},p}(x,y) \qquad (x, y \in \mathbb{R}^n),$$

then

$$\left(\int e^{-A}\, d\rho\right)\left(\int e^{-B}\, d\rho\right) \leq \left(\int e^{-U}\, d\rho\right)^2. \tag{10.3}$$

Equality holds for $A = U + a$ and $B = U - a$, $a \in \mathbb{R}$. As a universal assertion over finite families and arbitrarily large exponents p, this inequality is equivalent to Theorem A.

We omit the proof for the sake of brevity.

Corollary 10.2 (Radial cost Blaschke–Santaló inequality)**.** *For one body, write $V(x) = \|x\|_K^p$. Let A, B be Borel functions with positive finite exponential integrals. The cost becomes $c(x,y) = 2\sqrt{V(x)V(y)}$. If $A(x) + B(y) \geq 2\sqrt{V(x)V(y)}$ for every x, y, then*

$$\left(\int e^{-A}\, d\rho\right)\left(\int e^{-B}\, d\rho\right) \leq \left(\int e^{-V}\, d\rho\right)^2. \tag{10.4}$$

Question 2. Is there a better way to derive a sharper version of the Blaschke-Santaló inequality from the analytic input of Section 5?

10.2 Some unsolved conjectures

The first two questions are inspired by our work with Colesanti, Kolesnikov and Rotem [39]. Fix

$$p \geq 2, \qquad V_p(x) = \frac{1}{p}\sum_{j=1}^n |x_j|^p. \tag{10.5}$$

For a symmetric convex body K, write

$$\begin{aligned} Z_{p,K} &= \int_K e^{-V_p(x)}\, dx, \\ d\nu_{p,K}(x) &= Z_{p,K}^{-1}\mathbf{1}_K(x)e^{-V_p(x)}\, dx. \end{aligned} \tag{10.6}$$

Question 3 (Restricted Blaschke–Santaló)**.** For every $p \geq 2$, every symmetric convex body K, and every proper lower semicontinuous even convex function Φ whose displayed integrals are positive and finite, does the following inequality hold?

$$\left(\int_K e^{-\Phi(x)}\,dx\right)\left(\int_K e^{-\Phi^*(\nabla V_p(x))/(p-1)}\,dx\right)^{p-1} \leq \left(\int_K e^{-V_p(x)}\,dx\right)^p. \tag{10.7}$$

Here Φ^* is the Legendre transform on all of $\mathbb{R}^n$.

Since $V_p^*(\nabla V_p) = (p-1)V_p$, equality holds at $\Phi = V_p + a$ for any constant a. The whole-space assertion is a case of [39, Theorem A and Corollary 1.5]. For $p = 2$, even convex restrictions are allowed by the weighted functional Santaló inequality [81, Theorem 4.2]. Thus the question concerns $p > 2$.

Question 4 (Restricted strong Brascamp–Lieb)**.** For every $p \geq 2$, every symmetric convex body K, and every even smooth f with finite energy, does the following inequality hold?

$$\operatorname{Var}_{\nu_{p,K}} f \leq \frac{p-1}{p}\int \langle (D^2 V_p)^{-1}\nabla f, \nabla f\rangle\,d\nu_{p,K} \tag{10.8}$$

$$= \frac{1}{p}\int \sum_{j=1}^n |x_j|^{2-p}|\partial_j f|^2\,d\nu_{p,K}. \tag{10.9}$$

The whole-space inequality is contained in [39, Theorem C]. The Gaussian case, including an additional even convex potential, follows from the strengthened Gaussian inequality [46, 43].

For comparison, take $W = r^p/p$ for $r = \|\cdot\|_C$ in the even estimate of [155, Theorem 1.3], with $\Phi = r^2/2$ and $\alpha I \leq D^2\Phi \leq \beta I$ as in Section 3.3. For $p \geq 2$, convexity of r gives $\nabla r \otimes \nabla r \leq D^2\Phi$ and hence

$$D^2 W = r^{p-2}\Big(D^2\Phi + (p-2)\nabla r \otimes \nabla r\Big) \leq (p-1)r^{p-2}D^2\Phi.$$

Thus for $d\eta \propto \mathbf{1}_K e^{-W}dx$, approximation of I_K by even convex potentials gives

$$\operatorname{Var}_\eta f \leq \frac{p-1}{p-1+\alpha/\beta}\int \langle (D^2 W)^{-1}\nabla f, \nabla f\rangle\,d\eta \qquad (f \text{ even}).$$

For $C = B_2^n$ this gives the factor $(p-1)/p$ for $W = |x|^p/p$. Question 4 asks for that factor with $C = B_p^n$. When $n \geq 2$ and $p > 2$, $D^2(\|\cdot\|_p^2/2)(e_1)$ vanishes in the e_2 direction, so the positive lower pinching bound degenerates.

