# MATH-8803: Topics Course in Concentration of Measure Phenomena and Convexity, Fall 2023, Georgia Tech 

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#### Abstract

Disclaimer: these lecture notes are currently under construction, and may not be fully proofread yet. If you spot a typo please let me know! Also, the references are in the process of being matched to the text.


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## Notation

- $\mathbb{R}^{n}$ - the $n$-dimensional space
- $|\cdot|$ or $|\cdot|_{k}$ (for a set) - Lebesgue $k$-dimensional volume
- $|\cdot|$ (for a vector) - Euclidean length
- $\langle\cdot, \cdot\rangle$ - the scalar product in $\mathbb{R}^{n}$.
- For $x \in \mathbb{R}^{n}$, denote $x^{2}=\langle x, x\rangle=|x|^{2}$.
- $B_{2}^{n}$ - the Euclidean ball
- $\mathbb{S}^{n-1}$ - the unit sphere
- $A+B-$ Minkowski sum of sets
- $x^{\perp}=\left\{y \in \mathbb{R}^{n}:\langle x, y\rangle=0\right\}$ (for a vector $x \in \mathbb{R}^{n} \backslash\{0\}$ )


## 1 Volumes in High dimensions, hands-on computations and pretty pictures.

### 1.1 Introduction

The first lecture is all about some pretty pictures (including the one from the course website) and getting our hands onto some basic volume computations in high dimensions. We will generally work in $\mathbb{R}^{n}$ and assume that the dimension is very large (tends to infinity). Highdimensional phenomena manifests when complex systems that depend on many parameters actually behave in a simple way. Concentration of measure phenomena can be viewed as part of it. Roughly speaking, it tells that Lipshitz functions in certain high dimensional spaces behave similarly to constant functions. In high dimensions, if you walk a bit far, you quickly end up nowhere.

Where do the concentration phenomena stem from?
Example 1.1 (Central Limit Theorem). Let $X_{1}, \cdots, X_{n}$ be i.i.d. uniform random variables on the interval $[-1,1]$. Then as $n$ tends to infinity, the random variable

$$
n^{-1 / 2}\left(X_{1}+\cdots+X_{n}\right)
$$

converges, in distribution, to the Gaussian random variable with density $\frac{1}{\sqrt{2 \pi}} e^{-\frac{t^{2}}{2 c}}$, for the appropriate constant $c>0$.

To get a geometric interpretation of this, note that the vector $X:=\left(X_{1}, \cdots, X_{n}\right)$ is distributed uniformly over the unit cube $B_{n}^{\infty}:=[-1,1]^{n}$. Given the vector $\theta=(1 / \sqrt{n}, \cdots, 1 / \sqrt{n})$ the above example says that the random variable $\langle X, \theta\rangle$ is distributed roughly as a normal random variable.

What is the geometric meaning of the density $f(t)$ of $\langle X, \theta\rangle$ ? After thinking a little, we see that

$$
f(t)=\left|B_{n}^{\infty} \cap\{\langle x, \theta\rangle=t\}\right|_{n-1} .
$$

Thus $f(t)$ the $n-1$ dimensional area of the hyperplane section of the cube perpendicular to $\theta$, distance $t$ from the origin. Although sections of cubes are hard to compute exactly, as the dimension goes to infinity they resemble a normal random variable, which is a simple object.


This phenomenon appears to stem from independence. However, this fact is true not just about cubes! For any convex body there exists a direction $\theta$ (in fact, many of them) for which $\langle X, \theta\rangle$ behaves a bit like a Gaussian (in the appropriate sense). This is the content of the Central Limit Theorem for convex bodies from 2007 due to Bo'az Klartag.

In fact, it turns out that a lot of concentration type phenomena stems from independence, while some other concentration phenomena stems from convexity and isoperimetry. While concentration via independence is often simpler and better understood, it is also often less general. This semester, we will only discuss concentration stemming from convexity and isoperimetry, while concentration via independence will be discussed next semester in the HDP course.

### 1.2 Computing the volumes of the cube, the cross-polytope and the Euclidean ball.

We define an $L_{p}$-ball to be the set

$$
B_{p}^{n}=\left\{x \in \mathbb{R}^{n}: \sum_{i=1}^{n}\left|x_{i}\right|^{p} \leq 1\right\}
$$

The set $B_{2}^{n}$ is called the Euclidean ball, while the set $B_{1}^{n}$ is called the cross-polytope, or diamond. The set

$$
B_{\infty}^{n}=\left\{x \in \mathbb{R}^{n}: \max _{i=1, \ldots, n}\left|x_{i}\right| \leq 1\right\}
$$

is called the unit cube. Note that $B_{p}^{n} \subset B_{q}^{n}$ if $p \leq q$.
For a Borel-measurable set $A \subset \mathbb{R}^{n}$, denote by $|A|$ or $|A|_{n}$ its Lebesgue volume. In this course we always consider sets which are Borel-measurable. Note (a small home work) that

$$
\left|B_{\infty}^{n}\right|=2^{n},
$$

while

$$
\left|B_{1}^{n}\right|=\frac{2^{n}}{n!}
$$

Computing the volume of the ball $B_{2}^{n}$ is a bit harder. To get started we denote the unit sphere (the boundary of the unit ball) as

$$
\mathbb{S}^{n-1}=\left\{x \in \mathbb{R}^{n}:|x|=1\right\} .
$$

Lemma 1.2. $\left|\mathbb{S}^{n-1}\right|_{n-1}=n\left|B_{2}^{n}\right|$.
Proof. Use polar coordinates:

$$
\left|B_{2}^{n}\right|=\int_{B_{2}^{n}} d x=\int_{\mathbb{S}^{n}-1} \int_{0}^{\infty} \mathbf{1}_{B_{2}^{n}}(t \theta) t^{n-1} d t d \theta=\int_{\mathbb{S}^{n-1}} \int_{0}^{1} t^{n-1} d t d \theta=\frac{1}{n} \int_{\mathbb{S}^{n-1}} d \theta=\frac{1}{n}\left|\mathbb{S}^{n-1}\right|
$$

with the change of variables $x=t \theta, \theta \in \mathbb{S}^{n-1}, t \geq 0$.
Now we want to compute the area of $\mathbb{S}^{n-1}$. Building towards that, we show
Lemma 1.3. $\int_{\mathbb{R}} \exp \left(-x^{2} / 2\right) d x=\sqrt{2 \pi}$.
Proof. Let us consider the following two-dimensional integral:

$$
\iint_{\mathbb{R}^{2}} \exp \left(-\left(x^{2}+y^{2}\right) / 2\right)
$$

We first use the fact that it equals $\left(\int_{\mathbb{R}} \exp \left(-x^{2} / 2\right) d x\right)^{2}$ (follows from Fubini). On the other hand, rewriting $\int_{\mathbb{R}} \exp \left(-\left(x^{2}+y^{2}\right) / 2\right)$ in polar coordinates you get

$$
\int_{\mathbb{S}^{1}} \int_{0}^{\infty} t e^{-t^{2} / 2} d t d \theta=\left|\mathbb{S}^{1}\right|=2 \pi
$$

Now we tie this back to what we originally wanted.
Lemma 1.4.

$$
\left|\mathbb{S}^{n-1}\right|=\frac{(2 \pi)^{\frac{n}{2}}}{J_{n-1}}
$$

where $J_{n-1}:=\int_{0}^{\infty} t^{n-1} e^{-\frac{t^{2}}{2}} d t$.
Proof. Again by Fubini's theorem,

$$
\int_{\mathbb{R}^{n}} \exp \left(-x^{2} / 2\right) d x=\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{n} \exp \left(-x_{i}^{2} / 2\right) d x_{1} \cdots d x_{n}=\left(\int_{-\infty}^{\infty} e^{-\frac{t^{2}}{2}} d t\right)^{n}=\sqrt{2 \pi}^{n}
$$

Again we use polar coordinates to rewrite this as

$$
\int_{\mathbb{R}^{n}} e^{-x^{2} / 2} d x=\left|S^{n-1}\right|_{n-1} \cdot \int_{0}^{\infty} t^{n-1} e^{-t^{2} / 2} d t=\left|\mathbb{S}^{n-1}\right|_{n-1} \cdot J_{n-1}
$$

From which we conclude the Lemma.
Next, we need to evaluate $J_{n-1}$. One could reduce it to Gamma function, but we employ the so-called Laplace Method, which is a useful tool in handling integrals of this type.

Proposition 1.5 (An example of the use of the Laplace method). Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function, and $m$ be a positive number. We make the following assumptions.

- $F$ attains the absolute maximum at the point $s_{0}$, and for every $s \neq s_{0}$ we have $F(s)<$ $F\left(s_{0}\right)$.
- Further, assume that there exist numbers $a, b>0$ such that $F(s)<F\left(s_{0}\right)-b$ whenever $\left|s-s_{0}\right|>a$.
- Suppose that the integral $\int e^{F(s)} d s<\infty$.
- Suppose that $F$ is twice differentiable in some neighborhood of $s_{0}$.
- Suppose that $F^{\prime \prime}\left(s_{0}\right)<0$.

When $m \rightarrow \infty$, the integral

$$
\int e^{m F(s)} d s=(1+o(1)) e^{m F\left(s_{0}\right)} \frac{\sqrt{2 \pi}}{\sqrt{-m F^{\prime \prime}\left(s_{0}\right)}}
$$

Proof. See home work with a series of hints!
Consider

$$
F_{n}(t)=(n-1) \log (t)-t^{2} / 2 .
$$

Setting the $F_{n}(t)$ to 0 one can check that $F_{n}(t)$ has a maximum at $t_{0}=\sqrt{n-1}$. One can also check that $F_{n}^{\prime \prime}\left(t_{0}\right)=-(n-1) / t_{0}^{2}-1=-2$. Thus Proposition 1.5 gives

$$
\begin{aligned}
J_{n-1}= & \int_{0}^{\infty} e^{-t^{2} / 2} t^{n-1} d t=\int_{0}^{\infty} e^{-F_{n}(t)} d t \\
= & \left(1+o_{n}(1)\right) e^{-F_{n}\left(t_{0}\right)} \cdot \frac{\sqrt{2 \pi}}{\sqrt{-F_{n}^{\prime \prime}\left(t_{0}\right)}} \\
& (1+o(1)) e^{-\frac{n-1}{2}}(n-1)^{\frac{n-1}{2}} \cdot \frac{\sqrt{2 \pi}}{\sqrt{2}} .
\end{aligned}
$$

Combining the above with Lemma 1.4, we conclude

## Corollary 1.6.

$$
\left|\mathbb{S}^{n-1}\right|=\frac{(2 \pi)^{n / 2}}{J_{n-1}}=\left(1+o_{n}(1)\right) \cdot \frac{(2 \pi)^{n / 2} e^{\frac{n-1}{2}}}{(n-1)^{\frac{n-1}{2}} \sqrt{\pi}}
$$

Remark 1.7. The volume and surface area of euclidean balls tends to 0 as $n$ tends to infinity, but non-montonically $\left(\left|B_{2}^{n}\right|\right.$ is increasing in $n$ up to roughly $\left.n=16\right)$.

Remark 1.8. Note that

$$
\frac{\left|B_{2}^{n-1}\right|_{n-1}}{\left|B_{2}^{n}\right|_{n}}=(1+o(1)) \frac{\sqrt{n}}{\sqrt{2 \pi}}
$$

### 1.3 The first taste of the Concentration phenomena

Let $\epsilon>0$. Consider the problem of computing

$$
\left|B_{2}^{n} \cap\{|\langle x, \theta\rangle| \leq \varepsilon\}\right|,
$$

where $\theta \in \mathbb{S}^{n-1}$. We can write this volume (using Fubini and homogeneity of the volume) as

$$
\left|B_{2}^{n} \cap\{|\langle x, \theta\rangle| \leq \varepsilon\}\right|=\int_{-\varepsilon}^{\varepsilon}\left|\sqrt{1-t^{2}} B_{2}^{n-1}\right|_{n-1} d t=\int_{-\varepsilon}^{\varepsilon}\left(1-t^{2}\right)^{\frac{n-1}{2}}\left|B_{2}^{n-1}\right|_{n-1} d t .
$$

We could evaluate this using Laplace's method but instead we write out a coarse bound: by Remark 1.8, and since the function $\left(1-t^{2}\right)^{(n-1) / 2}$ decreases on $[0, \epsilon]$, we get

$$
\frac{\mid B_{2}^{n} \cap\{|\langle x, \theta\rangle|}{\left|B_{2}^{n}\right|}=\frac{\left|B_{2}^{n-1}\right|_{n-1}}{\left|B_{2}^{n}\right|_{n}} \cdot \int_{-\varepsilon}^{\varepsilon}\left(1-t^{2}\right)^{(n-1) / 2} d t \geq(1+o(1)) \frac{\sqrt{n}}{\sqrt{2 \pi}} \cdot 2 \epsilon \cdot\left(1-\epsilon^{2}\right)^{(n-1) / 2} .
$$

Now for $\varepsilon=1 / \sqrt{n}$ the above ratio is at least

$$
(1+o(1)) \frac{2}{e \sqrt{2 \pi}} \approx 0.294
$$

Suppose $n$ is very large. Then the slab of width $\frac{2}{\sqrt{n}}$ that we are considering is very thin. We see that he volume of the intersection of the ball with a very thin slab is more than $29 \%$ of the volume of the ball!

This means that for $n=1000000$, a constant fraction of the mass of the unit ball lies 0.001 away from the equator! Any equator!!!

Remark 1.9. A more honest estimate using the Laplace Method would give 99\%, provided that the slab has width $\frac{C}{\sqrt{n}}$ for an appropriate $C>0$.


This begs a question: maybe this means that most of the mass of the ball lies near the center? We will shortly see that the answer is "no", and it couldn't be more "no"!

### 1.4 About the thin annulus around the ball

Last time we concluded that the a sphere (even though it is convex) has its mass concentrated near the equator, so it "looks like a spinning top in every direction". That would suggest that most of the mass is contained in the center (because the center is contained in all the slabs). However it is actually all concentrated near the boundary of the ball.

Define the annulus $A_{\varepsilon}=B_{2}^{n} \backslash(1-\varepsilon) B_{2}^{n}$. Then

$$
\left|A_{\varepsilon}\right|=\left|B_{2}^{n}\right| \cdot(1-(1-\varepsilon))^{n}=\left|B_{2}^{n}\right| \cdot\left(n \varepsilon-O\left(\varepsilon^{2}\right)\right)
$$

assuming $\varepsilon \leq c / n$ for $c$ a constant say.
Conclusion: almost all the mass of the unit ball is located $1-c / n$ away from the origin!!!

### 1.5 First hand-wavy glance into the connection between isoperimetry and concentration: most of the mass of a convex body is near the boundary.

Definition 1.10. A set $K \subset \mathbb{R}^{n}$ is called convex if for every pair of $x, y \in \mathbb{R}^{n}$ and every $\lambda \in[0,1]$ one has $\lambda x+(1-\lambda) y \in K$.

Remark 1.11. We note that the interval connecting vectors $x$ and $y$ in $\mathbb{R}^{n}$ can be written as

$$
[x, y]=\{\lambda x+(1-\lambda) y: \lambda \in[0,1]\} .
$$

Definition 1.12. We say that $K$ is a convex body if it is a compact convex set with nonempty interior.

Note that the definition of convex body doesn't allow for infinite cylinders or disks (In the former case because it is not compact and the latter because it has empty interior), even though both sets are convex.

We now want to argue that most of the mass of a convex body is near its boundary. For a convex body $K$ we consider an annulus $A_{\varepsilon}$

$$
A_{\varepsilon}=\{x \in K: \quad: \operatorname{dist}(x, \partial K) \leq \varepsilon\}
$$

Recall also
Definition 1.13 (Minkowski sum). For $A, B \subset \mathbb{R}^{n}$ define their Minkowski sum as

$$
A+B=\{x+y: x \in A, y \in B\}
$$

Definition 1.14 (Perimeter). For a Borel measurable set $A \subset \mathbb{R}^{n}$ define perimeter as

$$
|\partial A|=\lim \inf _{\varepsilon \rightarrow 0} \frac{\left|A+\varepsilon B_{2}^{n} \backslash A\right|}{\varepsilon} .
$$



Claim 1.15 (a very hand-wavy part). When $\varepsilon$ is sufficiently small (depending on $K, n$ ) we have $\left|A_{\varepsilon}\right| \approx \varepsilon|\partial K|_{n-1}$.

A nice way to lower bound the perimeter is via the isoperimetric inequality:
Theorem 1.16 (The isoperimetric inequality). Balls minimize perimeter relative to volume. In other words, for any Borel-measurable set $K$ in $\mathbb{R}^{n}$ such that $|K|=\left|R B_{2}^{n}\right|$, for the right $R>0$, one has

$$
|\partial K| \geq\left|\partial\left(R B_{2}^{n}\right)\right|
$$

A fable has it that Roman soldiers wanted to maximize the amount of land they received as payment for military service:)


Note that the expression $\frac{|\partial K|}{|K|^{\frac{n-1}{n}}}$ is invariant under dilations of $K$ : indeed, the volume is $n$-homogeneous while the perimeter is $(n-1)$-homogeneous, and therefore for any $t>0$ :

$$
\frac{|\partial(t K)|}{|t K|^{\frac{n-1}{n}}}=\frac{t^{n-1}|\partial(K)|}{t^{\frac{n-1}{n}}|K|^{\frac{n-1}{n}}}=\frac{|\partial(K)|}{|K|^{\frac{n-1}{n}}} .
$$

Therefore, the isoperimetric inequality is equivalent to the fact that for any Borelmeasurable set $K$ in $\mathbb{R}^{n}$,

$$
\frac{|\partial K|}{|K|^{\frac{n-1}{n}}} \geq \frac{\left|\partial B_{2}^{n}\right|}{\left|B_{2}^{n}\right|^{\frac{n-1}{n}}}
$$

Thus by taking $\varepsilon=c / \sqrt{n}$ for $c$ a sufficiently large constant we conclude that $\left|A_{\varepsilon}\right| /|K| \geq$ $99 \%$ (we leave out some details).

Remark 1.17. Please note that the above is not a mathematical argument but merely a hand waving. The point is to provide the very first illustration of concentration arising from isoperimetry!

