Andrea Colesanti
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Brunn-Minkowski type inequalities and affine surface area

Online Asymptotic Geometric Analysis Seminar
May 5, 2020

The results presented in this talk were obtained in collaboration with Karoly Böröczky, Monika Ludwig and Thomas Wannerer.

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This talk is related to several previous lectures in this AGA series: the one by Elisabeth Werner (via the affine surface area), the one by Karoly Böröczky (via the Brunn-Minkowski inequality), and the one by Ilaria Fragalà, who spoke about various functionals which will be mentioned in the sequel.

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Equivalently: the volume raised to the power $1/n$ (which is the reciprocal of its homogeneity order), is **concave** in the family $\mathcal{K}^n$ of convex bodies, endowed with the Minkowski addition.

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- The survey: *The Brunn-Minkowski inequality* by R. Gardner (Bull. A.M.S., 2002), describes the state of the art on (BM) at the time of its publication. Probably at that time it would have been difficult to predict the quantity of new results and developments concerning (BM), that appeared since then.

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- replacing the ambient space $\mathbb{R}^n$ by a different one (e.g. the Gauss space, or more generally $\mathbb{R}^n$ with a different measure);
- replacing the volume by a different functional – we will focus on this aspect.

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We say that $\mathbf{F}$ verifies a Brunn-Minkowski type inequality, if for every $K_0, \, K_1 \in \mathcal{K}^n$, and for every $t \in [0, 1]$:

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- Several ("non-linear") generalisations of the last three examples.

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- The **affine surface area**, in dimension $n = 3$ and higher, as we will see in this talk.

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A step in this direction was made by D. Hug, E. Saorín-Gómez and A.C. [2012], by a characterisation of some specific valuations verifying a Brunn-Minkowski type inequality.

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where f_K is the *curvature function* of K – i.e. the density of the absolutely continuous part of the area measure of K, with respect to $\mathcal{H}^{n-1}$.

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- Ω is a valuation [Schütt]. In fact, Ludwig and Reitzner proved a remarkable characterisation of upper semicontinuous and equi-affinely invariant valuations, involving Ω.
- $\Omega(K)$ is related to the *floating body* of K; in fact, one of the equivalent definitions of $\Omega(K)$ is based on this link [Shütt & Werner].

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By the solution to the Minkowski problem, $\alpha \cdot K \,\#\, \beta \cdot L$ is well defined and uniquely determined up to translations.

A concavity inequality for Ω w.r.t. the Blaschke addition

Blaschke addition. For a convex body K, we denote by S_K its area measure. Let $K, L \in \mathcal{K}^n$, and $\alpha, \beta > 0$.

$$\alpha \cdot K \# \beta \cdot L$$

is the convex body such that

$$S_{\alpha \cdot K \# \beta \cdot L} = \alpha S_K + \beta S_L.$$

By the solution to the Minkowski problem, $\alpha \cdot K \# \beta \cdot L$ is well defined and uniquely determined up to translations. This body is called the *Blaschke linear combination* of K and L, with coefficients α and β.

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The proof is based the definition of the affine surface area due to Lutwak, which identifies $\Omega^{(n+1)/n}$ as the infimum of Blaschke linear functionals.

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(this does not work for $n = 2$).

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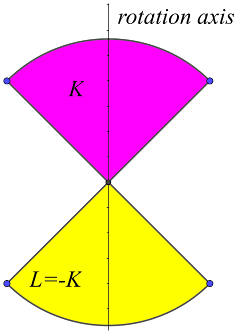

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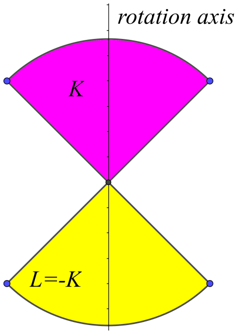

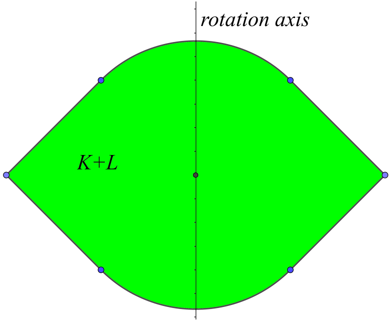

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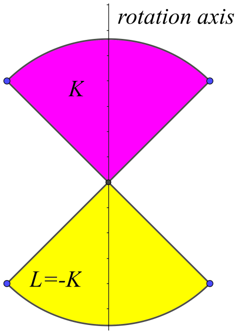

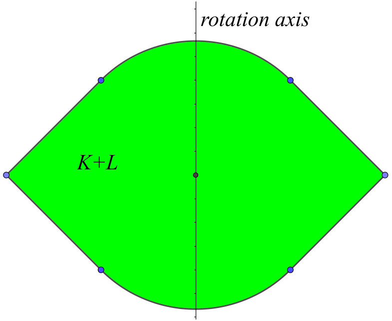

Recall that

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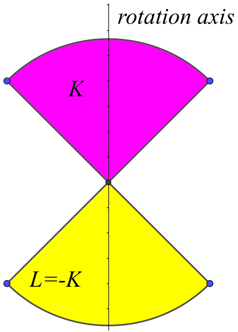

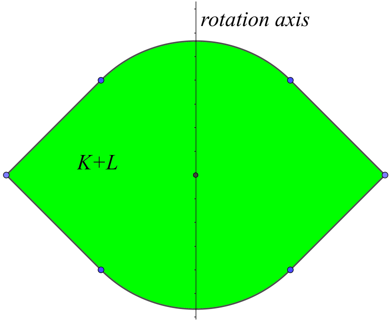

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Then it is easy to verify that

$$\Omega(K+L) = \Omega(K) + \Omega(L).$$

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which are centrally symmetric as well.

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can be written as an inequality involving the derivatives of h_N, up to the fourth order.

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- The geominimal surface area (being the infimum of Minkowski linear functionals) verifies a Brunn-Minkowski inequality.

□

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In particular:

Corollary 2. *Ω verifies a Brunn-Minkowski type inequality in a suitable C^4-neighbourhood of the unit ball of $\mathbb{R}^n$.*