Remark 10.3. *An affirmative answer to Question 3 implies an affirmative answer to Question 4 by second variation. This would provide an alternative route to the B-conjecture. See [39] for the details.*

In fact, these are similar to the following classical quesiton:

Question 5 (Cordero-Erausquin [45])**.** For every even log-concave measure μ on $\mathbb{R}^n$ and every symmetric convex body K, does

$$\mu(K)\mu(K^\circ) \leq \mu(B_2^n)^2$$

hold? In particular, for symmetric convex bodies K, T, does

$$|K \cap T|\,|K^\circ \cap T| \leq |B_2^n \cap T|^2$$

hold?

Cordero-Erausquin formulated it in [45], proving a complex case when the restricting body is also invariant under complex conjugation. Klartag [81, Corollary 1.1 and the discussion following it] obtained the general set inequality with a factor 2^n on the right. The sharp bound for unconditional log-concave measures is discussed by Milman and Yehudayoff [117].

Question 6 (Dar's conjecture)**.** For convex bodies $K, L \subset \mathbb{R}^n$, put

$$M(K, L) = \max_{z \in \mathbb{R}^n} |K \cap (z + L)|. \tag{10.10}$$

Does the strengthened Brunn–Minkowski inequality

$$|K + L|^{1/n} \geq M(K, L)^{1/n} + \frac{|K|^{1/n}|L|^{1/n}}{M(K, L)^{1/n}} \tag{10.11}$$

hold? This is Dar's conjecture [51], proved in the plane by Xi and Leng [151]. For origin-symmetric convex bodies, $M(K, L) = |K \cap L|$: the intersection-volume function is even and log-concave, so its maximum occurs at the origin. Can the moment-map method, or the equivalent functional and intersection inequalities above, prove (10.11) in higher dimensions?

10.3 Vague questions

Question 7 (Geometric consequences on the boundary)**.** The study of LogBM has produced many advances concerning geometry on the boundary of convex sets [40, 92, 89, 149, 121]. Section 9 identifies the homogeneous moment Poincaré inequality with the centroaffine one by E. Milman. Can the compensated estimate for the iterated operator in Proposition 5.17 also be expressed directly on the boundary? What geometric inequality would correspond to Theorem C under this passage, and does its nonnegative remainder have a geometric meaning?

For Lebesgue measure the boundary form in [155, Theorem 1.1] has $b = 0$ and $V_{K,\mu} = \sigma_K$. Thus the local form of Corollary 1.2, in the normalization (9.8), gives $-L_K \geq n/(n-1)$ on centered even functions. By spectral calculus this yields $P_K = L_K(L_K + I) \geq n/(n-1)^2$ on the same subspace. Can one recover this gap directly from Proposition 5.17 and interpret the individual squares in (5.51) on ∂K?

Question 8 (A weak Caffarelli-type transport principle)**.** Let $d\mu = Z^{-1}e^{-V}dx$ and $d\mu_W = Z_W^{-1}e^{-V-W}dx$, where V, W are even and convex. Caffarelli's theorem gives an odd contracting Brenier map from a Gaussian reference to its even convex perturbations [32]. Klartag's contraction theorem for the moment map is a counterpart of this result [83]. These results provide a strong conclusion (an L_2 Lipschitz transport map) under a strong assumption (strong log-concavity or Gaussianity). Could one prove a similar theorem with a weaker assumption and a weaker conclusion which would be sufficient to deduce Theorem B, or at least the B-conjecture (now theorem)? Might there be a contraction principle which would be applicable for any even log-concave measure as a source, with a more log-concave target, and would provide the existence of a transport map which satisfies some weakly contracting properties? Maybe not in L_2, or maybe not pointwise?

Note that Theorems B and C are identities when $m = 1$ and $V_0 = 0$. Perhaps at least when $m = 1$, these results are but manifestations of a general contraction principle? Note that Theorems B and C follow if they are proved for measures $e^{-\|x\|_p^p/p}dx$. Perhaps there is an L_p version of Caffarelli's theorem which yields all these phenomena? Several natural guesses turn out to be false.

Question 9 (A flow for logarithmic addition)**.** Can one define an interpolation $\mathcal{L}_t(\varphi_0, \varphi_1)$ of proper lower semicontinuous even convex potentials which is compatible with moment measures and satisfies

$$\mathcal{L}_t(I_K, I_L) = I_{(1-t)\cdot K + 0 t\cdot L},$$

$$\int e^{-\mathcal{L}_t(\varphi_0,\varphi_1)}\, d\rho \geq \left(\int e^{-\varphi_0}\, d\rho\right)^{1-t} \left(\int e^{-\varphi_1}\, d\rho\right)^{t}$$

for even log-concave backgrounds ρ? Might there be an operation for which this universal functional inequality is equivalent to (or stronger than) Theorems A, B, and C, and whose linearization explains the moment operator and the second-order estimate? Many natural guesses run into counterexamples, but this doesn't rule out this possibility.

Perhaps all these phenomena have an explanation as simple as a proof by inclusion, or by a lower bound of a mass transport Jacobian? As we discussed,

Bochner formula and integration by parts follow by differentiating some constant flows. We then apply them to study a second order version of our geometric phenomena. But why do we need to go two (in fact, four) derivatives down, and then go four derivatives back up, to prove a simple-sounding geometric fact?