### 1.6 Some more fun regarding metric estimates in high dimension

Consider the cube $B_{\infty}^{n}$ and place a copy of $B_{2}^{n}$ centered at each vertex of $B_{\infty}^{n}$ (see the Figure below). Note that the largest ball you can place at the center of $B_{\infty}$ without intersecting any of the copies of $B_{2}^{n}$ will have radius $\sqrt{n}-1$. This might be somewhat surprising, since in many directions the ball extends out much further from the origin than the cube does, when $n$ is large.


### 1.7 Home work

Question 1.18 (1 point). Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function, and $m$ be a positive number. We make the following assumptions.

- $F$ attains the absolute maximum at the point $s_{0}$, and for every $s \neq s_{0}$ we have $F(s)<$ $F\left(s_{0}\right)$.
- Further, assume that there exist numbers $a, b>0$ such that $F(s)<F\left(s_{0}\right)-b$ whenever $\left|s-s_{0}\right|>a$.
- Suppose that the integral $\int e^{F(s)} d s<\infty$.
- Suppose that $F$ is twice differentiable in some neighborhood of $s_{0}$.
- Suppose that $F^{\prime \prime}\left(s_{0}\right)<0$.

Prove that when $m \rightarrow \infty$, the integral

$$
\int e^{m F(s)} d s=(1+o(1)) e^{m F\left(s_{0}\right)} \frac{\sqrt{2 \pi}}{\sqrt{-m F^{\prime \prime}\left(s_{0}\right)}}
$$

Hint 1: Observe that WLOG $s_{0}=F\left(s_{0}\right)=0$, and that $F$ is equal to $-\infty$ outside of the support.

Hint 2: Pick any $\epsilon>0$ and note that one may find a $\delta>0$ so that for all $s \in(-\delta, \delta)$ we have

$$
\left|F(s)-\frac{F^{\prime \prime}(0) s^{2}}{2}\right| \leq \epsilon
$$

Hint 3: Find an estimate for

$$
\int_{-\delta}^{\delta} e^{m F(s)} d s
$$

Hint 4: Note that the assumptions imply that for every $\delta>0$ there is $\eta(\delta)>0$ such that $F(s)<F\left(s_{0}\right)-\eta(\delta)$;

Hint 5: Find an estimate for $\int_{\delta}^{\infty} e^{m F(s)} d s$ and $\int_{-\infty}^{-\delta} e^{m F(s)} d s$; to do that, use the previous hint, and also note that $e^{m F(s)}=e^{(m-1) F(s)} e^{F(s)}$. Use the assumption about the converging integral as well.

Hint 6: Carefully make sure that the assumptions allow you to let $m \rightarrow \infty$ and $\epsilon \rightarrow 0$.
Question 1.19 (1 point). All the questions below require an answer up to a multiplicative factor of $1+o(1)$, when $n \rightarrow \infty$.
a) Find $\frac{\left|B_{2}^{n}\right|_{n}}{\left|B_{2}^{n-1}\right|_{n-1}}$.

Hint: Use the formula from Question 1 and the Fubbini theorem. Note that this method is alternative to the one we used in class to express $\left|B_{2}^{n}\right|_{n}$.
b) Find the volume of

$$
\left\{x \in \mathbb{R}^{n}:|x| \leq 2, x_{1} \in[a, b]\right\}
$$

where b1) $a=0, b=0.1$; b2) $a=-\frac{1}{\sqrt{n} \log n}, b=\frac{1}{n}$.
Hint: Use the expression for $\left|B_{2}^{k}\right|_{k}$ which we derived in class.
c) Using any method you like, find the volume of

$$
\operatorname{conv}\left(\left\{x \in \mathbb{R}^{n}:|x|<3, x_{2}=0\right\} \cup\left\{x \in \mathbb{R}^{n}:\left|x-e_{2}\right|<1, x_{2}=1\right\}\right) .
$$

d) Let $\gamma$ be the standard Gaussian measure on $\mathbb{R}^{n}$ with density $\frac{1}{\sqrt{2 \pi}^{n}} e^{-\frac{|x|^{2}}{2}}$. For each $t \in(0, \infty)$, find $\gamma(\{x:|x|>t\})$, depending on $t$ (find the best approximation you can for each range).
e) Let $\mu$ be the probability measure with density $C(n) e^{-|x|^{3}}$. Find $C(n)$.
f) Let $\mu$ be as above. Let $R \in(0, \infty)$ be such that $\mu\left(R B_{2}^{n}\right)=\frac{1}{2}$. Find $R$.

Question 1.20 (1 point). Let $A$ be a convex set in $\mathbb{R}^{n}$ satisfying $x_{1}=0$ for all $x \in A$. Find the volume of $\operatorname{conv}\left(A, R e_{1}\right)$, in terms of $|A|_{n-1}, R$ and $n$

Question 1.21 (1 point). a) Using Laplace's method, prove that at least $99 \%$ of the volume of the $n$-dimensional Euclidean ball is contained in a strip of width $\frac{100}{\sqrt{n}}$ around any equator, for a sufficiently large $n$.
b) Prove the same fact on the sphere. (hint: use Fubbini's theorem directly on the sphere, but be careful about how the curvature of the sphere impacts your integral.)

Question 1.22 (2 points). Find a function $F: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$such that for every symmetric convex body $K$ in $\mathbb{R}^{n}$ with $|K|_{n}=1$, there exists a vector $u \in \mathbb{S}^{n-1}$ (possibly depending on the body), such that $\left|K \cap u^{\perp}\right|_{n-1} \geq F(n)$. Acceptable answers could be $F(t)=20 t^{-t}, F(t)=5^{-t}$, $F(t)=3 t^{-2}, F(t)=\frac{1}{t}, F(t)=\frac{10}{\sqrt{t}}, F(t)=100 t^{-\frac{1}{4}}, F(t)=\frac{1}{\log t}, F(t)=0.00001, F(t)=\sqrt{2}$, etc.

## 2 Background from Convexity

### 2.1 Convexity: basic concepts

We start with some more definitions.
Definition 2.1 (Convex and concave functions). A function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ is called convex if for every pair of $x, y \in \mathbb{R}^{n}$ and every $\lambda \in[0,1]$ one has $f(\lambda x+(1-\lambda) y) \leq \lambda f(x)+(1-\lambda) f(y)$. Similarly, a function $f$ is concave if $-f$ is convex.

Recall that we also defined a convex set and a convex body in subsection 3.2. Given a convex set $K$, one example of a convex function is the function

$$
\mathbb{1}_{K}^{\infty}(x)= \begin{cases}0 & x \in K \\ +\infty & x \notin K\end{cases}
$$

Note also that a hypergraph of any convex function is a convex set (recall that a hypergraph of a function on $\mathbb{R}^{n}$ is the set of points in $\mathbb{R}^{n+1}$ which are located above the graph of $f$ ).

Recall that if a convex $f \in C^{2}(\mathbb{R})$ then $f^{\prime \prime}(x) \geq 0$. The Hessian of a $C^{2}$ function $f$ is the matrix

$$
\nabla^{2} f=\left(\frac{\partial^{2} f}{\partial x_{i} \partial x_{j}}\right)
$$

Recall also that a matrix $A$ is called non-negative definite if for all $x \in \mathbb{R}^{n}$ we have

$$
\langle A x, x\rangle \geq 0
$$

Claim 2.2. If $f \in C^{2}\left(\mathbb{R}^{n}\right)$ is convex then the Hessian of $f$ is non-negative definite.
Proof. See home work with hints!

This fact is classical and important:
Theorem 2.3 (Jensen's inequality). For any convex function $F$ and probability measure $\mu$, we have

$$
F\left(\int f d \mu\right) \leq \int F(f) d \mu
$$

### 2.2 Minkowski functional of a convex body and the radial function

Definition 2.4 (origin-symmetry). We say that a convex body $K$ is origin symmetric if $x \in K \Longrightarrow-x \in K$.

Definition 2.5 (Minkowksi functional of a convex body). Given a convex body $K \subseteq \mathbb{R}^{n}$ we define the associated Minkowski functional $\|\cdot\|_{K}: \mathbb{R}^{n} \rightarrow[0, \infty)$ according to

$$
\|x\|_{K}=\inf \{\lambda>0: x \in \lambda K\} .
$$

Definition 2.6 (Radial function of a convex body). We can define the radial function $\rho_{K}$ of a convex body $K$ according to $\rho_{K}(x)=\|x\|_{K}^{-1}$.
Remark 2.7. Pick $\theta \in \mathbb{S}^{n-1}$. Note that $\rho_{K}(\theta)$ equals the distance between the origin and the furthest point from the origin inside $K$, in the direction $\theta$.


We now list some properties of the Minkowski functional:

- If $t \geq 0$ then $\|t x\|_{K}=t\|x\|_{K}$.
- $\|x\|_{K} \geq 0$ and $\|x\|_{K}=0 \Longleftrightarrow x=0$.
- If $K$ is symmetric then $\|x\|_{K}=\|-x\|_{K}$.
- triangle inequality: $\|x+y\|_{K} \leq\|x\|_{K}+\|y\|_{K}$.

Note that if $K$ is a symmetric convex body then its associated Minkowski functional is a norm. Conversely, given a norm $\|\cdot\|$ in $\mathbb{R}^{n}$, the unit ball with respect to this norm is a symmetric convex body. Recall that the unit ball of a norm $\|\cdot\|$ is defined to be

$$
\{\|x\| \leq 1\}
$$

### 2.3 Hahn-Banach theorem and the supporting hyperplanes

For $\theta \in \mathbb{S}^{n-1}$, we denote the proper subspace perpendicular to $\theta$ as

$$
\theta^{\perp}=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle=0\right\},
$$

and the affine subspace as

$$
\theta^{\perp}+t \theta=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle=t\right\} .
$$

More generally, for a proper subspace $H \subseteq \mathbb{R}^{n}$ of dimension $k$ we define

$$
H^{\perp}=\left\{x \in \mathbb{R}^{n}:\langle x, y\rangle=0 \forall y \in H\right\} .
$$

This set is a subspace of dimension $n-k$.
Definition 2.8 (supporting hyperplane of a convex body). Given a convex body $K$ and a direction $\theta \in \mathbb{S}^{n-1}$, a supporting hyperplane is the affine hyperplane $\left\{\theta^{\perp}+t \theta\right\}$, such that $K$ is fully contained in the half-space $\{\langle x, \theta\rangle \leq t\}$, while no points of $K$ are contained in the half-space $\{\langle x, \theta\rangle>t\}$.


Now, we state:
Theorem 2.9 (a version of Hahn-Banach in $\mathbb{R}^{n}$ ). Let $K$ be a convex body in $\mathbb{R}^{n}$. Then for all $\theta \in \mathbb{S}^{n-1}$ there exists $t_{\theta} \in \mathbb{R}$ such that the hyperplane $\theta^{\perp}+t_{\theta} \theta$ is the supporting hyperplane of $K$ in the direction $\theta$.

Proof. Homework!
Note that the intersection of convex sets is convex. In particular, intersection of halfspaces is always a convex set.

Remark 2.10. For any convex body $K$ we have

$$
K=\bigcap_{\theta \in \mathbb{S}^{n-1}}\left\{x:\langle x, \theta\rangle \leq t_{\theta}\right\}
$$

where $\theta^{\perp}+t_{\theta} \theta$ is the supporting hyperplane.
Let us give a name to this $t_{\theta} \ldots$

### 2.4 Support function of a convex set

Definition 2.11 (Support function of a convex body). The support function $h_{K}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ of a convex body $K \subseteq \mathbb{R}^{n}$ is defined as

$$
h_{K}(x)=\sup _{y \in K}\langle x, y\rangle
$$

Remark 2.12. Note the geometric meaning of the support function when $\theta \in \mathbb{S}^{n-1}$ is a unit vector: $h_{K}(\theta)$ is the (sometimes signed) distance from the origin to the supporting hyperplane of $K$ which is orthogonal to $\theta$.


The support function has a few nice properties:

- $h_{K}$ is 1-homogeneous: $h_{K}(t x)=t h_{K}(x)$ for all $t \geq 0$.
- For all $\theta \in \mathbb{S}^{n-1}, \theta^{\perp}+h_{K}(\theta) \theta$ is the support hyperplane for $K$.
- $h_{K}$ is a convex function.
- If $K$ is symmetric, then $h_{K}$ is a norm.


### 2.5 Duality/Polarity

Definition 2.13 (Duality/Polarity). Suppose $K$ is a set in $\mathbb{R}^{n}$. The polar of $K$ is

$$
K^{\circ}=\left\{x \in \mathbb{R}^{n}: \forall y \in K,\langle x, y\rangle \leq 1\right\}
$$

Letting the half-space $H_{y}=\left\{x \in \mathbb{R}^{n}:\langle x, y\rangle \leq 1\right\}$, we see that $K^{o}=\cap_{y \in K} H_{y}$. Therefore, the polar set is always convex (even if $K$ is not). Note also the following:

Claim 2.14. $K^{\circ}$ is the unit ball under the norm $h_{K}$. In other words, $h_{K}$ and $\|\cdot\|_{K}$ are dual norms: for all $x \in \mathbb{R}^{n}$ we have $h_{K}(x)=\|x\|_{K^{\circ}}$.

Example 2.15. For instance, a symmetric interval and a strip are polar to one another.

$$
[-a \theta, a \theta]^{\circ}=\left\{x \in \mathbb{R}^{n}:|\langle x, \theta\rangle| \leq 1 / a\right\}
$$

Definition 2.16. The convex hull of a set $\Omega \subseteq \mathbb{R}^{n}$ is

$$
\operatorname{conv} \Omega=\left\{\sum_{i=1}^{k} \lambda_{i} x^{i}: \sum_{i=1}^{k} \lambda_{i}=1, \lambda_{i} \geq 0, x^{i} \in \Omega\right\}
$$

Definition 2.17. A polytope $K \subseteq \mathbb{R}^{n}$ is the convex hull of finitely many points: There exist $\theta_{1}, \ldots, \theta_{k} \in \mathbb{S}^{n-1}, a_{1}, \ldots, a_{k} \in \mathbb{R}$, such that

$$
K=\operatorname{conv}\left\{a_{1} \theta_{1}, \ldots, a_{k} \theta_{k}\right\}
$$



Equivalently, one may define a polytope as an intersection of finitely-many half-spaces (see home work).

Remark 2.18. For a polytope $K=\operatorname{conv}\left\{a_{1} \theta_{1}, \ldots, a_{k} \theta_{k}\right\}$ we have

$$
K^{\circ}=\bigcap_{i=1}^{k}\left\{x \in \mathbb{R}^{n}:\left\langle x, \theta_{i}\right\rangle \leq 1 / a_{i}\right\}
$$



Some other examples:

- $K^{\circ}=K$ if and only if $K=B_{2}^{n}$.
- $\left(10 B_{2}^{n}\right)^{\circ}=0.1 B_{2}^{n}$.
- $\left(B_{\infty}^{n}\right)^{\circ}=B_{1}^{n}$, and more generally $\left(B_{p}^{n}\right)^{\circ}=B_{q}^{n}$ where $p^{-1}+q^{-1}=1$.

A few more definitions:
Definition 2.19 (Section of a convex body). Let $H$ be an affine $k$-dimensional hyperplane. A section of $K$ with $H$ is $K \cap H$.

Definition 2.20 (Projection of a convex body). Let $H$ be an affine $k$-dimensional hyperplane. The projection of $K$ onto $H$ is

$$
K \mid H=\left\{x \in H: \exists y \in H^{\perp}, x+y \in K\right\} .
$$

Now, some more properties of polarity:
(a) $\left(K^{\circ}\right)^{\circ}=K$.
(b) Let $T$ be an invertible linear operator. Then $(T K)^{\circ}=\left(T^{\top}\right)^{-1} K^{\circ}$. A special case is $(a K)^{\circ}=a^{-1} K^{\circ}$. Also, ellipsoids are closed under taking polarity.
(c) $(K \mid H)^{\circ} \cap H=K^{\circ} \cap H$ for all proper hyperplanes $H$.
(d) $K \subseteq L$ implies that $L^{\circ} \subseteq K^{\circ}$.

Recall the Minkowski sum of $K, L \subseteq \mathbb{R}^{n}$ is

$$
K+L=\{x+y: x \in K, y \in L\}
$$

Claim 2.21. $h_{K+L}=h_{K}+h_{L}$
Proof. Homework!
Equivalently,

$$
\rho_{K^{\circ}+L^{\circ}}=\left(\frac{1}{\rho_{K^{\circ}}}+\frac{1}{\rho_{L^{\circ}}}\right)^{-1} .
$$

Later in the course, we will discuss the following related notion:
Definition 2.22 (Legendre transform). Let $\phi: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$. The Legendre transform of $\phi$ is the function $\phi^{*}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ defined as

$$
\phi^{*}(x)=\sup _{y \in \mathbb{R}^{n}}(\langle x, y\rangle-\phi(y))
$$

The following connection between the Legendre transform and polarity is important to remember: for a convex body $K, h_{K}^{*}=\mathbb{1}_{K^{\circ}}^{\infty}$ where, as before,

$$
\mathbb{1}_{K}^{\infty}(x)= \begin{cases}0 & x \in K \\ +\infty & x \notin K\end{cases}
$$

### 2.6 Home work

Question 2.23 (1 point). Prove that for any convex body $K$ in $\mathbb{R}^{n}$ and for any point $x \in \mathbb{R}^{n} \backslash K$, there exists a vector $\theta \in \mathbb{S}^{n-1}$ and a number $\rho \in \mathbb{R}$ such that $\langle x, \theta\rangle>\rho$ and for all $y \in K,\langle y, \theta\rangle<\rho$. (this is a finite-dimensional version of the Khan-Banach Theorem)

Question 2.24 (1 point). Prove that a convex hull of a finite number of points in $\mathbb{R}^{n}$ either has an empty interior, or can be expressed as an intersection of a finite number of half spaces.

Question 2.25 (1 point). Show that the Minkowski functional of a symmetric convex body is a norm on $\mathbb{R}^{n}$.