The success of the moment map approach shows that when trying to invent a flow, what matters is not creating an unusual flow (the moment map is after all a Brenier map) but being flexible about the target. Can LogBM be proven by a variational approach, considering the usual Brenier map to all possible symmetric convex bodies and showing that its jacobian is smallest when the target is a dilate of the source (in the smooth case)?

Question 10 (Other operators)**.** For which other second-order operators, choices of a norm, or iterations does the L^2 method produce interesting inequalities?

Question 11 (A vague reflection on atomic moment measures)**.** In section 5 we discussed an interesting rigidity property of piece-wise affine potentials: their moment measures are purely atomic. Does this allow for some transfer of properties of point configurations to log-concave measures, and vice versa, similarly to how energy minimization questions about finite point arrangements sometimes translate to the geometry of convex polytopes with the corresponding atomic surface area measures? Beyond that, is this rigidity property a limiting case of a more general principle: relating "peakedness" of moment measures to curvature of the target potential? Klartag's contraction theorem is a manifestation of this phenomenon but its assumptions are very strong. Perhaps a more general principle of this type would allow us to study various extremality properties of exponential log-concave measures.

Question 12 (that continues to tantalize many experts)**.** Is there a good functional form of the Log-Brunn-Minkowski inequality? A version was proposed in [115, Theorem 1.3] but the construction is rather involved, due to the fact that log-addition is non-local (the same issue prevents natural attempts to construct monotonic transport which would capture log-addition, because log-addition does not respect straight lines). Is there a functional inequality which extends LogBM without having to involve the geometric log-addition? In some sense, Theorem B could be viewed in this light, and its equivalence to LogBM follows from the sublevel-set argument in the introduction. In short, maybe someone can come up with something better?

Question 13 (about dimensional liftings)**.** Is there a way to substantially simplify the proof of Log-Brunn-Minkowski? For instance, it suffices to prove Theorem B for the Gaussian measure (or any other given even log-concave measure). Might this be in the family of the Gaussian phenomena? If strong

B-property were true for non-standard Gaussian, then Theorem B would follow by viewing it as an inequality on $\mathbb{R}^{nm}$. But strong B-property is not true for non-standard Gaussian, we (as a field) crossed this bridge several years ago already [44]. Perhaps we missed something? Maybe a weaker version of the strong B-property is true for the general Gaussian, in such a form which would imply the main results of these notes on one hand, and would have a short and elegant proof on the other? Beautiful new machinery by E. Milman, Nakamura, Tsuji is available now for studying Gaussian inequalities [126, 127], but neither Astra nor I know how to deduce Theorem B from it.

A Removing the bounded-Hessian assumption and reducing regularity

The upper bound on $(D^2\varphi^*)^{-1}$ makes the Euclidean cutoff proof of Lemma 5.14 particularly short. It is not needed in Theorem 5.1. We give the additional global argument, since local convergence of coefficients alone would not justify passage of the two possibly noncompact energies in (5.3).

Proposition A.1 (The global reduction)**.** *Under the assumptions of Theorem 5.1, $\varphi \in C^{4,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, its gradient is a diffeomorphism of $\mathbb{R}^n$, and the metric $D^2_x\varphi^*$ is complete. For the self-adjoint realization $N = -L^m_\mu$, compact smooth functions form an operator core for both N and N^2. Moreover,*

$$\overline{\{Pu : u \in C^\infty_c(\mathbb{R}^n)\}}^{L^2(\mu)} = \mathrm{Aff}^\perp .$$

Consequently the compact calculation proving (5.54) gives the theorem with its stated regularity and without a bound $(D^2\varphi^*)^{-1} \le CI$.

Proof. Local regularity and global coordinates. The change-of-variables equation for the convex gradient transport is

$$\det D^2_y\varphi(y) = Z \exp\Big(V(\nabla\varphi(y)) - \varphi(y)\Big).$$

It holds almost everywhere initially. Since $\varphi \in C^2$ and V is finite and continuous, both sides are continuous, so it holds everywhere. The right side is positive. Thus $D^2_y\varphi > 0$, and the gradient is injective and locally invertible. Its image is dense, because its pushforward has a positive density on all of $\mathbb{R}^n$. For any $x_0 \in \mathbb{R}^n$, choose finitely many image points whose convex hull contains x_0 in its interior. The corresponding supporting planes make $\varphi(y) - \langle x_0, y\rangle$ coercive. Its minimum supplies a point with $\nabla\varphi = x_0$. Hence the gradient is onto.

On every compact source set the Hessian now has positive lower and finite upper bounds. The logarithmic transport equation is locally uniformly elliptic. Interior regularity first gives $C^{2,\beta}$ regularity for some $\beta > 0$; differentiating the equation and applying local Schauder estimates then gives $\varphi \in C^{4,\alpha}_{\mathrm{loc}}$ [60]. In this bootstrap, the first gain to $C^{3,\beta}$ makes the coefficients locally Lipschitz, after which one can use the prescribed exponent α. In particular,

$$M(x) = (D_x^2 \varphi^*(x))^{-1} \in C^{2,\alpha}_{\mathrm{loc}}.$$

These are the coefficient derivatives used in the compact calculations: the differentiated transport equation uses four derivatives of φ and two of V. The compact pairing uses only one derivative of f and two of g.

Approximation by bounded moment Hessians. Choose smooth convex potentials V_j such that $D^2 V_j \geq \varepsilon_j I$, where $\varepsilon_j \downarrow 0$, the probabilities $\mu_j = Z_j^{-1} e^{-V_j}\,dx$ are centered, and

$$V_j \longrightarrow V \quad \text{in } C^2_{\mathrm{loc}}, \qquad \int |x|^2\,d\mu_j \longrightarrow \int |x|^2\,d\mu.$$

One may convolve V with compactly supported smooth kernels, add $\varepsilon_j |x|^2/2$, and translate the resulting probabilities by their barycenters. The linear lower bound for an integrable convex potential gives a common exponential majorant. Dominated convergence then gives convergence of the normalizations, barycenters and moments; the translations tend to zero. Theorem 5.6, applied to these centered targets, supplies smooth normalized potentials φ_j with $0 < D_y^2 \varphi_j \leq C_j I$ and global gradient coordinates; see also [58, Corollary 2.4].

The convergence of the measures and their first moments gives $W_1(\mu_j, \mu) \to 0$. The moment-source stability theorem of Bonnet–Rubinstein [18, Theorem 1.3] supplies source translations such that

$$\|e^{-\varphi_j} - e^{-\varphi}\|_{L^1(dy)} \longrightarrow 0.$$

Convexity and positivity of the limiting density imply local uniform convergence $\varphi_j \to \varphi$ on $\mathbb{R}^n$. Their gradients also converge locally uniformly, since φ is C^1.

Here is the additional regularity needed to pass from potentials to metrics. Fix a compact source set and cover it by smaller sections contained in compact sections of the strictly convex function φ. Local uniform convergence gives uniformly controlled sections for φ_j. On these sections,

$$\det D_y^2 \varphi_j = Z_j \exp\Big(V_j(\nabla \varphi_j) - \varphi_j\Big)$$

has right-hand side bounded above and bounded away from zero, uniformly in j. The interior $C^{1,\beta}$ estimate on sections first gives a uniform Hölder modulus for

$\nabla\varphi_j$. The displayed right-hand sides are consequently uniformly Hölder, and the interior $C^{2,\beta'}$ estimate gives equicontinuity of the Hessians on the smaller sections, for some $\beta' > 0$. These are the usual interior Monge–Ampère estimates, followed by the local elliptic bootstrap [61, Chapters 3–4]. Compactness and uniqueness of the limit therefore give $D_y^2\varphi_j \to D_y^2\varphi$ locally uniformly. The inverse gradient maps converge on compact target sets as well. Consequently

$$M_j(x) := (D_x^2\varphi_j^*(x))^{-1} \longrightarrow M(x) \quad \text{locally uniformly.} \tag{A.1}$$

Source translations add linear functions to the dual potentials and therefore do not change these matrices. No uniform global Hessian bound is asserted here.

Completeness of the limiting metric. Write d_j for the distance of the metric M_j^{-1} and T_t^j for its weighted heat semigroup. Each metric is complete because $M_j \le C_j I$. Euclidean cutoffs also show that this semigroup is conservative and that the coordinate functions belong to its generator domain, with $L_{\mu_j}^m x_i = -x_i$: the cutoff errors are bounded multiples of the tails of $1 + |x|$ in $L^2(\mu_j)$. Thus $T_t^j x_i = e^{-t} x_i$.

The moment curvature calculation gives

$$\mathrm{Ric}_{M_j^{-1},\mu_j} \ge \tfrac{1}{2} M_j^{-1};$$

this is the moment specialization of [86, Theorem 4.3], also discussed in [58, Section 3.1]. Wang's dimension-free Harnack inequality [150], at time one and exponent two, gives

$$(T_1^j a(x))^2 \le T_1^j(a^2)(z)\exp(c\, d_j(x,z)^2), \qquad a \ge 0,$$

with a numerical constant c independent of j. The statement for nonnegative L^2 functions follows from bounded truncations. Apply it to $a = |x_i|$. Positivity and the coordinate eigenfunction identity give $T_1^j|x_i|(x) \ge e^{-1}|x_i|$.

Fix a Euclidean ball B about zero. Its μ_j-mass is bounded below, and (A.1) gives $\sup_{z\in B} d_j(0,z) \le D$ with D independent of j. Integrating the Harnack inequality over $z \in B$, using invariance of μ_j, and summing over i gives

$$|x|^2 \le C\exp\Big(c(d_j(x,0) + D)^2\Big). \tag{A.2}$$

The uniform second-moment bound makes C independent of j.