Question 2.26 (1 point). Show that for a convex set $Q$ containing the origin we have

$$
|Q|=\frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_{Q}^{n}(\theta) d \theta=\frac{1}{n} \int_{\mathbb{S}^{n-1}}\|\theta\|_{Q}^{-n} d \theta
$$

Question 2.27 (1 point). a) Show that for any pair of convex bodies $K, L$ we have

$$
h_{K+L}(x)=h_{K}(x)+h_{L}(y) .
$$

b) Show that for $a>0, h_{K}(a x)=a h_{K}(x)=h_{a K}(x)$.
c) Pick $v \in \mathbb{R}^{n}$. Show that $h_{[-v, v]}(x)=|\langle v, x\rangle|$. Here $[-v, v]$ is the interval connecting vectors $-v$ and $v$.

## 3 Brunn-Minkowski inequality and friends

### 3.1 Brunn-Minkowski inequality and the Isoperimetric inequality

Theorem 3.1. Let $K, L \subseteq \mathbb{R}^{n}$ be Borel-measurable sets. Then

$$
|K+L|^{1 / n} \geq|K|^{1 / n}+|L|^{1 / n}
$$

Some remarks:

- Note that for $\lambda>0$ we have $|\lambda K|=\lambda^{n}|K|$. So, the theorem is equivalent to the statement that for all $\lambda \in[0,1]$,

$$
|\lambda K+(1-\lambda) L|^{1 / n} \geq \lambda|K|^{1 / n}+(1-\lambda)|L|^{1 / n}
$$

In other words, the Lebesgue measure to the power $1 / n$ is "concave" under Minkowski addition.

- The theorem admits a dimension-free reformulation: For all $\lambda \in[0,1]$,

$$
\begin{equation*}
|\lambda K+(1-\lambda) L| \geq|K|^{\lambda}|L|^{1-\lambda} . \tag{1}
\end{equation*}
$$

Note that for all $a, b \geq 0, \lambda \in[0,1], p>0$,

$$
\left(\lambda a^{p}+(1-\lambda) b^{p}\right)^{1 / p} \geq a^{\lambda} b^{1-\lambda}
$$

by which the Theorem 3.1 implies (1). The other direction is a homework problem using the homogeneity of the Lebesgue measure.

We now define log-concavity.
Definition 3.2 (Log-concave function). A function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is $\log$-concave if $\log f$ is concave:

$$
\log f(\lambda x+(1-\lambda) y) \geq \lambda \log f(x)+(1-\lambda) \log f(y)
$$

for all $x, y \in \mathbb{R}^{n}, 0 \leq \lambda \leq 1$. Equivalently,

$$
f(\lambda x+(1-\lambda) y) \geq f^{\lambda}(x) f^{1-\lambda}(y)
$$

Definition 3.3 (Log-concave measure). A measure $\mu$ on $\mathbb{R}^{n}$ is log-concave if supp $\mu$ has nonempty interior and for all Borel-measurable $K, L$ and $0 \leq \lambda \leq 1$,

$$
\mu(\lambda K+(1-\lambda) L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}
$$

Recall once again
Theorem 3.4 (the Isoperimetric inequality). For all $K \subseteq \mathbb{R}$,

$$
\frac{|\partial K|_{n-1}}{|K|^{\frac{n-1}{n}}} \geq \frac{\left|\partial B_{2}^{n}\right|_{n-1}}{\left|B_{2}^{n}\right|^{\frac{n-1}{n}}}
$$

Proof. (Using Brunn-Minkowski) We have

$$
\begin{aligned}
|\partial K|_{n-1} & =\liminf _{\varepsilon \rightarrow 0} \frac{\left|K+\varepsilon B_{2}^{n}\right|-|K|}{\varepsilon} \\
& \geq \liminf _{\varepsilon \rightarrow 0} \frac{\left(|K|^{1 / n}+\left|\varepsilon B_{2}^{n}\right|^{1 / n}\right)^{n}-|K|}{\varepsilon} \\
& =\liminf _{\varepsilon \rightarrow 0} \frac{\left(|K|^{1 / n}+\varepsilon\left|B_{2}^{n}\right|^{1 / n}\right)^{n}-|K|}{\varepsilon} \\
& =\liminf _{\varepsilon \rightarrow 0} \frac{|K|+n|K|^{\frac{n-1}{n}} \varepsilon\left|B_{2}^{n}\right|^{1 / n}+O\left(\varepsilon^{2}\right)-|K|}{\varepsilon} \\
& =\liminf _{\varepsilon \rightarrow 0} \frac{n|K|^{\frac{n-1}{n}} \varepsilon\left|B_{2}^{n}\right|^{1 / n}}{\varepsilon} \\
& =n|K|^{\frac{n-1}{n}}\left|B_{2}^{n}\right|^{1 / n} .
\end{aligned}
$$

Rearranging, we have

$$
\frac{|\partial K|_{n-1}}{|K|^{\frac{n-1}{n}}} \geq n\left|B_{2}^{n}\right|^{1 / n}=\frac{\left|\partial B_{2}^{n}\right|_{n-1}}{\left|B_{2}^{n}\right|^{\frac{n-1}{n}}}
$$

Claim 3.5. We have the equality

$$
|K+L|^{1 / n}=|K|^{1 / n}+|L|^{1 / n}
$$

if and only if $K=t L+v, t \geq 0, v \in \mathbb{R}^{n}$.

### 3.2 Proof of the Brunn-Minkowski inequality, due to Lazar Lyusternik in 1935.

Remark 3.6. Unfortunately, Lazar Lyusternik is also known for his actively negative role in the Luzin affair.

Step 1. Suppose $K$ and $L$ are coordinate boxes, i.e. $K=\left[0, a_{1}\right] \times \cdots \times\left[0, a_{n}\right], L=$ $\left[0, b_{1}\right] \times \cdots \times\left[0, b_{n}\right]$. We have

$$
K+L=\left[0, a_{1}+b_{1}\right] \times \cdots \times\left[0, a_{n}+b_{n}\right],
$$

so

$$
|K+L|=\prod_{i=1}^{n}\left(a_{i}+b_{i}\right)
$$

Hence, it will suffice to show that

$$
\prod_{i=1}^{n}\left(a_{i}+b_{i}\right)^{1 / n} \geq \prod_{i=1}^{n} a_{i}^{1 / n}+\prod_{i=1}^{n} b_{i}^{1 / n}
$$

Using AM-GM (the arithmetic-geometric mean inequality),

$$
\begin{gathered}
\prod_{i=1}^{n}\left(\frac{a_{i}}{a_{i}+b_{i}}\right)^{1 / n}+\prod_{i=1}^{n}\left(\frac{b_{i}}{a_{i}+b_{i}}\right)^{1 / n} \leq \frac{1}{n} \sum_{i=1}^{n} \frac{a_{i}}{a_{i}+b_{i}}+\frac{1}{n} \sum_{i=1}^{n} \frac{b_{i}}{a_{i}+b_{i}} \\
=\frac{1}{n} \sum_{i=1}^{n} \frac{a_{i}+b_{i}}{a_{i}+b_{i}}=1
\end{gathered}
$$

from which the claim follows.
Step 2. Suppose $K$ and $L$ are finite unions of disjoint boxes. We will proceed by induction on the total number of boxes comprising $K$ and $L$. The base case with 2 boxes was proved in step 1. So, suppose the Brunn-Minkowski inequality holds for $N$ boxes. Let $H=\theta^{\perp}+t \theta$ be a hyperplane which does not intersect at least one of the boxes that form $K$, and set

$$
H^{+}=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle>t\right\}
$$

and $H^{-}=\mathbb{R}^{n} \backslash H^{+}$. Note that on each of the sides of $H$ there is no more than $N$ boxes. Next, by shifting $L$ we may ensure that

$$
\frac{\left|K \cap H^{+}\right|}{|K|}=\frac{\left|L \cap H^{+}\right|}{|L|}=a \in[0,1] .
$$



By induction, we have

$$
\begin{aligned}
|K+L| & \geq\left|K \cap H^{+}+L \cap H^{+}\right|+\left|K \cap H^{-}+L \cap H^{-}\right| \\
& \geq\left(\left|K \cap H^{+}\right|^{1 / n}+\left|L \cap H^{+}\right|^{1 / n}\right)^{n}+\left(\left|K \cap H^{-}\right|^{1 / n}+\left|L \cap H^{-}\right|^{1 / n}\right)^{n} \\
& =\left(a^{1 / n}|K|^{1 / n}+a^{1 / n}|L|^{1 / n}\right)^{n}+\left((1-a)^{1 / n}|K|^{1 / n}+(1-a)^{1 / n}|L|^{1 / n}\right)^{n} \\
& =a\left(|K|^{1 / n}+|L|^{1 / n}\right)^{n}+(1-a)\left(|K|^{1 / n}+|L|^{1 / n}\right)^{n} \\
& =\left(|K|^{1 / n}+|L|^{1 / n}\right)^{n}
\end{aligned}
$$

which gives the result.
Step 3. The statement follows by approximation, using the definition of Borel-measurable sets.

### 3.3 Steiner symmetrization

Throughout this subsection, we assume that the sets involved are convex.
Steiner symmetrization is a technique used by Jacob Steiner to prove the isoperimetric inequality in 1837. Loosely, given a hyperplane $\theta^{\perp}$ and a set $K$, we will take every interval of $K$ along $\theta$ and shift them to be symmetric about $\theta^{\perp}$.

Definition 3.7 (Steiner symmetrization). Let $\theta \in \mathbb{S}^{n-1}$, and $K \subseteq \mathbb{R}^{n}$ Borel-measurable. The Steiner symmetrization of $K$ is

$$
S_{\theta}(K)=\bigcup_{y \in \theta^{\perp}} S_{\theta}(K \cap\{y+t \theta: t \in \mathbb{R}\})
$$

where $S_{\theta}(K \cap\{y+t \theta: t \in \mathbb{R}\})$ is an interval symmetric about $\theta^{\perp}$, contained in $\{y+t \theta: t \in \mathbb{R}\}$, and of length $|K \cap\{y+t \theta: t \in \mathbb{R}\}|_{1}$.


Claim 3.8. $|K|=\left|S_{\theta}(K)\right|$
Proof. Using Fubini,

$$
|K|=\int_{\theta^{\perp}} \int_{-\infty}^{\infty}|K \cap\{y+t \theta: t \in \mathbb{R}\}|_{1} d t=\int_{\theta^{\perp}} \int_{-\infty}^{\infty}\left|S_{\theta}(K) \cap\{y+t \theta: t \in \mathbb{R}\}\right|_{1} d t=\left|S_{\theta}(K)\right|
$$

Now we define a notion of Hausdorff distance between convex bodies.
Definition 3.9 (Hausdorff distance). The Hausdorff distance between convex bodies $K, L \subseteq$ $\mathbb{R}^{n}$ is

$$
d_{H}(K, L)=\inf \{t>0: \exists \alpha>0 \text { s.t. } K \subseteq \alpha L \subseteq t \alpha K\}
$$

## Properties of the Steiner symmetrization

- $S_{\theta}(K)$ is convex whenever $K$ is convex.
- $\operatorname{circ} K \geq \operatorname{circ} S_{\theta}(K)$ where the circum-radius

$$
\operatorname{circ} L=\inf \left\{s>0: \exists y \in \mathbb{R}^{n}, L \subseteq s B_{2}^{n}+y\right\}
$$

- inrad $K \leq \operatorname{inrad} S_{\theta}(K)$ where the inradius

$$
\operatorname{inrad} K=\sup \left\{t>0: \exists y \in \mathbb{R}^{n}, t B_{2}^{n}+y \subseteq K\right\}
$$

- $\lambda S_{\theta}(K)=S_{\theta}(\lambda K)$ for all $\lambda \geq 0$.
- $S_{\theta}$ is continuous in the Hausdorff metric.
- $S_{\theta}(K)+S_{\theta}(L) \subseteq S_{\theta}(K+L)$.
- $\left|\partial S_{\theta}(K)\right|_{n-1} \leq|\partial K|_{n-1}$.
- $\operatorname{diam} K \geq \operatorname{diam} S_{\theta}(K)$, where the diameter of a set $A$ is

$$
\operatorname{diam}(A)=\sup _{x, y \in A}|x-y|
$$

Basic trick: The distance from $\theta^{\perp}$ to the boundary of $K$ on each of the sides of $\theta$ is a concave function. Denote them, say $f, g: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. For $S_{\theta}(K)$, these functions are both $\frac{f+g}{2}$.
Claim 3.10. There exists a sequence $\left\{\theta_{k}\right\}_{k} \subseteq \mathbb{S}^{n-1}$ such that for all convex bodies $K$,

$$
K, S_{\theta_{1}}(K), S_{\theta_{2}}\left(S_{\theta_{1}}(K)\right), \ldots \rightarrow R B_{2}^{n}
$$

where $R=\frac{|K|^{1 / n}}{\left|B_{2}^{n}\right|^{1 / n}}$, and the convergence is in the Hausdorff distance.
Sketch proof. Since $K$ is compact, there exists $t>0$ such that $K \subseteq t B_{2}^{n}$. Consider the family $\Omega$ of all successive Steiner symmetrals of $K$. Note that by our properties, all of these symmetrals are also contained in $t B_{2}^{n}$. Let $r=\inf _{L \in \Omega} \operatorname{circ}(L)$ (where once again $\operatorname{circ}(L)$ stads for the circum-radius), and consider a sequence of radii $r_{k} \rightarrow r$, with the corresponding bodies $Q_{k}$.

The Blaschke selection theorem provides that for any family of convex bodies contained in $t B_{2}^{n}$, there exists a convergent subsequence. Since $\operatorname{circ}(K) \geq \operatorname{circ}\left(S_{\theta}(K)\right)$, there exists a sequence $\left\{L_{k}\right\} \subseteq \Omega$ such that $L_{k}$ converges to $L$ with $\operatorname{circ}(L)=r>0$.

We claim that $L$ is a ball. Suppose not. Then $L$ misses a cap of the ball $r B_{2}^{n}$. By compactness we may cover the boundary of the ball $r B_{2}^{n}$ with rotations of this cap, corresponding to directions $\theta_{1}, \ldots, \theta_{m}$. Then, symmetrizing $L$ with respect to $\theta_{1}, \ldots, \theta_{m}$ one may get a body with a strictly smaller in-radius, which contradicts our choice of $L$.

### 3.4 Proof of the Brunn-Minkowski inequality via Steiner symmetrizations, valid for convex bodies only

Using the properties of the Steiner symmetrization, we see

$$
|K+L|=\left|S_{\theta_{1}}(K+L)\right| \geq\left|S_{\theta_{1}}(K)+S_{\theta_{1}}(L)\right|
$$

Next, for another direction $\theta_{2}$,

$$
\left|S_{\theta_{1}}(K)+S_{\theta_{1}}(L)\right| \geq\left|S_{\theta_{2}}\left(S_{\theta_{1}}(K)+S_{\theta_{1}}(L)\right)\right| \geq\left|S_{\theta_{2}} S_{\theta_{1}}(K)+S_{\theta_{2}} S_{\theta_{1}}(L)\right| .
$$

By iterating symmetrizations so that the symmetrals of $K$ and $L$ converge to balls $R_{1} B_{2}^{n}, R_{2} B_{2}^{n}$ respectively, we obtain

$$
|K+L| \geq\left(R_{1}+R_{2}\right)^{n}\left|B_{2}^{n}\right|
$$

By the volume preservation property of the Steiner symmetrization, we see that $|K|=$ $R_{1}^{n}\left|B_{2}^{n}\right|,|L|=R_{2}^{n}\left|B_{2}^{n}\right|$, and this yields the Brunn-Minkowski inequality.

We proceed with a couple essential applications of Brunn-Minkowski inequality.

### 3.5 Mixed volumes, Minkowski's first inequality

Definition 3.11. Let $K$ and $L$ be convex bodies in $\mathbb{R}^{n}$. Define mixed volumes

$$
V_{j}(K, L)=\frac{(n-j)!}{n!}|K+t L|_{t=0}^{(j)}, \text { for } j \in\{0,1,2, \cdots, n\}
$$

We briefly outline the following facts about mixed volumes:

- When $j=0$, we have $V_{0}(K, L)=|K|$.
- $V_{j}\left(K, B_{2}^{n}\right)$ are called intrinsic volumes. When $K$ is a polytope, those are multiples of the total $j$-dimensional area of its $j$-dimensional facets.
- We have $V_{1}\left(K, B_{2}^{n}\right)=\frac{1}{n}|\partial K|$, i.e. the first intrinsic volume is proportional to the perimeter of $K$; more generally, $V_{1}(K, L)$ is called anisotropic perimeter.
- $|K+t L|$ is a polynomial in $t$ of degree $n$, called Steiner polynomial:

$$
|K+t L|=\sum_{j=0}^{n} \frac{1}{j!}|K+t L|_{t=0}^{(j)} t^{j}=\sum_{j=0}^{n} \frac{1}{j!} \frac{n!}{(n-j)!} V_{j}(K, L) t^{j}=\sum_{j=0}^{n}\binom{n}{j} V_{j}(K, L) t^{j} .
$$

- $V_{n}\left(K, B_{2}^{n}\right)$ is a multiple of so-called mean width $w(K)$ of a convex body $K$ :

$$
w(K)=\int_{\mathbb{S}^{n-1}} h_{K}(\theta) d \theta
$$

- There is a surprising symmetry property: $V_{j}(K, L)=V_{n-j}(L, K)$.
- Note that the above implies that on the plane $V_{1}(K, L)=V_{1}(L, K)$. Therefore in $\mathbb{R}^{2}$ there is only three mixed volumes for a pair $K$ and $L$, and

$$
|K+t L|=|K|+2 t V_{1}(K, L)+t^{2}|L|
$$

Lemma 3.12 (Minkowski's first inequality).

$$
V_{1}(K, L) \geq|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}
$$

Proof. Home work! Similar to how we deduced the isoperimetric inequality from BrunnMinkowski.

Remark 3.13 (concerning Minkowski's second inequality which appears in the home work). The direct application of Brunn-Minkowski inequality implies that for any pair of convex bodies $K$ and $L$, the function $|K+t L|^{\frac{1}{n}}$ is concave in $t$.

Recall that if a function $F$ is concave, then

1. $\left.F^{\prime \prime}(t)\right|_{t=0} \leq 0$;
2. $\forall x, y$, letting $F((1-t) x+t y)-(1-t) F(x)-t F(y)=\alpha(t)$, we have $\alpha(t) \geq 0$, and $\alpha(0)=0$ and therefore $\alpha^{\prime}(0) \geq 0$.