For any fixed piecewise smooth path from zero to x, its length in M_j^{-1} converges to its length in M^{-1} by (A.1). Bound $d_j(x,0)$ by this path length in (A.2), pass to the limit, and then take the infimum over paths. The same bound follows for the distance d of M^{-1}. Every bounded d-ball is therefore

Euclidean bounded. Local positive definiteness identifies the two topologies, so closed d-balls are compact and the metric is complete. This step uses the full target space $\mathbb{R}^n$.

Compact cores at the stated regularity. Let N be the nonnegative self-adjoint operator associated with the closure of $\int \langle M\nabla u, \nabla u\rangle \, d\mu$ on compact smooth tests. Completeness supplies compactly supported Lipschitz cutoffs η_R, equal to one on the d-ball of radius R, with

$$0 \le \eta_R \le 1, \qquad \Gamma(\eta_R) := \langle M\nabla\eta_R, \nabla\eta_R\rangle \le C/R^2.$$

They may be used in weak identities by local smooth approximation. If $z \in L^2(\mu)$ annihilates $(N+I)(C_c^\infty)$, then $L_\mu^m z = z$ distributionally. Local elliptic regularity and testing against $\eta_R^2 \overline{z}$ give

$$\int \eta_R^2 |z|^2 \, d\mu + \frac{1}{2}\int \eta_R^2 \langle M\nabla z, \nabla\overline{z}\rangle \, d\mu \le \frac{C}{R^2}\|z\|_2^2.$$

Letting $R \to \infty$ gives $z = 0$. The dense-range criterion for a nonnegative symmetric operator proves essential self-adjointness on C_c^∞, so this is an operator core for N.

The energy argument for the wave equation gives finite propagation speed for $\cos(t\sqrt{N})$ in the metric d; see the method of [36, 48]. This argument uses the principal energy and local regularity, which are available for the present $C^{2,\alpha}$ coefficients. We give the remaining core step explicitly, so that no theorem requiring infinitely differentiable coefficients is being applied to N^2.

Choose an even Schwartz function χ with $\chi(0) = 1$ and compactly supported Fourier transform. Spectral calculus gives

$$\chi(\delta\sqrt{N})v \longrightarrow v \quad \text{in the graph norm of } N^2, \qquad v \in \mathcal{D}(N^2), \quad \delta \downarrow 0.$$

For fixed δ, this spectral multiplier maps L^2 continuously into $\mathcal{D}(N^k)$ for every integer k. Approximate its input by compactly supported L^2 functions. The Fourier representation through $\cos(t\sqrt{N})$ and finite propagation make the outputs compactly supported: finite metric neighborhoods of compact sets are compact. Such an output q belongs to H^4_{loc}, by applying local elliptic regularity first to Nq and then to q. Here two locally bounded derivatives of M suffice. Euclidean mollification therefore approximates q by compact smooth functions in H^4, on a common compact set, and hence in the graph norm of N^2. The differential expression for N^2 has order four and involves at most two derivatives of M. Thus C_c^∞ is a core for N^2, and also for $P = N(N-I)$, whose graph norm is equivalent to that of N^2.

The kernel and dense range. The Stein identity extends to linear tests and gives

$$\int M\,d\mu = \int xx^T\,d\mu.$$

To justify this without a bound on M, integrate $\varphi_{ii}e^{-\varphi}$ with source cutoffs. The error tends to zero because $\varphi_i \in L^2(e^{-\varphi}dy)$, by the pushforward relation and the finite second moment of μ. Nonnegativity of φ_{ii} first gives its integrability; one can then pass to the limit in the identity. Polarization gives the off-diagonal entries.

The intrinsic cutoffs above now put constants and linear functions in the closed form domain: their gradient energies are integrable, and the additional cutoff term is bounded by $CR^{-2}\int(1+|x|^2)\,d\mu$. Their weak equations extend by form density, giving $N1 = 0$ and $Nx_i = x_i$ in the operator domain. In particular the diffusion is conservative. Moment Brascamp–Lieb gives $\sigma(N) \subset \{0\} \cup [1,\infty)$.

Approximate any $q \in \mathcal{D}(N)$ in the graph norm of N. Applying integrated moment Bochner to differences of compact approximants shows that their weighted Hessians are Cauchy. Local ellipticity identifies the limit with the distributional Hessian, and therefore

$$\|Nq\|_2^2 - \langle q, Nq\rangle_{L^2(\mu)} = \int \|M^{1/2}D^2qM^{1/2}\|_{\mathrm{HS}}^2\,d\mu.$$

Since $P = N(N-I)$, this is its closed quadratic-form energy; on $\mathcal{D}(P)$ it equals $\int qPq\,d\mu$. Its zero-energy functions are precisely affine functions, so $\ker P = \mathrm{Aff}$. Self-adjointness and the compact core for P give $\overline{P(C_c^\infty)} = \mathrm{Aff}^\perp$.

Finally, all integrations by parts in (5.54) have a compactly supported factor. The established coefficient regularity, together with $f \in C^1$ and $g \in C^2$, justifies these pairings. Taking the supremum over compact u proves Theorem 5.1, without passing noncompact weighted energies through the approximating measures. □

Remark A.2 (Scope of the global operator argument). *The completeness and core arguments above use convexity of V, but not the strict inequality $D^2V > 0$. They therefore apply to any smooth centered log-concave moment pair on $\mathbb{R}^n$ with positive moment Hessian and a global moment diffeomorphism, including the fixed pairs used in Section 7. Strict positivity of D^2V is needed for the inverse matrix in Theorem 5.1. For an even pair, the operator commutes with reflection, so compact approximants may be symmetrized whenever even tests are required.*

B Approximation and passage to convex sets: a detailed deduction of Theorem A from Theorem C

The introduction sketched how Theorem C implies Theorem A. We now give the approximation argument.

Let $V_0, \dots, V_m$ be proper lower semicontinuous even convex potentials, and assume

$$0 < \int_{\mathbb{R}^n} e^{-V_0 - \sum_i V_i}\, dx < \infty.$$

Consider

$$(t_1, \dots, t_m) \longmapsto \log \int_{\mathbb{R}^n} \exp\left(-V_0(x) - \sum_i V_i(e^{t_i} x)\right) dx. \tag{B.1}$$

The smooth concavity follows from Theorem C and the differentiation formula (1.21). We pass to the limit in the multiplicative inequality and do not differentiate nonsmooth potentials.

Subtract the finite minima so that every potential is nonnegative and vanishes at zero. For each potential use the Moreau envelope and then an even smoothing kernel [128, 137]:

$$\boxed{\begin{aligned} V^{[j]}(x) &= \inf_z \left\{ V(z) + \frac{j}{2}|x - z|^2 \right\}, \\ V_j &= V^{[j]} * \rho_{j^{-2}}. \end{aligned}} \tag{B.2}$$

The kernels are smooth even probability densities supported in balls of radius j^{-2}. The approximants are smooth, even and convex, with $D^2 V_j \le jI$ and derivatives of at most polynomial growth. Convexity, the zero first moment of the kernel, and the Hessian bound for the Moreau envelope give

$$0 \le V_j - V^{[j]} \le \frac{1}{2j^3}, \qquad V^{[j]}(x) \uparrow V(x).$$

Add $\delta|x|^2/2$ to the smoothed background. For fixed $\delta > 0$, Gaussian domination passes the smooth log-concavity inequality to the limit $j \to \infty$. Monotone convergence as $\delta \downarrow 0$ removes this Gaussian. Restoring the subtracted constants does not change log-concavity.

For completeness, positivity and finiteness at one parameter imply the same at every parameter. After subtracting minima, put

$$U = V_0 + \sum_i V_i, \qquad c_- = \min\{1, e^{t_1}, \dots, e^{t_m}\}, \qquad c_+ = \max\{1, e^{t_1}, \dots, e^{t_m}\}.$$

Each potential is nondecreasing along rays from the origin. Thus

$$U(c_-x) \le V_0(x) + \sum_i V_i(e^{t_i}x) \le U(c_+x).$$

Writing $Z(t)$ for the integral in (B.1), changes of variables give

$$0 < c_+^{-n} Z(0) \le Z(t) \le c_-^{-n} Z(0) < \infty. \tag{B.3}$$

This also gives a common integrable bound for the densities when t ranges over a compact set.

For convex indicators, take $V_0 = V$, $V_i = I_{K_i}$, and replace t by $-t$. Then the integral in (B.1) is

$$\int e^{-V(x)-\sum_i I_{K_i}(e^{-t_i}x)}\,dx = \mu\left(\bigcap_i e^{t_i}K_i\right).$$

This completes the proof of Theorem A and therefore of the geometric corollaries in the introduction.

The passage from Theorem A to Theorem B. We justify the supporting-strip argument for (∗) and the bounded-sublevel reduction used in the introduction. First suppose that all positive sublevel sets $K_i(r_i)$ are bounded. Positivity of the normalizing integral, evenness and convexity put zero in the interior of every effective domain. After subtracting the minima, each $K_i(r_i)$ with $r_i > 0$ is therefore a symmetric convex body.

Choose increasing finite sets of directions, containing a basis, whose union is dense in $\mathbb{S}^{n-1}$. For each such finite family, apply Theorem A to the strips $\{x : |\langle x, \theta\rangle| \le 1\}$, using all indices i and the endpoint widths

$$e^{-s_i}h_{K_i(r_i)}(\theta), \qquad e^{-t_i}h_{K_i(u_i)}(\theta).$$

The two endpoint intersections decrease to $C(s, r)$ and $C(t, u)$. The intersections with geometrically interpolated widths also decrease; their limit is contained in $C((1-\lambda)s + \lambda t, (1-\lambda)r + \lambda u)$ by the support-function bound in the proof. Each initial intersection is bounded, because a basis was included and all widths are positive. Thus continuity of measure from above passes the finite-family inequality to the limit and proves (∗).