In order to obtain Minkowski's first inequality, we basically used idea (2). Your home work Question 2.15 is asking to use idea (1) to deduce some other inequality, which is called Minkowski's second.

Remark 3.14. More generally, so-called Alexandrov-Fenchel inequalities state that mixed volumes form a log-concave sequence (a log-concave sequence is a function on $\mathbb{N}$ which is log-concave). The normalization for the mixed volumes is chosen deliberately so that this happens (without any extra coefficients).

### 3.6 Brunn's concavity principle

Definition 3.15 (section function). Let $\theta \in \mathbb{S}^{n-1}$. We define the section function of a convex body $K \subset \mathbb{R}^{n}$ in the direction $\theta$, to be the following function on $\mathbb{R}$ :

$$
A_{\theta, K}(t)=\left|K \cap\left(\theta^{\perp}+t \theta\right)\right|_{n-1} .
$$



We recall that support of a function is the closure of the set of points where this function is not zero.

Theorem 3.16 (Brunn). For any convex body $K$, the function $A_{\theta, K}^{\frac{1}{n-1}}(t)$ is concave on its support.

Proof. We aim to show for any $\lambda \in[0,1]$ and any $s, t$ in the support of $A_{\theta, K}$,

$$
\left|K \cap\left(\theta^{\perp}+(\lambda s+(1-\lambda) t) \theta\right)\right|^{\frac{1}{n-1}} \geq \lambda\left|K \cap\left(\theta^{\perp}+s \theta\right)\right|^{\frac{1}{n-1}}+(1-\lambda)\left|K \cap\left(\theta^{\perp}+t \theta\right)\right|^{\frac{1}{n-1}}
$$

Since $K$ is convex, for all $x, y \in K$ we have $\lambda x+(1-\lambda) y \in K$ since $K$. Also

$$
\left(\theta^{\perp}+s \theta\right)+\left(\theta^{\perp}+t \theta\right)=\theta^{\perp}+(\lambda s+(1-\lambda) t) \theta .
$$

Therefore, we get

$$
\begin{equation*}
\lambda K \cap\left(\theta^{\perp}+s \theta\right)+(1-\lambda) K \cap\left(\theta^{\perp}+t \theta\right) \subset K \cap\left(\theta^{\perp}+(\lambda s+(1-\lambda) t) \theta\right) . \tag{2}
\end{equation*}
$$

We conclude

$$
\begin{gathered}
\left.\mid K \cap\left(\theta^{\perp}+(\lambda s+(1-\lambda) t) \theta\right)\right)\left.\right|^{\frac{1}{n-1}} \geq\left|\lambda K \cap\left(\theta^{\perp}+s \theta\right)+(1-\lambda) K \cap\left(\theta^{\perp}+t \theta\right)\right|^{\frac{1}{n-1}} \\
\geq \lambda\left|K \cap\left(\theta^{\perp}+s \theta\right)\right|^{\frac{1}{n-1}}+(1-\lambda)\left|K \cap\left(\theta^{\perp}+t \theta\right)\right|^{\frac{1}{n-1}},
\end{gathered}
$$

where the last inequality is obtained by applying the Brunn-Minkowski inequality in $\mathbb{R}^{n-1}$.

Remark 3.17. We remark that for any subspace $H$ of dimension $k$, the function $F: H^{\perp} \rightarrow$ $\mathbb{R}$ given by $F(y)=|K \cap(H+y)|^{\frac{1}{k}}$ is concave on its support. The proof is exactly the same.

### 3.7 Log-concave functions and measures, Borell's theorem and the Prekopa-Leindler inequality

Recall that a function $f$ on $\mathbb{R}^{n}$ is called log-concave if $f(\lambda x+(1-\lambda) y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$. In other words, $f(x)=e^{-V(x)}$ where $V$ is convex. Note that if a function is log-concave then its support is necessarily a convex set. Note also that if $f$ and $g$ are log-concave then $f g$ is also log-concave. Some examples of log-concave functions:

- $f(x)=1_{K}(x)$ where $K$ is a convex set
- $f(x)=e^{-\frac{x^{2}}{2}}$
- $f(x)=e^{-\|x\|_{M}^{q}} \cdot 1_{K}(x)$ for some convex sets $M$ and $K$.

Recall that we say that a measure $\mu$ on $\mathbb{R}^{n}$ is log-concave if for all Borel-measurable sets $K, L$ and any $\lambda \in[0,1]$, we have

$$
\mu(\lambda K+(1-\lambda) L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}
$$

Theorem 3.18 (Christer Borell). A measure $\mu$ on $\mathbb{R}^{n}$ is log-concave if and only if it has a density $f$ with respect to Lebesgue measure (possibly with respect to Lebesgue measure on some affine subspace), and $f$ is a log-concave function.

Remark 3.19. Borell's theorem is more general than the Brunn-Minkowski inequality: indeed, the density of Lebesgue measure is 1 , which is indeed a log-concave function, and thus Borell's theorem implies that Lebesgue measure is log-concave, which is equvivalent to the Brunn-Minkowski inequality.

But more than that, Borell's theorem implies, for instance, that for $\gamma$ (the standard Gaussian measure), $\gamma(\lambda K+(1-\lambda) L) \geq \gamma(K)^{\lambda} \gamma(L)^{1-\lambda}$.

Theorem 3.20 (Prekopa-Leindler inequality 1970, the functional version of Brunn-Minkowski). Fix $\lambda \in[0,1]$. Let $f, g, h \in L^{1}\left(\mathbb{R}^{n}\right)$. Suppose for all $x, y \in \mathbb{R}^{n}$,

$$
h(\lambda x+(1-\lambda) y) \leq \lambda f(x)+(1-\lambda) g(y)
$$

Then

$$
\int e^{-h} \geq\left(\int e^{-f}\right)^{\lambda}\left(\int e^{-g}\right)^{1-\lambda}
$$

Remark 3.21. Equivalently, let $H=e^{-h}, F=e^{-f}$ and $G=e^{-g}$. If

$$
H(\lambda x+(1-\lambda) y) \geq F(x)^{\lambda} G(x)^{1-\lambda}
$$

we get $\int H \geq\left(\int F\right)^{\lambda}\left(\int G\right)^{1-\lambda}$.

## Proof of Theorem 3.18.

- The forward direction - home work!
- The backward direction (if $f$ is a log-concave function, then $d \mu(x)=f(x) d x$ is a $\log$-concave measure) follows from Prekopa-Leindler inequality. Indeed, let $f(x)$ be some log-concave function, let $K$ and $L$ be Borel-measurable sets, and let $\lambda \in[0,1]$. We let

$$
\begin{gathered}
H(z)=f(z) 1_{\lambda K+(1-\lambda) L}(z), \\
F(x)=f(x) 1_{K}(x), \\
G(y)=f(y) 1_{L}(y) .
\end{gathered}
$$

One may check that

$$
H(\lambda x+(1-\lambda) y) \geq F(x)^{\lambda} G(x)^{1-\lambda}
$$

and by Prekopa-Leindler inequality we get $\int H \geq\left(\int F\right)^{\lambda}\left(\int G\right)^{1-\lambda}$, which amounts to

$$
\mu(\lambda K+(1-\lambda) L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}
$$

Thus Prékopa-Leindler inequality implies the main (backward) direction of Borell's theorem.

### 3.8 Proof of Prékopa-Leindler inequality

Lemma 3.22 (the layer-cake formula). Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a non-negative continuous function. Then for any measure $\mu$ on $\mathbb{R}^{n}$ we have

$$
\int_{\mathbb{R}^{n}} f(x) d \mu(x)=\int_{0}^{\infty} \mu\left(\left\{x \in \mathbb{R}^{n}: f(x)>t\right\}\right) d t
$$

Proof. One can use Fubbini's theorem in dimension $\mathbb{R}^{n+1}$ and write the measure of the sub-graph of $f$, i.e. the set $\left\{(x, t) \in \mathbb{R}^{n+1}: t \leq f(x)\right\}$, in two different ways.

Recall the alternative statement of the Prékopa-Leindler inequality.
Theorem 3.23 (a restatement of Theorem 3.20.). Fix $\lambda \in[0,1]$. Let $F, G, H$ be non-negative function in $L^{1}\left(\mathbb{R}^{n}\right)$. If, for all $x, y \in \mathbb{R}^{n}$,

$$
H(\lambda x+(1-\lambda) y) \geq F(x)^{\lambda} G(y)^{1-\lambda}
$$

then

$$
\int H \geq\left(\int F\right)^{\lambda}\left(\int G\right)^{1-\lambda}
$$

As remarked in the previous lecture, this is equivalent the the original form.
Proof. The proof is by induction on the dimension.
Step 1: $\mathbf{n}=\mathbf{1}$. First, we rewrite the left-hand side of the desired inequality using the layer-cake formula Lemma 3.22:

$$
\int_{\mathbb{R}} H(t) d t=\int_{0}^{\infty}|\{t \in \mathbb{R}: H(t)>s\}| d s
$$

Notice that

$$
\begin{equation*}
\{H>s\} \supseteq \lambda\{F>s\}+(1-\lambda)\{G>s\} . \tag{3}
\end{equation*}
$$

Using (3) together with the one-dimensional Brunn-Minkowski inequality, we get:

$$
\begin{aligned}
\int_{\mathbb{R}} H(t) d t & =\int_{0}^{\infty}|\{t \in \mathbb{R}: H(t)>s\}| d s \\
& \geq \int_{0}^{\infty}|\lambda\{F>s\}+(1-\lambda)\{G>s\}| d s \\
& \geq \lambda \int_{0}^{\infty}|\{F>s\}| d s+(1-\lambda) \int_{0}^{\infty}|\{G>s\}| d s
\end{aligned}
$$

From here, we can use the layer-cake formula Lemma 3.22 in reverse, and finish with the AM-GM inequality:

$$
\begin{aligned}
\int_{\mathbb{R}} H(t) d t & \geq \lambda \int_{0}^{\infty}|\{F>s\}| d s+(1-\lambda) \int_{0}^{\infty}|\{G>s\}| d s \\
& =\lambda \int_{\mathbb{R}} F(t) d t+(1-\lambda) \int_{\mathbb{R}} G(t) d t \\
& \geq\left(\int_{\mathbb{R}} F(t) d t\right)^{\lambda}\left(\int_{\mathbb{R}} G(t) d t\right)^{1-\lambda} .
\end{aligned}
$$

Step 2: induction. We proceed by induction on the dimension. Assume inductively that $n>1$ and that the claim has been verified for smaller dimensions. Define the one-dimensional function

$$
H_{n}\left(x_{n}\right):=\int_{\mathbb{R}^{n-1}} H\left(x_{1}, \ldots, x_{n}\right) d x_{1} \cdots d x_{n-1}
$$

and define $F_{n}$ and $G_{n}$ similarly. Fubini-Tonelli implies that $F_{n}, G_{n}, H_{n} \in L^{1}(\mathbb{R})$. Also, for fixed $x_{n}, y_{n} \in \mathbb{R}$, we can still see that

$$
H\left(\lambda\left(\bar{x}, x_{n}\right)+(1-\lambda)\left(\bar{y}, y_{n}\right) \geq F\left(\bar{x}, x_{n}\right)^{\lambda} G\left(\bar{y}, y_{n}\right)^{1-\lambda}\right.
$$

where $\bar{x}=\left(x_{1}, \ldots, x_{n-1}\right)$ and $\bar{y}=\left(y_{1}, \ldots, y_{n-1}\right)$. When viewed as functions only of $\bar{x}$ and $\bar{y}$, the induction hypothesis implies that

$$
H_{n}\left(\lambda x_{n}+(1-\lambda) y_{n}\right) \geq F_{n}\left(x_{d}\right)^{\lambda} G\left(y_{d}\right)^{1-\lambda}
$$

We now apply the one-dimensional version of the Prékopa-Leindler inequalit and obtain

$$
\int_{\mathbb{R}} H_{n}(t) d t \geq\left(\int_{\mathbb{R}} F_{n}(t) d t\right)^{\lambda}\left(\int_{\mathbb{R}} G_{n}(t) d t\right)^{1-\lambda}
$$

and another application of Fubini's theorem gives the desired result.

### 3.9 Log-concavity of marginals and convolutions of log-concave measures

Definition 3.24 (marginal). Let $\mu$ be a Borel measure on $\mathbb{R}^{n}$, and $H$ a subspace of $\mathbb{R}^{n}$ or dimension $k$. Define $\pi_{\mu}(H)$ (the marginal measure of $\mu$ ) to be the measure on $H^{\perp}$ given by

$$
\pi_{\mu}(H)(\Omega)=\int_{\Omega} \int_{H} d \mu(x+y), \text { for } \Omega \subseteq H^{\perp}
$$

Proposition 3.25. If $\mu$ has a density $f$, and $H$ is a subspace, then $\pi_{\mu}(H)$ has density on $H^{\perp}$ given by the section function:

$$
f_{\mu, H}(x)=\int_{x+H} f(y) d y
$$

Example 3.26. Let $A \subseteq \mathbb{R}^{n}$ be a Borel set, and consider $\mu$ to be the uniform distribution over $A$. Then, for any subspace $H$ of dimension $k$, the marginal measure $\pi_{\mu}(H)$ has density

$$
f_{\mu, H}(x)=|A \cap(x+H)|_{k}
$$

We can also give a probabilistic meaning to the marginal measures. Suppose that we have a random vector $X$ which is distributed according to $\mu$. Then the projection of $X$ onto $H^{\perp}\left(\operatorname{proj}\left(X \mid H^{\perp}\right)=Y\right.$ where $Y \in H^{\perp}$ and $\left.X-Y \in H\right)$ will be distributed according to the marginal measure $\pi_{\mu}(H)$.

As a consequence of Prékopa-Leindler inequality, we get the following fact:

Theorem 3.27. If $\mu$ is a log-concave measure on $\mathbb{R}^{n}$ and $H$ is a subspace, then the marginal measure $\pi_{\mu}(H)$ is also a log-concave measure on $H^{\perp}$.

Proof. Let $f$ be the density for the measure $\mu$, and $f_{H}(x)=\int_{x+H} f(z) d z$ be the density for the marginal measure $\pi_{\mu}(H)$. By making an affine change of variables we can also write that $f_{H}(x)=\int_{H} f(x+z) d z$. Our final goal is to show that $f_{H}\left(\lambda x+(1-\lambda y) \geq f_{H}(x)^{\lambda} f_{H}(y)^{1-\lambda}\right.$ for all $x, y \in H^{\perp}$ and $\lambda \in[0,1]$. For now, fix $x, y \in H^{\perp}$ and $\lambda \in[0,1]$, and we define three functions $A, B, C: H \rightarrow \mathbb{R}$ as follows:

$$
\begin{aligned}
& A(z)=f(x+z) \\
& B(z)=f(y+z) \\
& C(z)=f(\lambda x+(1-\lambda) y+z)
\end{aligned}
$$

Now we can verify that these three functions satisfy the hypothesis of Prékopa-Leindler. For $u, v \in H$

$$
\begin{aligned}
C(\lambda u+(1-\lambda) v) & =f(\lambda(x+u)+(1-\lambda)(y+v)) \\
& \geq f(x+u)^{\lambda} f(y+v)^{1-\lambda} \\
& =A(u)^{\lambda} B(v)^{1-\lambda},
\end{aligned}
$$

where in the middle we use the assumption that $f$ is log-concave. Thus, we can apply Prékopa-Leindler and conclude that

$$
\begin{aligned}
f_{H}(\lambda x+(1-\lambda) y) & =\int_{H} C(z) d z \\
& \geq\left(\int_{H} A(u) d u\right)^{\lambda}\left(\int_{H} B(v) d v\right)^{1-\lambda} \\
& =f_{H}(x)^{\lambda} f_{H}(y)^{1-\lambda},
\end{aligned}
$$

which shows that the marginal measure is also log-concave.
By looking at convolutions as an affine marginal in a higher-dimensional space, we get the following corollary.

Corollary 3.28. If $f$ and $g$ are log-concave functions, then the convolution is also logconcave.

Proof. Fill in the details: home work.
We conclude that the class of log-concave random vectors is closed under sums (convolutions) and projections (marginals).

### 3.10 Borell-Brascamp-Lieb inequality

Definition 3.29. We say a function $F$ on $\mathbb{R}^{n}$ is $p$-concave if $F^{p}$ is concave.
Theorem 3.30 (Borell-Brasscamp-Lieb). Suppose $p \in(-1 / n, \infty)$ and $f, g, h \geq 0$ and fix $\lambda \in[0,1]$. If $p \geq 0$ and $h(\lambda x+(1-\lambda y) \geq \lambda f(x)+(1-\lambda) g(y)$ then

$$
\left(\int h^{1 / p}\right)^{\frac{p}{n p+1}} \geq \lambda\left(\int f^{1 / p}\right)^{\frac{p}{n p+1}}+(1-\lambda)\left(\int g^{1 / p}\right)^{\frac{p}{n p+1}} .
$$

If $p<0$ and $h(\lambda x+(1-\lambda y) \leq \lambda f(x)+(1-\lambda) g(y)$ then

$$
\left(\int h^{1 / p}\right)^{\frac{p}{n p+1}} \leq \lambda\left(\int f^{1 / p}\right)^{\frac{p}{n_{p+1}}}+(1-\lambda)\left(\int g^{1 / p}\right)^{\frac{p}{n p+1}} .
$$

Using this theorem in place of the Prékopa-Leindler inequality we can get similar corollaries.

Corollary 3.31. If $F$ is $p$-concave for $p \geq-1 / n$ then its $k$-dimensional marginals are $\frac{p}{k p+1}$-concave.

Note that when $p \nearrow 0$ we recover the Prékopa-Leindler inequality, and when $p \rightarrow-1 / n$ the result becomes a tautology.

### 3.11 Linearizations of geometric and functional inequalities

Idea: Suppose that $\mathcal{F}$ is a functional on some reasonable class of functions, such that $\mathcal{F}$ is concave:

$$
\mathcal{F}((1-t) f+t g) \geq(1-t) \mathcal{F}(f)+t \mathcal{F}(g)
$$

Then, for fixed $f$ and $g$, we can define the univariate function

$$
\alpha(t)=\mathcal{F}((1-t) f+t g)-(1-t) \mathcal{F}(f)-t \mathcal{F}(g),
$$

and notice the following properties.