For general sublevels, replace V_i by $V_i + I_{RB_2^n}$, $1 \le i \le m$, leaving the background measure ρ fixed. The positive sublevels become $K_i(r_i) \cap RB_2^n$, so

the preceding argument and the layer-cake and Prékopa steps apply. For every fixed s,

$$\int_{\mathbb{R}^n} e^{-\sum_i V_i(e^{s_i}x)} \prod_{i=1}^m \mathbf{1}_{RB_2^n}(e^{s_i}x)\, d\rho(x)$$
$$\uparrow \int_{\mathbb{R}^n} e^{-\sum_i V_i(e^{s_i}x)}\, d\rho(x), \qquad R \to \infty.$$

Monotone convergence in the three integrals of (1.15) proves the assertion for the original potentials. Their positivity and finiteness follow from (B.3). The layer-cake identity includes extended-valued potentials, since $e^{-\infty} = 0$, and no differentiation is used here. Restoring the subtracted minima multiplies both sides of (1.15) by the same constant.

C Analytic details for the moment semigroups

We justify the domain and limiting steps in Section 7. Throughout, μ is a smooth even log-concave probability measure on $\mathbb{R}^n$, and its smooth even moment potential has positive Hessian and a global gradient diffeomorphism. The diffusion matrix is $(D^2\varphi^*)^{-1}$, with ordinary x-derivatives, by (2.17). Appendix A, including Remark A.2, supplies completeness of the moment metric, compact operator cores, and density of the compact-test range of P. Thus no upper bound on $(D^2\varphi^*)^{-1}$ is needed here.

C.1 The closed Hessian energy and the gap

Let L_μ^m be the nonpositive self-adjoint generator of the closed diffusion form

$$(u, v) \longmapsto \int \langle (D^2\varphi^*)^{-1}\nabla u, \nabla v\rangle\, d\mu.$$

The global argument in Appendix A puts constants and linear functions in its operator domain, with $L_\mu^m 1 = 0$ and $L_\mu^m x_i = -x_i$. In particular, $T_s = e^{sL_\mu^m}$ is conservative. Moment Brascamp–Lieb gives

$$\sigma(-L_\mu^m) \subset \{0\} \cup [1, \infty), \qquad \ker L_\mu^m = \operatorname{span}\{1\},$$

so spectral calculus defines the nonnegative self-adjoint operator $P = L_\mu^m(L_\mu^m + I)$.

For compact smooth q, integrated moment Bochner identifies the Hessian energy with $\|L_\mu^m q\|_2^2 + \int q L_\mu^m q\, d\mu$. Since $1 + \lambda(\lambda - 1)$ and $1 + \lambda^2$ are comparable on $\{0\} \cup [1, \infty)$, the energy norm is equivalent to the graph norm of L_μ^m. Its completion therefore has domain $\mathcal{D}(L_\mu^m)$. Applying Bochner to differences of

compact approximants shows that their weighted Hessians are Cauchy; local ellipticity identifies their limit with the distributional Hessian. Consequently

$$\begin{aligned}\mathcal{E}(q) &= \|L_\mu^m q\|_2^2 + \int q L_\mu^m q \, d\mu \\ &= \int \mathrm{Tr}((D^2\varphi^*)^{-1} D^2 q (D^2\varphi^*)^{-1} D^2 q) \, d\mu, \qquad q \in \mathcal{D}(L_\mu^m).\end{aligned} \tag{C.1}$$

This is the closed quadratic form associated with P; on $\mathcal{D}(P)$ it equals $\int qPq \, d\mu$. Its zero-energy functions are exactly the affine functions, so $\ker P = \mathrm{Aff}$.

For compact smooth q, u, the Hessian pairing and (7.6) give

$$\left| \int qPu \, d\mu \right| \le \mathcal{E}(q)^{1/2} \mathcal{E}(u)^{1/2} \le 2\mathcal{E}(q)^{1/2} \|Pu\|_2.$$

The compact-test range is dense in $\mathrm{Aff}^\perp$. Taking the supremum over $\|Pu\|_2 \le 1$ and then closing the form gives

$$\|q - \ell_q\|_2^2 \le 4\mathcal{E}(q), \qquad q \in \mathcal{D}(L_\mu^m),$$

where ℓ_q is the affine projection in (7.2). Hence $P \ge \frac{1}{4} I$ on $\mathrm{Aff}^\perp$. On centered even functions, the spectral values of $-L_\mu^m$ satisfy $\lambda \ge 1$ and $\lambda(\lambda - 1) \ge 1/4$, and thus $\lambda \ge 1 + \delta$, with $\delta = (\sqrt{2} - 1)/2$. Including the constant mode yields

$$P \ge -\delta L_\mu^m \qquad \text{on even functions.} \tag{C.2}$$

In particular, $R_t = e^{-t(P + \delta L_\mu^m)}$ is an L^2 contraction on that subspace. Reflection commutes with both operators, so all compact approximations may be symmetrized when even tests are needed.