1. $\alpha(t) \geq 0$ on $[0,1]$ and $\alpha(0)=0$, which implies $\alpha^{\prime}(0) \geq 0$;
2. $\alpha^{\prime \prime}(0) \leq 0$ and $\frac{d^{2}}{d t^{2}} \mathcal{F}(f+t g) \leq 0$,
3. If $\mathcal{F}(f) \leq \mathcal{F}\left(f_{0}\right)$ then $\left.\frac{d}{d \epsilon} \mathcal{F}\left(f_{0}+\epsilon f\right)\right|_{\epsilon=0}=0$, and $\left.\frac{d^{2}}{d \epsilon^{2}} \mathcal{F}\left(f_{0}+\epsilon f\right)\right|_{\epsilon=0} \leq 0$.

Our first goal will be to understand in what sense the Prékopa-Leindler inequality can be understood as a statement about concave functionals. The hypothesis of Prékopa-Leindler is that

$$
h((1-t) x+t y) \leq(1-t) f(x)+t g(y) .
$$

What is the best possible $h$ in this inequality?

Definition 3.32 (Infimal convolution). Given functions $f, g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and $t \in[0,1]$ we define the infimal convolution

$$
f \square_{t} g(z)=\inf _{(1-t) x+t y=z}\{(1-t) f(x)+t g(y)\} .
$$

We will also write

$$
f \square g(z)=\inf _{x+y=z}\{f(x)+g(y)\} .
$$

Note that $f \square_{t} g=h$ satisfies the condition, so Prékopa-Leindler implies that

$$
\int e^{f \square_{t} g} \geq\left(\int e^{-f}\right)^{1-t}\left(\int e^{-g}\right)^{t}
$$

We also can note that $f \square_{0} g=f$ and $f \square_{1} g=g$, so in some sense the infimal convolution interpolates between the two functions $f$ and $g$. What we want, is to find a functional $\mathcal{F}$ such that

$$
\int e^{-\mathcal{F}((1-t) f+t g)} \geq\left(\int e^{-\mathcal{F}(f)}\right)^{1-t}\left(\int e^{-\mathcal{F}(g)}\right)^{t}
$$

Example 3.33. When $K, L$ are convex bodies, consider the infmal convolution of their convex indicator functions:

$$
\mathbb{1}_{K}^{\infty} \square \mathbb{1}_{L}^{\infty}(z)=\inf _{x+y=z}\left\{\mathbb{1}_{K}^{\infty}(x)+\mathbb{1}_{L}^{\infty}(y)\right\}=\mathbb{1}_{K+L}^{\infty}(z)
$$

Recall that $h_{K+L}=h_{K}+h_{L}$, so our goal will be to find a functional $\mathcal{F}$ such that $\mathcal{F}(f \square g)=\mathcal{F}(f)+\mathcal{F}(g)$, and a good hint for what we are looking for is that we want $\mathcal{F}: \mathbb{1}_{K}^{\infty} \mapsto h_{K}$.

### 3.12 Legendre Transform

Recall the definition
Definition 3.34. For $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}=\mathbb{R} \cup\{\infty\}$ we define

$$
f^{*}(x)=\sup _{y \in \mathbb{R}^{n}}\{\langle x, y\rangle-f(y)\}
$$

Example 3.35. 1. For a convex body $K$

$$
\begin{aligned}
\left(\mathbb{1}_{K}^{\infty}\right)^{*}(x) & =\sup _{y \in \mathbb{R}^{n}}\left\{\langle x, y\rangle-\mathbb{1}_{K}^{\infty}(y)\right\} \\
& =\sup _{y \in K}\{\langle x, y\rangle\} \\
& =h_{K}(x) .
\end{aligned}
$$

2. Suppose $f(x)=|x|: \mathbb{R}^{n} \rightarrow \mathbb{R}$. Then

$$
\begin{aligned}
f^{*}(x) & =\sup _{y \in \mathbb{R}^{n}}\{\langle x, y\rangle-|y|\} \\
& =\sup _{t \geq 0}\{t|x|-t\} \\
& =\left\{\begin{array}{ll}
0 & \text { if }|x| \leq 1 \\
\infty & \text { if }|x|>1
\end{array}=\mathbb{1}_{B_{2}^{n}}^{\infty}(x) .\right.
\end{aligned}
$$

3. When if $f^{\star}=f$ ? This happens when $f(x)=\frac{1}{2}|x|^{2}$. Indeed

$$
\begin{aligned}
f^{*}(x) & =\sup _{y \in \mathbb{R}^{n}}\left\{\langle x, y\rangle-|y|^{2} / 2\right\} \\
& =\sup _{t \geq 0}\left\{t|x|^{2}-t^{2}|x|^{2} / 2\right\} .
\end{aligned}
$$

The function inside the supremum is a quadratic in $t$ and is maximized when $t=1$. Thus

$$
f^{*}(x)=|x|^{2} / 2
$$

4. Consider $f(x)=C$ a constant function. Then

$$
f^{*}(x)=\sup _{y \in \mathbb{R}^{n}}\{\langle x, y\rangle-C\}=\infty
$$

Note that this does not depend on the choice of constant $C$.
5. Consider $f(y)= \begin{cases}-\sqrt{1-|y|^{2}} & \text { if }|y| \leq 1 \\ \infty & \text { if }|y|>1 .\end{cases}$

$$
\begin{aligned}
f^{*}(x) & =\sup _{y \in \mathbb{R}^{n}}\{\langle x, y\rangle-f(y)\} \\
& =\sup _{|y| \leq 1}\left\{\langle x, y\rangle+\sqrt{1-|y|^{2}}\right\} \\
& =\sup _{t \in\left[0,|x|^{-1}\right]}\left\{t|x|^{2}+\sqrt{1-t^{2}|x|^{2}}\right\} .
\end{aligned}
$$

Once, again, we can try to optimize by hand:

$$
\frac{d}{d t}\left(t|x|^{2}+\sqrt{1-t^{2}|x|^{2}}\right)=|x|^{2}-\frac{t|x|^{2}}{\sqrt{1-t^{2}|x|^{2}}}
$$

This is zero when

$$
\frac{t}{\sqrt{1-t^{2}|x|^{2}}}=1
$$

which rearranges to $t=\frac{1}{\sqrt{1+|x|^{2}}}$ (which is in the correct range). Now we can plug this back in to see that

$$
\begin{aligned}
f^{*}(x) & =\frac{|x|^{2}}{\sqrt{1+|x|^{2}}}+\sqrt{1-\frac{|x|^{2}}{1+|x|^{2}}} \\
& =\frac{|x|^{2}}{\sqrt{1+|x|^{2}}}+\sqrt{\frac{1+|x|^{2}-|x|^{2}}{1+|x|^{2}}} \\
& =\frac{|x|^{2}+1}{\sqrt{1+|x|^{2}}}=\sqrt{|x|^{2}+1}
\end{aligned}
$$

which is the top branch of a hyperbola with asymptotes $y= \pm x$.
Lemma 3.36 (Properties of the Legendre transform). There are a few important properties of the Legendre transform:

1. For any $\phi: \mathbb{R}^{n} \rightarrow \mathbb{R}, \phi^{*}$ is convex (since it is the supremum of linear functions.)
2. If $\phi$ is convex then $\left(\phi^{*}\right)^{*}=\phi$.
3. For any $a \in \mathbb{R},(\phi+a)^{*}(x)=\phi^{*}(x)-a$.
4. $f(x) \leq g(x)$ implies that $f^{*}(x) \leq g^{*}(x)$.
5. $(a f)^{*}(x)=a f^{*}\left(\frac{x}{a}\right)$.

Proof. Home work!
We outline another key property:
Proposition 3.37 (Legendre transform linearizes infimal convolution). For $f, g$ convex

$$
(f \square g)^{*}=f^{*}+g^{*} .
$$

Proof. This is a direct calculation:

$$
\begin{aligned}
(f \square g)^{*}(x) & =\sup _{y \in \mathbb{R}^{n}}\{\langle x, y\rangle-f \square g(y)\} \\
& =\sup _{y \in \mathbb{R}^{n}}\left\{\langle x, y\rangle-\inf _{a, b, a+b=y}\{f(a)+g(b)\}\right\} \\
& =\sup _{a, b \in \mathbb{R}^{n}}\{\langle x, a+b\rangle-f(a)-g(b)\} \\
& =f^{*}(x)+g^{*}(x)
\end{aligned}
$$

Remark 3.38. Consider what happens to the epigraph when taking the infimal convolution.
The following are further important properties of the Legendre transform.
Proposition 3.39 (Legendre transform of smooth functions). Let $V$ be a strictly convex $C^{2}$ function $\mathbb{R}^{n} \rightarrow \mathbb{R}$.

1. $V(x)+V^{*}(\nabla V(x))=\langle x, \nabla V(x)\rangle$,
2. $\nabla V\left(\nabla V^{*}(x)\right)=x$, in other words $\nabla V \circ \nabla V^{*}=I d$.
3. $\nabla^{2} V^{*}(\nabla V(x))=\left(\nabla^{2} V\right)^{-1}(x)$.

Proof. 1. For all $x, y \in \mathbb{R}^{n}$ we know that $V(x)+V(y) \geq\langle x, y\rangle$, just from the definition of Legendre transform. Now

$$
V^{*}(y)=\sup _{z \in \mathbb{R}^{n}}\{\langle y, z\rangle-V(z)\}
$$

where at teh optimal point, $\nabla_{z}[\langle y, z\rangle-V(z)]=0$. This implies that the optimal $y=\nabla V(z)$, and equality holds.
2. Take the gradient of both sides in 1 .

$$
\nabla V(x)+\nabla^{2} V(x) \cdot \nabla V^{*}(\nabla V(x))=\nabla V(x)+\nabla^{2} V(x) \cdot x
$$

Rearranging and cancelling terms using the fact that $V$ strictly convex implies $\nabla^{2} V$ is invertable, and we conclude

$$
\nabla V^{*}(\nabla V(x))=x
$$

3. This is equivalent to 2 . since the Hessian is the Jacobian of the gradient map.

Remark 3.40. Note that the relation (2) from Proposition 3.39 implies that $V^{* *}=V$ (i.e. property (2) from Lemma 3.36) under the assumption that $V$ is $C^{2}$ and only takes finite values.

### 3.13 Generalized Log-Sobolev Inequality

Recall that Prékopa-Leindler inequality can be written as

$$
\int e^{-f \square_{t} g} \geq\left(\int e^{-f}\right)^{t}\left(\int e^{-g}\right)^{1-t}
$$

By replacing functions with their Legendre transform we get the following inequality (for which, in fact, convexity is not needed):

$$
\int e^{-(t f+(1-t) g)^{*}} \geq\left(\int e^{-f^{*}}\right)^{t}\left(\int e^{-g^{*}}\right)^{1-t}
$$

This convenient formulation was noted by Cordero-Erausquin and Klartag [51]. In other words, $\log \int e^{-f^{*}}$ is a concave functional on the space of reasonable functions (for which the corresponding integrals exist).

Remark 3.41. Note also the dual fact: the functional $\log \int e^{-f}$ is convex by Hölder's inequality.

Now, let

$$
\alpha(t)=\log \int e^{-((1-t) f+t g)^{*}}-(1-t) \log \int e^{-f^{*}}-t \log \int e^{-g^{*}}
$$

We know that $\alpha(t) \geq 0$, and $\alpha(0)=0$, so we can conclude that $\alpha^{\prime}(0) \geq 0$. In order to compute what this means, we shall need:

Lemma 3.42. Let $V_{t}(x)$ be a family of functions on $\mathbb{R}^{n}$ for $t \in[0,1]$ such that $V_{t} \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ and $V_{t}(x)$ is convex for each $t$. Then

$$
\begin{aligned}
\frac{d}{d t} V_{t}^{*}(x) & =-\dot{V}_{t}\left(\nabla V_{t}^{*}(x)\right) \\
\frac{d^{2}}{d t^{2}} V_{t}^{*}(x) & =-\ddot{V}_{t}\left(\nabla V_{t}^{*}(x)\right)+\left\langle\left(\nabla^{2} V_{t}(x)\right)^{-1} \nabla\left[\left.\dot{V}_{t}\right|_{\nabla V_{t}^{*}(x)}\right], \nabla\left[\left.\dot{V}_{t}\right|_{\nabla V_{t}^{*}(x)}\right]\right\rangle
\end{aligned}
$$

Proof. We will only show the first identity

$$
\frac{d}{d t} V_{t}^{*}(x)=-\dot{V}_{t}\left(\nabla V_{t}^{*}(x)\right)
$$

as the second identity will appear in one homework problem (Question 4.12).
Recall the following duality formula for Legendre transform:

$$
V_{t}(x)+V_{t}^{*}\left(\nabla V_{t}\right)=\left\langle\nabla V_{t}, x\right\rangle
$$

Differentiating with respect to $t$ on both sides, we get

$$
\dot{V}_{t}(x)+\frac{d}{d t} V_{t}^{*}\left(\nabla V_{t}\right)+\left\langle\nabla V_{t}^{*}\left(\nabla V_{t}\right), \nabla \dot{V}_{t}\right\rangle=\left\langle\nabla \dot{V}_{t}, x\right\rangle
$$

Now using $\nabla V_{t}^{*} \circ \nabla V_{t}=x$, the above identity is rewritten as

$$
\dot{V}_{t}(x)+\frac{d}{d t} V_{t}^{*}\left(\nabla V_{t}\right)+\left\langle x, \nabla \dot{V}_{t}\right\rangle=\left\langle\nabla \dot{V}_{t}, x\right\rangle
$$

which yields

$$
\frac{d}{d t} V_{t}^{*}\left(\nabla V_{t}\right)=-\dot{V}_{t}(x)
$$

It remains to set $y=\nabla V_{t}$ and use $\nabla V_{t}^{*} \circ \nabla V_{t}=x$ again to complete the proof.

We deduce:

$$
\begin{aligned}
\alpha^{\prime}(0) & =\frac{1}{\int e^{-f^{*}}} \cdot \int-e^{-f^{*}} \cdot \frac{d}{d t}\left(\left.((1-t) f+t g)^{*}\right|_{t=0}+\log \frac{\int e^{-f^{*}}}{\int e^{-g^{*}}}\right. \\
& =\frac{1}{\int e^{-f^{*}}} \cdot \int-e^{-f^{*}} \cdot(f-g)\left(\nabla f^{*}\right)+\log \frac{\int e^{-f^{*}}}{\int e^{-g^{*}}} .
\end{aligned}
$$

When $f$ and $g$ are convex, we let $F=f^{*}$ and $G=g^{*}$, and using the fact that $\alpha^{\prime}(0) \geq 0$ we get the inequality

$$
\int-e^{-F} \cdot\left(F^{*}-G^{*}\right)(\nabla F)+\int e^{-F} \cdot \log \frac{\int e^{-F}}{\int e^{-G}} \geq 0
$$

This can be seen as a version of Minkoswki's first inequality
Theorem 3.43 (Minkowski's first inequality for functions). Suppose $F, G$ are convex and $\int e^{-F}=\int e^{-G}$. Then

$$
\int G^{*}(\nabla F) e^{-F} \geq \int F^{*}(\nabla F) e^{-F}
$$

and the left-hand side is minimized when $G=F$.
Remark 3.44. Compare this to the inequality $V_{1}(K, L) \geq|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}$ in the case that $|K|=$ $|L|$.

Continuing with our previous inequality before the theorem. We can use the lemma to make some substitutions:

$$
\int F^{*}(\nabla F) e^{-F}=\int(\langle\nabla F, x\rangle-F(x)) e^{-F}
$$

Note that $\int\langle\nabla F(x), x\rangle e^{-F}=-\int\left\langle\nabla e^{-F}, x\right\rangle$, so we use integration by parts

$$
\int\langle\nabla F(x), x\rangle e^{-F}=-\int\left\langle\nabla e^{-F}, x\right\rangle=\int e^{-F} \cdot \Delta \frac{x^{2}}{2}=n \int e^{-F} .
$$

Plugging all this back in, we get the following inequality
Theorem 3.45 (Generalized log-Sobolev inequality). If $F, G$ are convex functions, then

$$
\int G^{*}(\nabla F) e^{-F} \geq n \int e^{-F}-\int F e^{-F}+\int e^{-F} \log \frac{\int e^{-F}}{\int e^{-G}}
$$

Corollary 3.46. If $F, G$ are convex functions and $\int e^{-F}=\int e^{-G}$, then

$$
\int G^{*}(\nabla F) e^{-F} \geq n \int e^{-F}-\int F e^{-F}
$$

Remark 3.47. For the first time, the derivation of the Log-Sobolev inequality using linearization of Prékopa-Leindler inequality was done by Bobkov and Ledoux.
Remark 3.48. Recall that for any function $F$ that is nice enough (convex, smooth and does not take infinity values), $\nabla F^{*} \circ \nabla F=x$. Note that this in fact already implies $F^{* *}=F$ for such functions.

### 3.14 Reformulations and notable partial cases of the Generalized Log-Sobolev Inequality

We continue with a few remarks on Theorem 3.45:
Remark 3.49. The equality holds if $F=G$.
Remark 3.50. Theorem 3.43 (equivalent to Theorem 3.45) implies Minkowski's first inequality Lemma 3.12. To see this, let $F=\mathbb{1}_{K}^{\infty}(x)$ for some set $K$ and $G=\mathbb{1}_{L}^{\infty}(x)$ for some set $L$. Then $\int e^{-F}=\int e^{-G}$ is equivalent to $|K|=|L|$. We get, using that $G^{*}=h_{L}$ :

$$
\int_{\mathbb{R}^{n}} G^{*}(\nabla F) e^{-F}=\int_{K} G^{*}(\nabla F)=\int_{K} h_{L}\left(\nabla \mathbb{1}_{K}^{\infty}\right)=\int_{\partial K} h_{L}\left(n_{x}\right),
$$

where $n_{x}$ is the outer unit normal vector to $\partial K$. We have seen in a homework problem that the last quantity is actually proportional to the mixed volume $n V_{1}(K, L)=|K+t L|_{t=0}^{\prime}$. By Theorem 3.43, we get

$$
|K+t L|_{t=0}^{\prime} \geq|K+t K|_{t=0}^{\prime}
$$

which is precisely $V_{1}(K, L) \geq|K|=|K|^{\frac{n-1}{n}} \cdot|L|^{\frac{1}{n}}$, in view of the volume restriction.
Now let us state a reformulation of Theorem 3.45. Consider $\phi=e^{-F}$ for some convex $F$. Then $F=-\log \phi$ and $\nabla F=-\nabla \phi / \phi$. Assume $G$ is convex and $\int e^{-G}=1$, then Theorem 3.45 can be rewritten as

$$
\int G^{*}\left(-\frac{\nabla \phi}{\phi}\right) \phi \geq n \int \phi+\int \phi \log \phi-\int \phi \log \int \phi .
$$

Definition 3.51 (Entropy). If $d \mu$ is a measure on $\mathbb{R}^{n}$, then the entropy with respect to measure $\mu$ of a function $\phi$ is defined as

$$
\operatorname{Ent}_{\mu}(\phi):=\int \phi \log \phi d \mu-\left(\int \phi d \mu\right) \log \left(\int \phi d \mu\right) .
$$

When $\mu$ is Lebesgue we write $\operatorname{Ent}(\phi)$ for simplicity.
Remark 3.52. By Jensen's inequality, we get, using that $t \log t$ is convex: $\operatorname{Ent}_{\mu}(\phi) \geq 0$ for any probability measure $\mu$.

Theorem 3.53 (Reformulation of the Generalized Log-Sobolev Inequality). For any logconcave function $\phi$ and convex function $G$ with $\int e^{-G}=1$,

$$
\int G^{*}\left(-\frac{\nabla \phi}{\phi}\right) \phi \geq n \int \phi+\operatorname{Ent}(\phi)
$$

We derive, by plugging in $G^{*}(x)=|x|-\log \left|B_{2}^{n}\right|$ :

Corollary 3.54 ( $L_{1}$-Sobolev inequality, Bobkov-Ledoux, 2000). For log-concave function $\phi$,

$$
\operatorname{Ent}(\phi)+C_{n} \int \phi \leq \int|\nabla \phi|
$$

where $C_{n}=n+\log \left|B_{2}^{n}\right|$.
Next, we get, by plugging in $G^{*}(x)=\frac{|x|^{2}}{2}-n \log \sqrt{2 \pi}$ :
Corollary 3.55 (Classical Lebesgue Log-Sobolev inequality, first form).

$$
\text { (1st form) } \quad \operatorname{Ent}(\phi)+n \log (\sqrt{2 \pi} e) \int \phi \leq \frac{1}{2} \int \frac{|\nabla \phi|^{2}}{\phi}
$$

Note that $\int \frac{|\nabla \phi|^{2}}{\phi}$ is called the (Lebesgue) Fisher information. By substituting $\phi=f^{2}$, one can also write

Corollary 3.56 (Classical Lebesgue Log-Sobolev inequality, second form).

$$
\text { (2nd form) } \quad \operatorname{Ent}\left(f^{2}\right)+n \log (\sqrt{2 \pi} e) \int f^{2} \leq 2 \int|\nabla f|^{2} \text {. }
$$

Remark 3.57. In the Corollaries 3.55 and 3.56 one does not need to assume that $\phi$ is log-concave - see home work.

We also note by plugging $G^{*}=|x|^{p}+C(n, p)$ :
Corollary 3.58 ( $L_{p}$-Sobolev inequality).

$$
\operatorname{Ent}\left(f^{p}\right)+C_{n, p} \int f^{p} \leq p^{p-1} \int|\nabla f|^{p},
$$

where $C_{n, p}=n-\log \left(\left|S^{n-1}\right| \cdot \int_{0}^{\infty} t^{n-1} e^{-t^{p} / q} d t\right), 1 / p+1 / q=1$.
Finally, we formulate
Theorem 3.59 (Gaussian Log-Sobolev inequality). Let d $\gamma$ be the standard Gaussian measure in $\mathbb{R}^{n}$, and let $g \in W^{1,2}(d \gamma)$, then

$$
E n t_{\gamma}\left(g^{2}\right) \leq 2 \int|\nabla g|^{2} d \gamma
$$

Remark 3.60. Log-concavity assumption on $g$ is not needed; see home work.
Remark 3.61. Theorem 3.59 is equivalent to the Corollary 3.56 by choosing $g=(2 \pi)^{\frac{n}{4}} e^{\frac{|x|^{2}}{4}} f$. See home work.

Remark 3.62. $g=1$ gives the equality case in Theorem 3.59.

Remark 3.63. Theorem 3.59 (or Corollary 3.55, since they are equivalent) implies the Lebesgue Sobolev inequality:

$$
n\left|B_{2}^{n}\right|^{\frac{1}{n}}\left(\int_{\mathbb{R}^{n}}|f|^{\frac{n-1}{n}} d x\right)^{\frac{n}{n-1}} \leq \int_{\mathbb{R}^{n}}|\nabla f| d x
$$

which holds for all smooth $f$ such that the integral converges.
Remark 3.64. The inequality also holds on the sphere $S^{n-1}$, which actually implies Theorem 3.59:

$$
\int_{S^{n-1}} f^{2} \log f^{2}-\int_{S^{n-1}} f^{2} \log \int_{S^{n-1}} f^{2} \leq 2 \int_{S^{n-1}}\left|\nabla_{S^{n-1}} f\right|^{2}
$$

Lastly, we mention the following fact (whose prove is left as a home work problem):
Theorem 3.65 (Generalized Log-Sobolev inequality for log-concave measures). Let $d \mu=$ $e^{-V} d x$ be a log-concave measure and $F, G$ be convex functions such that $\int e^{-G} d \mu=1$. Then

$$
\operatorname{Ent}_{\mu}\left(e^{-F}\right)+n \int e^{-F} d \mu-\int\langle\nabla V, x\rangle e^{-F} d \mu \leq \int G^{*}(\nabla F) e^{-F} d \mu
$$

Proof. Home work!

### 3.15 The $p$-Beckner Inequality

We mention, without proof:
Theorem 3.66 ( $p$-Beckner inequality). For $f \in W^{1,2}\left(\mathbb{R}^{n}, \gamma\right)$ and $p \in[1,2)$,

$$
\int f^{2} d \gamma-\left(\int|f|^{p} d \gamma\right)^{\frac{2}{p}} \leq(2-p) \int|\nabla f|^{2} d \gamma
$$

This result implies the so-called Gaussian Poincare's inequality when $p=1$ :

$$
\int f^{2} d \gamma-\left(\int f d \gamma\right)^{2} \leq \int|\nabla f|^{2} d \gamma
$$

We will formally prove this fact soon.
Also, Beckner's inequality implies the Gaussian Log-Sobolev: one can obtain

$$
\operatorname{Ent}_{\gamma}(f) \leq 2 \int|\nabla f|^{2} d \gamma
$$

by letting $p \rightarrow 2$ in Theorem 3.66 and taking the derivative; see home work.
Remark 3.67. The $p$-Beckner inequality is stronger when $p$ is bigger. In other words, the Gaussian Log-Sobolev Inequality is the strongest inequality in the family of all p-Beckner inequalities.

Remark 3.68. The inequality is also known to hold on the sphere $S^{n-1}$.
Question (Siva): Is there a Lebesgue Beckner inequality?
Question 2 (Siva): What will happen if we plug $\phi=f^{2} e^{-V}$ into Theorem 8.4?
Below we fix a log-concave probability measure $d \mu=e^{-G} d x$ on $\mathbb{R}^{n}$ : that is, we assume that $G$ is a smooth convex function and $\int e^{-G}=1$.

Definition 3.69 (Variance). The variance of a function $\phi$ w.r.t. $\mu$ is defined as

$$
\operatorname{Var}_{\mu}(\phi):=\int \phi^{2} d \mu-\left(\int \phi d \mu\right)^{2}
$$

Note that variance of a constant function is zero. Also, variance is invariant under adding constants to $\phi$. In a sense, variance measures "how far is $\phi$ from a constant function".

### 3.16 A few words about the Laplace operator with respect to logconcave measures

Definition 3.70 (Laplace operator associated to $\mu$ ). For a "reasonable" function $u$ (to be discussed in more detail in November),

$$
L_{\mu} u:=\Delta u-\langle\nabla G, \nabla u\rangle .
$$

Example 3.71. 1. If $\mu$ is Lebesgue then $L_{\mu} u=\Delta u$.
2. If $\mu$ is Gaussian then $L_{\mu} u=\Delta u-\langle x, \nabla u\rangle$, which is called the Ornstein-Uhlenbeck operator.

Lemma 3.72 (Integration by parts). If $u, v \in C^{2}\left(\mathbb{R}^{n}\right)$ such that the integrals converge, then

$$
\int u L_{\mu} v d \mu=-\int\langle\nabla u, \nabla v\rangle d \mu
$$

Proof. Using the classical integration by parts $\int f \Delta g=-\int\langle\nabla f, \nabla g\rangle$, we get

$$
\begin{aligned}
\int u L_{\mu} v d \mu & =\int\left(u e^{-G}\right) \Delta v-\int u\langle\nabla G, \nabla v\rangle e^{-G} \\
& =-\int\left\langle\nabla\left(u e^{-G}\right), \nabla v\right\rangle-\int u\langle\nabla G, \nabla v\rangle e^{-G} \\
& =-\int\langle\nabla u, \nabla v\rangle e^{-G}+\int u\langle\nabla G, \nabla v\rangle e^{-G}-\int u\langle\nabla G, \nabla v\rangle e^{-G} \\
& =-\int\langle\nabla u, \nabla v\rangle d \mu
\end{aligned}
$$

### 3.17 A short and non-standard proof sketch of the integration by parts

In fact, we can prove the integration by parts formula for Lebesgue measure (as well as for any Log-concave measure) in the following simple and unusual way. Here is the sketch.

Lemma 3.73 (Integration by parts). Let $\mathrm{d} \mu=e^{-v} \mathrm{~d} x$, with $L_{\mu} u=\Delta u-\langle\nabla v, \nabla u\rangle$, with $f, g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that $g$ is smooth and bounded. Then

$$
\int f \cdot L_{\mu} g \mathrm{~d} \mu=-\int\langle\nabla f, \nabla g\rangle \mathrm{d} x
$$

$A$ special case for $v=0$ is

$$
\int f \cdot \Delta g \mathrm{~d} x=-\int\langle\nabla f, \nabla g\rangle \mathrm{d} x
$$

Proof. Consider the change of variable $x=y+t \nabla g(y)$, which has Jacobean $\operatorname{det}\left(\operatorname{Id}+t \nabla^{2} g\right)$. Then:

$$
\int f(x) \mathrm{d} \mu(x)=\int f(y+t \nabla g(y)) e^{-v(y+t \nabla g(y))} \operatorname{det}\left(\mathrm{Id}+t \nabla^{2} g\right) \mathrm{d} y
$$

Noting that the LHS does not depend on $t$, differentiating gives

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} t} \int f(y+t \nabla g(y)) e^{-v(y+t \nabla g(y))} \operatorname{det}\left(\mathrm{Id}+t \nabla^{2} g\right) \mathrm{d} x\right|_{t=0}=0
$$

Differentiating under the integral,

$$
0=\int\left(\langle\nabla f, \nabla g\rangle e^{-v}-f\langle\nabla v, \nabla g\rangle e^{-v}+f \cdot \Delta g e^{-v}\right) \mathrm{d} y
$$

where we used the fact that $\frac{\mathrm{d}}{\mathrm{d} t} \operatorname{det} \operatorname{Id}+t A=\operatorname{tr}(A)$. Then it follows that

$$
\int f \cdot L_{\mu} g \mathrm{~d} \mu=-\int\langle\nabla f, \nabla g\rangle \mathrm{d} \mu
$$

### 3.18 A word about eigen-functions

An interesting topic related to this operator is its eigen-functions. Define the first eigenvalue $\lambda_{1}$ to be the smallest $\lambda>0$ such that there exists a $u \neq 0$ such that $L_{\mu} u=-\lambda u$. It can be shown that $1 / \lambda_{1}$ is actually the Poincaré constant, i.e.,

$$
\lambda_{1}=\inf _{u \neq 0} \frac{\int|\nabla u|^{2} d \mu}{\int u^{2} d \mu-\left(\int u d \mu\right)^{2}}=\inf _{u \neq 0} \frac{\int|\nabla u|^{2} d \mu}{\int u^{2} d \mu}
$$

where the last equality follows from $\operatorname{Var}_{\mu}(f)=\inf _{a} \int(f-a)^{2} d \mu$. Note that in general it is nontrivial to have $\lambda_{1}>0$, but this is true for log-concave measures. We will discuss this in more detail in November. The main takeaway for now is that we always have the Poincaré type inequality for log-concave measures, i.e., for any function $u$,

$$
\operatorname{Var}_{\mu}(u) \leq \frac{1}{\lambda_{1}} \int|\nabla u|^{2} d \mu
$$

### 3.19 The derivation of the Brascamp-Lieb Inequality from the Generalized Log-Sobolev inequality

The idea: Recall that we obtained the Generalized Log-Sobolev inequality from PrékopaLeindler inequality by taking the first derivative near the point of the minimum. In this lecture, we will continue with this approach: Derive new inequalities by taking further derivatives of Prékopa-Leindler (or Generalized Log-Sobolev) around the point of maximum.


Recall the Generalized Log-Sobolev inequality: For convex functions $F, G$ with $\int e^{-G}=1$,

$$
\int G^{*}(\nabla F) e^{-F} \geq n \int e^{-F}-\int F e^{-F}-\int e^{-F} \log \int e^{-F},
$$

where the equality is attained when $F=G$. To take the derivative around this point, let $F=G+t \phi$ and denote
$\beta(t):=\int G^{*}(\nabla G+t \nabla \phi) e^{-G-t \phi}-n \int e^{-G-t \phi}+\int(G+t \phi) e^{-G-t \phi}+\int e^{-G-t \phi} \log \int e^{-G-t \phi}$.
We have that, $\beta \geq 0$ and $\beta(0)=0$. We will see that also $\beta^{\prime}(0)=0$ (as is customary at a point of minimum), and this will imply that $\beta^{\prime \prime}(0) \geq 0$, which will amount to a nice inequality called the Brascamp-Lieb inequality.

1. Write the Taylor expansion of $G^{*}$ up to the second term:

$$
G^{*}(\nabla G+t \nabla \phi)=G^{*}(\nabla G)+t\left\langle\nabla G^{*}(\nabla G), \nabla \phi\right\rangle+\frac{t^{2}}{2}\left\langle\nabla^{2} G^{*}(\nabla G) \nabla \phi, \nabla \phi\right\rangle+o\left(t^{2}\right)
$$

We will drop all $o\left(t^{2}\right)$ terms from now on.
2. Use $e^{-\delta}=1-\delta+\frac{\delta^{2}}{2}$ to obtain

$$
e^{-G-t \phi}=e^{-G} \cdot e^{-t \phi}=e^{-G}\left(1-t \phi+\frac{t^{2}}{2} \phi^{2}\right) .
$$

Combining this with $\log (1+\delta)=\delta-\frac{\delta^{2}}{2}$ and note that $\int e^{-G}=1$, we get

$$
\begin{align*}
\log \left(\int e^{-G-t \phi}\right) & =\log \left(\int e^{-G} \cdot\left(1-t \phi+\frac{t^{2}}{2} \phi^{2}\right)\right) \\
& =\log \left(1-t \int \phi e^{-G}+\frac{t^{2}}{2} \int \phi^{2} e^{-G}\right) \\
& =-t \int \phi e^{-G}+\frac{t^{2}}{2} \int \phi^{2} e^{-G}-\frac{t^{2}}{2}\left(\int \phi e^{-G}\right)^{2} . \tag{*}
\end{align*}
$$

With the variance notation, (*) becomes

$$
\log \left(\int e^{-G-t \phi}\right)=-t \int \phi d \mu+\frac{t^{2}}{2} \operatorname{Var}_{\mu}(\phi)
$$

Going back to our differentiation, the first term of $\beta(t)$ is

$$
\begin{aligned}
& \int G^{*}(\nabla G+t \nabla \phi) e^{-G-t \phi} \\
&= \int\left(G^{*}(\nabla G)+t\left\langle\nabla G^{*}(\nabla G), \nabla \phi\right\rangle+\frac{t^{2}}{2}\left\langle\nabla^{2} G^{*}(\nabla G) \nabla \phi, \nabla \phi\right\rangle\right)\left(1-t \phi+\frac{t^{2}}{2} \phi^{2}\right) d \mu \\
&=\int G^{*}(\nabla G) d \mu+t \int\left(\left\langle\nabla G^{*}(\nabla G), \nabla \phi\right\rangle-\phi G^{*}(\nabla G)\right) d \mu \\
& \quad+\frac{t^{2}}{2} \int\left(\left\langle\nabla^{2} G^{*}(\nabla G) \nabla \phi, \nabla \phi\right\rangle+\phi^{2} G^{*}(\nabla G)-2 \phi\left\langle\nabla G^{*}(\nabla G), \nabla \phi\right\rangle\right) d \mu \\
&=\int G^{*}(\nabla G) d \mu+t \int\left(\langle\nabla \phi, x\rangle-\phi G^{*}(\nabla G)\right) d \mu \\
& \quad+\frac{t^{2}}{2} \int\left(\left\langle\left(\nabla^{2} G\right)^{-1} \nabla \phi, \nabla \phi\right\rangle+\phi^{2} G^{*}(\nabla G)-2 \phi\langle\nabla \phi, x\rangle\right) d \mu,
\end{aligned}
$$

while the other terms are

$$
\begin{aligned}
- & n \int e^{-G-t \phi}+\int(G+t \phi) e^{-G-t \phi}+\int e^{-G-t \phi} \log \int e^{-G-t \phi} \\
= & -n \int\left(1-t \phi+\frac{t^{2}}{2} \phi^{2}\right) d \mu+\int(G+t \phi)\left(1-t \phi+\frac{t^{2}}{2} \phi^{2}\right) d \mu \\
& +\int(1-t \phi) d \mu\left(-t \int \phi d \mu+\frac{t^{2}}{2} \operatorname{Var}_{\mu}(\phi)\right) \\
= & -n+\int G d \mu+t\left(n \int \phi d \mu-\int \phi G d \mu\right)+\frac{t^{2}}{2}\left(\int(G-n) \phi^{2} d \mu-\operatorname{Var}_{\mu}(\phi)\right) .
\end{aligned}
$$