C.2 The diffusion inequality and the norm comparison

The gradient map is an isometry from the source metric $D_y^2\varphi$ to the target metric $D_x^2\varphi^*$, and transports $e^{-\varphi} dy$ to μ. Kolesnikov's curvature formula gives

$$\mathrm{Ric}_{D_y^2\varphi, e^{-\varphi} dy} \ge \frac{1}{2} D_y^2\varphi + \frac{1}{2} D_y^2\varphi \, D_x^2 V(\nabla\varphi) D_y^2\varphi \ge \frac{1}{2} D_y^2\varphi;$$

see [86, Theorem 4.3] and [93, Theorem 2.4]. Completeness permits the Bakry–Émery criterion [93, Theorem 2.1], giving

$$\mathrm{Ent}_\mu(h^2) \le 4 \int \langle (D^2\varphi^*)^{-1} \nabla h, \nabla h \rangle \, d\mu. \tag{C.3}$$

The inequality holds throughout the closed diffusion form domain, by truncation, form approximation, and lower semicontinuity of entropy.

For clarity, the Gross differentiation in (7.8) may first be performed for $h = \varepsilon + T_\tau k$, with $\varepsilon, \tau > 0$ and k bounded and nonnegative. The evolved functions remain bounded above and bounded away from zero. Spectral regularization puts them in the generator domain, and the closed-form chain rule justifies testing against their powers. Integrate the differential inequality, then let $\tau, \varepsilon \downarrow 0$ using strong continuity and Fatou's lemma. Truncation gives arbitrary nonnegative L^p data, and positivity gives $|T_s h| \le T_s|h|$ for signed data. Thus

$$\|T_s h\|_{1+(p-1)e^s} \le \|h\|_p, \qquad p > 1, \quad s \ge 0.$$

The spectral multipliers in the proof of (7.10) may first be restricted to bounded spectral intervals. Their L^2 bound on $\mathrm{Aff}^\perp$, together with the diffusion estimate above, permits passage to the full spectral measure. For $p < 2$, bounded truncations converge in L^p. The affine projection is bounded from L^p into every finite L^q, since μ has all finite moments. Applying the affine estimate to differences therefore makes their images under S_t Cauchy in L^q and defines a unique bounded extension. On $L^p \cap L^2$ it agrees with the spectral semigroup, since the truncations converge in both spaces.

C.3 Relaxation and integration in time

The spectral theorem gives the strongly continuous L^2 contraction semigroup $S_t = e^{-tP}$ [79, Chapters V and IX]. The gap gives

$$\|S_t f - \ell_f\|_2 \le e^{-t/4}\|f - \ell_f\|_2.$$

For $t > 0$, $f_t = S_t f$ belongs to $\mathcal{D}(P^k)$ for every integer $k \ge 1$, and $\partial_t f_t = -P f_t$ in L^2. The Bochner integral $u_t = \int_0^t S_r f \, dr$ has spectral multiplier $(1 - e^{-t\eta})/\eta$, with value t at $\eta = 0$. Multiplication by η gives the bounded multiplier $1 - e^{-t\eta}$, proving

$$u_t \in \mathcal{D}(P), \qquad P u_t = f - f_t.$$

For centered even f, $f_t \to 0$, so these are approximate solutions of $Pu = f$.

C.4 Compact tests and the dilation argument

Fix z, a as in the semigroup derivation of Theorem B in Section 7, put $W_i(x) = V_i(e^{z_i} x)$, and use V, F, μ from (5.56) with V_i replaced by W_i. Set $f = \langle x, \nabla F\rangle - \int \langle x, \nabla F\rangle \, d\mu$. The compact estimate (5.54) gives

$$2\int f P v \, d\mu - \|Pv\|_2^2 \le Q(f, F), \qquad v \in C_c^\infty(\mathbb{R}^n), \quad v \text{ even}.$$

Since Pu_t is centered and even, compact-range density supplies even v_j with $Pv_j \to Pu_t$ in L^2. Passing the pairing and norm to the limit, and using (5.58) and (5.13), gives

$$\begin{aligned} 2\int f Pu_t \, d\mu - \|Pu_t\|_2^2 &\le Q(f,F) \\ &\le \sum_i a_i^2 \int \Big(\langle x, \nabla W_i \rangle + \langle D^2 W_i x, x \rangle \Big) \, d\mu. \end{aligned} \tag{C.4}$$

Substituting $Pu_t = f - f_t$ proves the finite-time estimate in that remark. For $0 < \tau < T < \infty$, differentiation in L^2 and (C.1) give

$$\|f_\tau\|_2^2 - \|f_T\|_2^2 = 2\int_\tau^T \int f_t P f_t \, d\mu \, dt = 2\int_\tau^T \mathcal{E}(f_t) \, dt.$$

Strong continuity at zero and relaxation at infinity allow $\tau \downarrow 0$ and $T \to \infty$. Uniform convexity of V_0 and the polynomial growth assumptions justify differentiation of the partition function locally uniformly in z.

Finally replace V_0 by $V_0 + \varepsilon|x|^2/2$ and let $\varepsilon \downarrow 0$. The original integrable log-concave density has an exponential tail, so polynomial growth gives dominated convergence for the normalizations, variances and the right-hand side of (C.4), locally uniformly in the dilation parameters. Appendix B then passes the multiplicative integral inequality to extended-valued convex potentials and sets.

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