Here we used that $\int e^{-G}=1$.
To proceed, we need the following observations:

1. Using the duality formula $G+G^{*}(\nabla G)=\langle\nabla G, x\rangle$ and $\int\langle\nabla G, x\rangle d \mu=n \int e^{-G}=n$, we can easily obtain

$$
\int G^{*}(\nabla G) d \mu=n-\int G d \mu
$$

so the constants cancel out, and as expected, confirm that $\beta(0)=0$.
2. By Lemma 3.72,

$$
\begin{aligned}
\int\langle\nabla \phi, x\rangle d \mu & =-\int \phi \cdot L_{\mu} \frac{x^{2}}{2} d \mu \\
& =-\int \phi(n-\langle\nabla G, x\rangle) d \mu \\
& =-n \int \phi d \mu+\int \phi\langle\nabla G, x\rangle d \mu
\end{aligned}
$$

Combining this with the duality formula, one can get

$$
\begin{aligned}
& \int\left(\langle\nabla \phi, x\rangle-\phi G^{*}(\nabla G)\right) d \mu+n \int \phi d \mu-\int \phi G d \mu \\
& =-n \int \phi d \mu+\int \phi\langle\nabla G, x\rangle d \mu-\int \phi(\langle\nabla G, x\rangle-G) d \mu+n \int \phi d \mu-\int \phi G d \mu \\
& =0
\end{aligned}
$$

so the first order terms also cancel out, and we have, as expected that $\beta^{\prime}(0)=0$.
3. (Conclusion) From the second order terms, we obtain

$$
\begin{equation*}
\int\left(\left\langle\left(\nabla^{2} G\right)^{-1} \nabla \phi, \nabla \phi\right\rangle+\phi^{2} G^{*}(\nabla G)-2 \phi\langle\nabla \phi, x\rangle\right) d \mu+\int(G-n) \phi^{2} d \mu-\operatorname{Var}_{\mu}(\phi) \geq 0 \tag{4}
\end{equation*}
$$

We are left with showing:

## Claim 3.74.

$$
\int\left(\phi^{2} G^{*}(\nabla G)-2 \phi\langle\nabla \phi, x\rangle\right) d \mu+\int(G-n) \phi^{2} d \mu=0 .
$$

Proof. Indeed, by the duality formula and Lemma 3.72, one has

$$
\begin{aligned}
& \int\left(\phi^{2} G^{*}(\nabla G)-2 \phi\langle\nabla \phi, x\rangle\right) d \mu \\
& =\int\left(\phi^{2}\langle\nabla G, x\rangle-\phi^{2} G\right) d \mu-\int\left\langle\nabla \phi^{2}, x\right\rangle d \mu \\
& =\int\left(\phi^{2}\langle\nabla G, x\rangle-\phi^{2} G\right) d \mu+\int \phi^{2} L_{\mu} \frac{x^{2}}{2} d \mu \\
& =\int\left(\phi^{2}\langle\nabla G, x\rangle-\phi^{2} G\right) d \mu+\int \phi^{2}(n-\langle\nabla G, x\rangle) d \mu \\
& =\int(n-G) \phi^{2} d \mu .
\end{aligned}
$$

Finally, by combining (4) and Claim 3.74 we arrive to
Theorem 3.75 (the Brascamp-Lieb inequality, 1976). Let $G$ be a strictly convex function with $\int e^{-G}=1$ and let $d \mu=e^{-G} d x$. Then for any locally Lipschitz function $\phi$, we have

$$
\operatorname{Var}_{\mu}(\phi) \leq \int\left\langle\left(\nabla^{2} G\right)^{-1} \nabla \phi, \nabla \phi\right\rangle d \mu,
$$

where $\operatorname{Var}_{\mu}(\phi)=\int \phi^{2} d \mu-\left(\int \phi d \mu\right)^{2}$.
Remark 3.76 (Brascamp-Lieb inequality is the local form of the Prekopa-Leindler inequality!). One can alternatively show directly that the Brascamp-Lieb inequality is equivalent to the following consequence of Prekopa-Leindler inequality:

$$
\frac{d^{2}}{d t^{2}} \log \int e^{-(f+t g)^{*}} \leq 0
$$

by substituting $G=f^{*}$ and $\phi=g\left(\nabla f^{*}\right)$. In fact, Brascamp-Lieb inequality also implies Prekopa-Leindler "by integration". See home work!

Remark 3.77. The Brascamp-Lieb inequality attains equality when $\phi=\langle\nabla G, \theta\rangle$ for any $\theta \in$ $\mathbb{R}^{n}$. Something cool happens here: For any measure $\mu$, one can obtain $L_{\mu}\langle x, \theta\rangle=-\langle\nabla G, \theta\rangle$.
Remark 3.78. In fact, it may be tempting to continue our explorations of the linearization idea, and set $\phi=\langle\nabla G, \theta\rangle+\epsilon f$ into the Brascamp-Lieb inequality, take derivatives in $\epsilon$ and see if a new inequality comes out for the function $f$. Unfortunately, it turns out the Brascamp-Lieb inequality is "the end of the line"; see home work.

For $d \mu=d \gamma$ being the Gaussian measure, one has $\nabla^{2} G=$ Id which yields
Corollary 3.79 (the Gaussian Poincaré inequality).

$$
\operatorname{Var}_{\gamma}(\phi) \leq \int|\nabla \phi|^{2} d \gamma
$$

Recall that we discussed it as a partial case of Beckner's inequality, but now we formally proved it.

Remark 3.80. By our earlier discussion about eigenfunctions, this inequality means that the first eigenvalue of $L_{\gamma}$ is 1. One can check that the corresponding eigenfunctions are the linear functions, i.e., $L_{\gamma}\langle x, \theta\rangle=-\langle x, \theta\rangle$ for all $\theta \in \mathbb{R}^{n}$.

Remark 3.81. In fact, one may deduce the Gaussian Poincare Inequality from the classical Gaussian Log-Sobolev inequality (rather than from the generalized one), in the same way as above. That computation however is markedly simpler; see home work.

Remark 3.82 (An important observation about Brascamp-Lieb). For $G=V+W$ and $d \nu=e^{-(V+W)} d x$, we have

$$
\int \phi^{2} d \nu-\left(\int \phi d \nu\right)^{2} \leq \int\left\langle\left(\nabla^{2}(V+W)\right)^{-1} \nabla \phi, \nabla \phi\right\rangle d \nu
$$

Suppose that $V, W$ are convex functions. Then clearly $\nabla^{2}(V+W) \geq \nabla^{2} V$, which yields

$$
\operatorname{Var}_{\nu}(\phi) \leq \int\left\langle\left(\nabla^{2} V\right)^{-1} \nabla \phi, \nabla \phi\right\rangle d \nu
$$

Let $K$ be a convex set and consider in the previous remark $W=\mathbb{1}_{K}^{\infty}(x)+C_{n}$ with some properly chosen constant $C_{n}$. We obtain the following automatic generalization:

Corollary 3.83. Let $d \mu=e^{-V} d x$ be a log-concave measure and $K$ a convex body. Then for any function $\phi$ that is nice enough,

$$
\frac{1}{\mu(K)} \int_{K} \phi^{2} d \mu-\left(\frac{1}{\mu(K)} \int_{K} \phi d \mu\right)^{2} \leq \frac{1}{\mu(K)} \int_{K}\left\langle\left(\nabla^{2} V\right)^{-1} \nabla \phi, \nabla \phi\right\rangle d \mu
$$

In other words, Brascamp-Lieb inequality can be automatically restricted to any convex set.

As another corollary, we obtain:
Corollary 3.84 (An extension of the Gaussian Poincare inequality). If $d \mu=e^{-V} d x$ is a probability measure and $\nabla^{2} V \geq k$. Id, then

$$
\operatorname{Var}_{\mu}(\phi) \leq \frac{1}{k} \mathbb{E}_{\mu}|\nabla \phi|^{2} .
$$

In this case, the Poincaré constant (or the inverse of the first eigenvalue of $L_{\mu}$ ) can be bounded by $k$.

### 3.20 Going back to Poincaré inequalities: Payne-Weinberger and Poincaré on the circle

Let $\mathrm{d} \mu$ be a log-concave density, $\lambda_{1}$ the first eigenvalue of $L_{\mu}$. The Poincaré inequality is

$$
\operatorname{Var}_{\mu}(f) \leq \frac{1}{\lambda_{1}} \int\|\nabla f\|^{2} \mathrm{~d} \mu
$$

Theorem 3.85 (Payne-Weinberger). Let $K$ be a convex body, with diam $K=R$. Then for all locally-Lipschitz $f$,

$$
\frac{1}{|K|} \int_{K} f^{2}-\left(\frac{1}{|K|} \int_{K} f\right)^{2} \leq \frac{R^{2}}{\pi^{2}|K|} \int_{K}\|\nabla f\|^{2}
$$

Theorem 3.86. Let $\phi \in C^{1}(-\pi, \pi)$. Then

$$
\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi^{2} \mathrm{~d} \theta-\left(\frac{1}{2 \pi} \int \phi \mathrm{~d} \theta\right)^{2} \leq \frac{1}{2 \pi} \int_{-\pi}^{\pi} \dot{\phi}^{2} \mathrm{~d} \theta
$$

Proof. Recall the Fourier series:

$$
\phi(\theta)=\sum_{n=-\infty}^{\infty} \hat{\phi}(n) e^{2 \pi i n \theta}
$$

where

$$
\hat{\phi}(0)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi \mathrm{d} \theta
$$

We have $\widehat{(\dot{\phi}})(n)=n \hat{\phi}(n)$. By Parseval's theorem,

$$
\int_{-\pi}^{\pi} \dot{\phi}^{2}=\sum_{n=-\infty}^{\infty} n^{2}|\hat{\phi}(n)|^{2}=\sum_{n \neq 0} n^{2}|\hat{\phi}(n)|^{2} \geq \sum_{n \neq 0}|\hat{\phi}(n)|^{2}
$$

Then observe that

$$
\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi^{2} \mathrm{~d} \theta-\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi \mathrm{d} \theta\right)^{2}=\sum_{n=-\infty}^{\infty}|\hat{\phi}(n)|^{2}-|\hat{\phi}(0)|^{2}=\sum_{n \neq 0}|\hat{\phi}(n)|^{2}
$$

which gives the inequality.
Remark 3.87. For $\phi(\theta)=a \cos \theta+b \sin \theta$ we have equality.
Remark 3.88. If $\phi$ is even on $[-\pi, \pi]$, then for all odd $k, \hat{\phi}(k)=0$. From this we obtain

$$
\sum_{n \neq 0}|\hat{\phi}(n)|^{2}=\sum_{|n| \geq 2}|\hat{\phi}(n)|^{2} \leq \frac{1}{4} \sum_{|n| \geq 2} n^{2}|\hat{\phi}(n)|^{2}
$$

so we may sharpen the above inequality to $\operatorname{Var} \phi \leq \frac{1}{4} \mathbb{E} \dot{\phi}^{2}$.

### 3.21 Colesanti's inequality via Brascamp-Lieb (Cordero-Erasquin's approach)

We have certain parallels between inequalities for functions and inequalities for volumes (and can often deduce the volume version from the functional one):

| Functional | Volumetric |
| :---: | :---: |
| $\log \int e^{(f+t g)^{*}}$ is concave | $\log \|K+t L\|$ is concave |
| Generalized log-Sobolev | Minkowski's first inequality |
| Brascamp-Lieb inequality | Minkowski's second inequality |

Explicitly, Minkowski's second inequality is

$$
\left.|K+t L|^{\prime \prime}\right|_{t=0} \cdot|K| \leq \frac{n-1}{n}\left(\left.|K+t L|^{\prime}\right|_{t=0}\right)
$$

The factor of $\frac{n-1}{n}$ arises when we differentiate (twice) $t^{1 / n}$, which differs from (but resembles) the $\log t$ involved in deriving Brascamp-Lieb.

Now, suppose we have smooth convex bodies $K$ and $L$, and we take $v=\mathbb{1}_{K}^{\infty}, \phi=h_{L}$ in Brascamp-Lieb. This will give

$$
\int \phi^{2} \cdot e^{-v}-\left(\int \phi \cdot e^{-v}\right)^{2} \leq \int\left\langle\left(\nabla^{2} \phi\right)^{-1} \nabla \phi, \nabla \phi\right\rangle e^{-v}
$$

We have $\nabla \mathbb{1}_{K}^{\infty}=n_{x}$ and $\nabla^{2} \mathbb{1}_{K}^{\infty}=\mathbb{I}$, the second fundamental form... It is difficult to see this through, however. The right approach, due to Cordero-Erasquin, is to consider $v=h_{K}^{2} / 2$. With some work, one deduces:

Theorem 3.89 (Colesanti). Let $\phi: \partial K \rightarrow \mathbb{R}$, where $K$ is a smooth convex body and $\phi \in$ $C^{1}(\partial K)$. Then $\int_{\partial K} \phi=0$ implies that

$$
\int_{\partial K} \operatorname{tr}(\mathbb{I}) \phi^{2}(x) \mathrm{d} x \leq \int_{\partial K}\left\langle\mathbb{I}^{-1} \nabla \phi, \nabla \phi\right\rangle
$$

Proof. Homework! Take $v=h_{K}^{2} / 2$ in the Brascamp-Lieb inequality, then integrate in polar coordinates.

### 3.22 Dimensional extensions of Generalized Log-Sobolev and BrascampLieb inequalities

Recall the following corollary of the Borell-Brascamp-Lieb inequality:
Corollary 3.90. For $p \in[-1 / n, 0]$ and $f, g$ convex, the function

$$
\left(\int_{\mathbb{R}^{n}}\left(f^{*}+t g^{*}\right)^{1 / p}\right)^{\frac{p}{n p+1}}
$$

is concave in $t$.

Corollary 3.91 (Bolley, Gentil, Guillin [29]). For $q \in(-\infty,-n]$, and a probability measure $\mathrm{d} \mu=e^{-v} \mathrm{~d} x$ satisfying

$$
\nabla^{2} v-\frac{\nabla v \otimes \nabla v}{q} \succeq 0
$$

for all locally-Lipschitz $g$, we have

$$
\int\left(g \cdot e^{v / q}\right)^{2} \mathrm{~d} \mu-\left(\int g \cdot e^{v / q} d \mu\right)^{2} \leq \frac{-q}{-q+1} \int\left\langle\left(\nabla^{2} v-\frac{\nabla v \otimes \nabla v}{q}\right)^{-1} \nabla g, \nabla g\right\rangle \mathrm{d} \mu
$$

Remark 3.92. When $q \rightarrow-\infty$, we recover Brascamp-Lieb. When $q=-n$, we obtain

$$
\int\left(g \cdot e^{-v / n}\right)^{2} \mathrm{~d} \mu-\left(\int g \cdot e^{-v / n} \mathrm{~d} \mu\right)^{2} \leq \frac{n}{n+1} \int\left\langle\left(\nabla^{2} v+\frac{\nabla v \otimes \nabla v}{n}\right)^{-1} \nabla g, \nabla g\right\rangle \mathrm{d} \mu
$$

Theorem 3.93 (Bolley, Cordero-Erasquinn, Fujita, Gentil, Guillin [30] extension of BBL). Let $h, g$, $w$ be Borel-measurable functions satisfying $\forall x, y \in R^{n}, t \in[0,1]$,

$$
h((1-t) x+t y) \leq(1-t) g(x)+t w(y)
$$

and $\int w^{-n}=\int g^{-n}=1$. Then

$$
\int h^{1-n} \geq(1-t) \int g^{1-n}+t \int w^{1-n}
$$

Corollary 3.94 (Convex Sobolev inequality extension of generalized log-Sobolev from [30]). Let $n \geq 2, w: \mathbb{R}^{n} \rightarrow(0, \infty)$ such that $\lim \inf _{x \rightarrow \infty} \frac{w(x)}{\|x\|^{\gamma}}>0$ for some $\gamma>\frac{n}{n-1}$, and nonnegative $g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ satisfying $\int g^{-n}=\int w^{-n}=1$. Then

$$
\int w^{*}(\nabla g) g^{-n} \geq \frac{1}{n-1} \int w^{1-n}
$$

Remark 3.95. Plug

$$
w(x)=\left(1+\frac{|x|^{q}}{q}\right) C_{q}
$$

and $g=f^{\frac{p}{p-n}}$ to recover the Sobolev inequality,

$$
\|f\|_{p^{*}} \leq \frac{\left\|h_{p}\right\|_{p^{*}}}{\left(\int\left\|\nabla h_{p}\right\|^{p}\right)^{1 / p}}\left(\int\|\nabla f\|^{p}\right)^{1 / p}
$$

where $h_{p}(x)=\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}$ and $p^{*}=\frac{n p}{n-p}$.

### 3.23 Home work

Question 3.96 (1 point). Below $S_{u}$ stands for Steiner symmetrization with respect to $u^{\perp}$; $K$ stands for a convex body in $\mathbb{R}^{n}$ with non-empty interior. Show that
a) $S_{u}(a K)=a S_{u} K$ for all $a>0$;
b) If $K \subset L$ then $S_{u}(K) \subset S_{u}(L)$; conclude that $S_{u}(K)$ is continuous with respect to Hausdorf metric;
c) $S_{u}(K)+S_{u}(L) \subset S_{u}(K+L)$.

Question 3.97 (1 point). Recall that for a compact set $A \subset \mathbb{R}^{n}$, the diameter

$$
\operatorname{diam}(A)=\max _{x, y \in A}|x-y|
$$

Prove that

$$
\operatorname{diam}\left(S_{u}(K)\right) \leq \operatorname{diam}(K)
$$

Conclude the isodiametric inequality: if the volume of a set is fixed, its diameter is minimized by a Euclidean ball.

Question 3.98 (1 point). Prove that the Steiner symmetrization decreases the perimeter of a convex set. Note that this gives another proof of the isoperimetric inequality for convex sets.

Question 3.99 (1 point). Recall that for a convex set $K \subset \mathbb{R}^{n}$, the in-radius of $K$ is

$$
r(K)=\sup \left\{t>0: \exists y \in \mathbb{R}^{n}: y+t B_{2}^{n} \subset K\right\}
$$

and the circum-radius of $K$ is

$$
R(K)=\inf \left\{t>0: \exists y \in \mathbb{R}^{n}: K \subset y+t B_{2}^{n}\right\}
$$

a) Prove that $r\left(S_{u}(K)\right) \geq r(K)$.
b) Prove that $R\left(S_{u}(K)\right) \leq R(K)$.

Conclude that the Euclidean ball maximizes the in-radius and minimizes the circum-radius when the volume is fixed.

Question 3.100 (2 points). Prove the Urysohn inequality. Define mean width of a convex body $K$ as

$$
w(K)=\frac{2}{\left|\mathbb{S}^{n-1}\right|} \int_{\mathbb{S}^{n-1}} h_{K}(\theta) d \theta
$$

Show that if $|K|=\left|B_{2}^{n}\right|$ then $w(K) \geq 2$.
Hint: use the Brunn-Minkowski inequality and Steiner symmetrizations.
Question 3.101 (1 point). Fix Borel measurable sets $K, L \subset \mathbb{R}^{n}$. Confirm what we discussed in class: the validity for every $\lambda \in[0,1]$ of the inequality

$$
|\lambda K+(1-\lambda) L| \geq|K|^{\lambda}|L|^{1-\lambda}
$$

implies the validity of

$$
|\lambda K+(1-\lambda) L|^{\frac{1}{n}} \geq \lambda|K|^{\frac{1}{n}}+(1-\lambda)|L|^{\frac{1}{n}} .
$$

Question 3.102 (1 point). Show that for $a, b>0$, one has $\left(\lambda a^{p}+(1-\lambda) b^{p}\right)^{\frac{1}{p}} \rightarrow_{p \rightarrow 0} a^{\lambda} b^{1-\lambda}$. Question 3.103 (2 points). a) Let $p \geq-\frac{1}{n}$, and suppose functions $f, g$ and $h$ on $\mathbb{R}^{n}$ satisfy

$$
h(\lambda x+(1-\lambda) y) \geq\left((1-\lambda) f^{p}(x)+\lambda g^{p}(y)\right)^{\frac{1}{p}} .
$$

Show that

$$
\int h \geq\left((1-\lambda)\left(\int f\right)^{\frac{p}{n p+1}}+\lambda\left(\int g\right)^{\frac{p}{n p+1}}\right)^{\frac{n p+1}{p}}
$$

Hint: try, for example, a similar proof to Lyusternik's proof of the Brunn-Minkowski inequality. b) Conclude that if a measure's density is supported on a convex set with non-empty interior and is $p$-concave, then the measure is $\frac{p}{n p+1}$-concave.
c) Deduce that if the density of a measure $\mu$ on $\mathbb{R}^{n}$ is $p$-concave, then the density of a marginal measure $\pi_{H}(\mu)$ is $\frac{p}{k p+1}$-concave, if $H$ is an $(n-k)$-dimensional subspace (note that this is a generalization of Brunn s principle).

Question 3.104 (2 points). We say that a function $f$ in $\mathbb{R}^{n}$ is unconditional if it invariant under coordinate reflections. That is, $f\left(\epsilon_{1} x_{1}, \ldots, \epsilon_{n} x_{n}\right)=f(x)$ for any choice of $\epsilon_{i} \in\{-1,1\}$. $A$ set $K$ is called unconditional if $1_{K}$ is an unconditional function.

Suppose $K$ is an unconditional convex body and $V$ is an unconditional convex function in $\mathbb{R}^{n}$. Denote $d \mu(x)=e^{-V(x)} d x$. Show that $\log \mu\left(e^{t} K\right)$ is a concave function in $t \in \mathbb{R}$.
Hint: pass the integration from $\mathbb{R}^{n}$ to the set $\left\{x \in \mathbb{R}^{n}: \forall i=1, \ldots, n, x_{i} \geq 0\right\}$, and make $a$ change of variables in the Prekopa-Leindler inequality given by $\left(x_{1}, \ldots, x_{n}\right)=\left(e^{t_{1}}, \ldots, e^{t_{n}}\right)$.

Question 3.105 (1 point). a) Prove Minkowski's first inequality: $V_{1}(K, L) \geq|K|^{\frac{n-1}{n}}|L|^{\frac{1}{n}}$ (similar to the isoperimetric inequality which we deduced in class.)
b) Prove Minkowski's quadratic inequality: for convex bodies $K$ and $L$ in $\mathbb{R}^{n}$,

$$
V_{2}(K, L)|K| \leq V_{1}(K, L)^{2}
$$

Hint: use the Brunn-Minkowski inequality to obtain some information about $\frac{d^{2}}{d t^{2}}|K+t L|^{\frac{1}{n}}$.
Question 3.106 (1 point). (this question is added upon Alex's request) Give an example of $a$ (rough, non-convex) set $K$ such that $\lim _{\epsilon \rightarrow 0} \frac{\left|K+\epsilon B_{2}^{n}\right|-|K|}{\epsilon}$ does not exist, and

$$
\liminf _{\epsilon \rightarrow 0} \frac{\left|K+\epsilon B_{2}^{n}\right|-|K|}{\epsilon}<\limsup _{\epsilon \rightarrow 0} \frac{\left|K+\epsilon B_{2}^{n}\right|-|K|}{\epsilon} .
$$

Question 3.107 (1 point). Show that any convex function $V: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ is a) continuous on the support of $e^{-V}$ (i.e. on the set where $V$ does not take infinite values) b) Of class $C^{2}$ almost everywhere on the support of $e^{-V}$.

Question 3.108 (1 point). a) Suppose $V \in C^{2}\left(\mathbb{R}^{n}\right)$. Show that for all $z_{1}, z_{2} \in \mathbb{R}^{n}$,

$$
\begin{equation*}
V\left(\frac{z_{1}+z_{2}}{2}\right)+\beta\left(z_{1}, z_{2}\right)=\frac{V\left(z_{1}\right)+V\left(z_{2}\right)}{2} \tag{5}
\end{equation*}
$$

where, letting $z(t)=\frac{(1-t) z_{1}+(1+t) z_{2}}{2}$, we have

$$
\begin{equation*}
\beta\left(z_{1}, z_{2}\right)=\frac{1}{8} \cdot \int_{-1}^{1}(1-|t|)\left\langle\nabla^{2} V(z(t))\left(z_{1}-z_{2}\right), z_{1}-z_{2}\right\rangle d t \geq 0 \tag{6}
\end{equation*}
$$

b) Conclude that convexity of a $C^{2}$-smooth function is equivalent to the non-negative definiteness of its Hessian.

Question 3.109 (2 points). a) Show that for any pair of convex bodies $K$ and $L$ the function $|K+t L|$ is a polynomial in $t$ of degree $n$.
b) Conclude that $|K+t L|=\sum_{k=0}^{n}\binom{n}{k} V_{k}(K, L) t^{k}$. This is called the Steiner polynomial.

Question 3.110 (2 points). For a convex set $K$ define the Gauss map $\nu_{K}: \partial K \rightarrow \mathbb{S}^{n-1}$ by $\nu_{K}(x)=\left\{n_{x}\right\}$ (the set of all outer normal vectors to $\partial K$ at $x$; it is a singleton almost everywhere). Define also a measure $S_{K}$ on the sphere $\mathbb{S}^{n-1}$ by letting, for every Borel measurable $\Omega \subset \mathbb{S}^{n-1}:$

$$
S_{K}(\Omega)=\left|\nu_{K}^{-1}(\Omega)\right|_{n-1} .
$$

Here $|\cdot|_{n-1}$ stands for the $(n-1)$-Hausdorff measure, i.e. for $M \subset \partial K$ we let $|M|_{n-1}=\int_{M}$ in the sense we usually do it in class. The measure $S_{K}$ is called the surface area measure of $K$.
a) Show that for a pair of convex bodies $K$ and $L$,

$$
V_{1}(K, L)=\frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{L}(\theta) d S_{K}(\theta)
$$

In particular,

$$
|K|=\frac{1}{n} \int_{\mathbb{S}^{n}-1} h_{K}(\theta) d S_{K}(\theta)
$$

b) Use Minkowski's first inequality to prove that the surface area measure determines a convex body uniquely up to shifts (i.e. if $d S_{K}=d S_{L}$ then $K=L+v$ for some vector $v$.)

Question 3.111 (2 points). Recall that the projection of a convex body $K$ onto a hyperplane $\theta^{\perp}$, for some $\theta \in \mathbb{S}^{n-1}$, is the set defined as

$$
K \mid \theta^{\perp}=\left\{x \in \theta^{\perp}: \exists t \in \mathbb{R}: x+t \theta \in K\right\} .
$$

a) Prove the Cauchy formula for a symmetric convex body $K$ :

$$
\left.|K| \theta^{\perp}\right|_{n-1}=\frac{1}{2} \int_{\mathbb{S}^{n}-1}|\langle\theta, u\rangle| d S_{K}(u) .
$$

Hint: option 1 - use elementary geometry and approximation by polytopes. option 2 - use Questions 2.27 part c) and 3.110 part a).
b) Suppose $K$ and $L$ are symmetric convex bodies such that for every $\theta \in \mathbb{S}^{n-1}$ one has $\left.|K| \theta^{\perp}\right|_{n-1}=\left.|L| \theta^{\perp}\right|_{n-1}$. Conclude that $K=L+v$ for some vector $v \in \mathbb{R}^{n}$.
(you don't want to me to add a hint here on which Question(s) to use, right?)
Question 3.112 (1 point). Prove that when $h \in C^{2}\left(\mathbb{R}^{2}\right)$ is a support function of a strictly convex compact region $K$ in $\mathbb{R}^{2}$, the surface area measure has a density expressible in the form

$$
f_{K}(u)=h(u)+\ddot{h}(u)
$$

for all $u \in \mathbb{S}^{1}$. Note that $h+\ddot{h}$ is translation invariant.
Question 3.113 (10 points). Prove (perhaps using elementary Harmonic Analysis?) that for every pair of $\pi$-periodic infinitely smooth functions $\psi$ and $h$ on $[-\pi, \pi]$, such that $h+\ddot{h}>0$ and $h>0$, one has

$$
\begin{equation*}
\left(\int_{-\pi}^{\pi}\left(h^{2}-\dot{h}^{2}\right) d u\right)\left(\int_{-\pi}^{\pi}\left(\psi^{2}-\dot{\psi}^{2}+\psi^{2} \frac{h+\ddot{h}}{h}\right) d u\right) \leq 2\left(\int_{-\pi}^{\pi}(h \psi-\dot{h} \dot{\psi}) d u\right)^{2} \tag{7}
\end{equation*}
$$

(I can provide explanation/motivation upon request. Note that the assumption is $\pi$-periodic rather than $2 \pi$-periodic.)

Question 3.114 (2 points). Prove the Rogers-Shepherd inequality. For a convex body in $\mathbb{R}^{n}$, define the difference body

$$
K-K=\{x-y: x, y \in K\} .
$$

Show that

$$
|K-K| \leq\binom{ 2 n}{n}|K|
$$

Hint: use the Brunn-Minkowski inequality to show that $|K \cap(x+K)|^{\frac{1}{n}}$ is a concave function supported on $K-K$, and therefore it can be estimated from below by $1-\rho_{K-K}(x)$. Using this estimate (among other considerations) show that

$$
|K|^{2}=\int_{K-K}|K \cap(x+K)| d x \geq\binom{ 2 n}{n}^{-1}|K| \cdot|K-K| .
$$

Question 3.115 (2 points). Prove the Grunbaum inequality: let $K$ be a convex body whose barycenter is at the origin (that is $\int_{K} x d x=0$.) Show that for any $\theta \in \mathbb{S}^{n-1}$, one has

$$
|\{x \in K:\langle x, \theta\rangle \geq 0\}| \geq\left(\frac{n}{n+1}\right)^{n}|K| \geq \frac{|K|}{e}
$$

Question 3.116 (3 points). Prove Busemann's theorem: given $x \in \mathbb{R}^{n} \backslash\{0\}$, the function $\frac{|x|}{\left|x^{\perp} \cap K\right|}$ is convex in $\mathbb{R}^{n}$. Conclude that it is a norm. The unit ball of this norm is called the intersection body of $K$.

Question 3.117. Derive the Santalo formula for the area of a convex region in $\mathbb{R}^{2}$ :

$$
|K|=\frac{1}{2} \int_{-\pi}^{\pi} h^{2}-\dot{h}^{2} d t
$$

where $h$ is the support function of $K$.
Hint: use Questions 3.112 and 3.110
Question 3.118 (2 points). Using elementary Harmonic Analysis, prove that for every pair of $C^{1}$ periodic functions on $[-\pi, \pi]$, one has

$$
\int_{-\pi}^{\pi} h^{2}-\dot{h}^{2} \cdot \int_{-\pi}^{\pi} \psi^{2}-\dot{\psi}^{2} \leq\left(\int h \psi-\dot{h} \dot{\psi}\right)^{2}
$$

Explain why this provides an alternative solution to Question 3.105 b) on the plane (hint: use Questions 3.117 and 3.110 for this explanation).

Question 3.119 (1 point). Prove the general version of Brunn's principle: for a convex body $K$ in $\mathbb{R}^{n}$ and a $k$-dimensional subspace $H$, the function $|K \cap(y+H)|^{\frac{1}{k}}$ is concave on its support (inside $H^{\perp}$.) Here $k \in\{1, \ldots, n-2\}$ (the case $k=n-1$ we did in class.)

Question 3.120. Show that the convolution of log-concave functions is log-concave. Hint: Use the fact that marginals of log-concave functions are log-concave, in dimension $\mathbb{R}^{2 n}$.

Question 3.121 (1 point). Provide an alternative proof (to what was done in class) of the Gaussian Poincare inequality

$$
\int_{\mathbb{R}^{n}} f^{2} d \gamma-\left(\int_{\mathbb{R}^{n}} f d \gamma\right)^{2} \leq \int_{\mathbb{R}^{n}}|\nabla f|^{2} d \gamma
$$

using the decomposition of $f$ into the series of Hermite polynomials (the orthonormal system with respect to the Gaussian measure - you can read about them e.g. in Wikipedia.)

Question 3.122 (1 point). As per our discussion in class, prove the following statement using the Borell-Brascamp-Lieb inequality (Question 3.103).

Fix $q \in(-\infty,-n]$. Let $d \mu=e^{-V} d x$ be a probability measure and $g$ be a $C^{1}$ function. Suppose $V \in C^{2}\left(\mathbb{R}^{n}\right)$ and $\nabla^{2} V-\frac{\nabla V \otimes \nabla V}{q} \geq 0$ (i.e. $V$ is $q$-concave.) Then, assuming all the integrals below exist,

$$
\int\left(g e^{\frac{V}{q}}\right)^{2} d \mu-\left(\int g e^{\frac{V}{q}} d \mu\right)^{2} \leq \frac{-q}{-q+1} \int\left\langle e^{-\frac{2 V}{-q}}\left(\nabla^{2} V+\frac{\nabla V \otimes \nabla V}{-q}\right)^{-1} \nabla g, \nabla g\right\rangle d \mu
$$

Question 3.123 (1 point). Deduce the Gaussian Poincare inequality from the Gaussian LogSobolev inequality via the linearization method (this is sort of a partial case of the argument we discuss in class).

Question 3.124 (1 point). Prove that the (classical) Gaussian Log-Sobolev inequality and the (classical) Lebesgue Log-Sobolev inequality (as stated in class) are indeed equivalent.

Question 3.125 (1 point). By differentiating the infimal convolution directly, prove the Gaussian Log-Sobolev inequality without the convexity assumption on $f$ :

$$
E n t_{\gamma}\left(f^{2}\right) \leq 2 \int|\nabla f|^{2}
$$

for any $f \in C^{1}\left(\mathbb{R}^{n}\right)$ for which the corresponding integrals converge.
Question 3.126 (1 point). Deduce the Sobolev inequality from the Log-Sobolev inequality for the Lebesgue measure.
Question 3.127 (1 point). Show that the Gaussian Beckner inequality implies the classical Gaussian Log-Sobolev when $p \rightarrow 2$.

Question 3.128 (1 point). Deduce Nash's inequality from the (classical) Lebesgue LogSobolev inequality: for any non-negative $f \in L^{2}\left(\mathbb{R}^{n}\right) \cap C^{1}\left(\mathbb{R}^{n}\right)$

$$
\left(\int f^{2} d x\right)^{1+\frac{2}{n}} \leq \frac{2}{\pi e n}\left(\int|\nabla f|^{2} d x\right)\left(\int f d x\right)^{\frac{4}{n}}
$$

Question 3.129 (1 point). Deduce the isoperimetric inequality from the Sobolev inequality for Lebesgue measure.

Question 3.130 (1 point). Prove the following variant of the Generalized Log-Sobolev inequality: given a log-concave measure $\mu$ on $\mathbb{R}^{n}$ with density $e^{-V}$, and any pair of smooth convex functions $f$ and $g$ with $\int e^{-f} d \mu=\int e^{-g} d \mu$, one has

$$
\int g^{*}(\nabla f) e^{-f} d \mu \geq n \int e^{-f} d \mu-\int\langle\nabla V, x\rangle e^{-f} d \mu-\int f e^{-f} d \mu
$$

Question 3.131 (3 points). Is it possible to obtain Gaussian Beckner inequalities for $p \in$ $[1,2)$ via linearizations of (some) geometric inequalities directly?

Question 3.132 (2 points). Prove the following extension of the Borell-Brascamp-Lieb inequality due to Bolley, Cordero-Erasquin, Fujita, Gentil, Guillin: for convex $f$ and $g$ on $\mathbb{R}^{n}$ with $n \geq 2$ :

$$
\int\left(((1-t) f+t g)^{*}\right)^{1-n} \geq(1-t) \int\left(f^{*}\right)^{1-n}+t \int\left(g^{*}\right)^{1-n}
$$

Question 3.133 (Generalized Sobolev, 2 points). Prove the following extension of the Sobolev inequality due to Bolley, Cordero-Erasquin, Fujita, Gentil, Guillin: for convex F and $G$ on $\mathbb{R}^{n}$ with $n \geq 2$ : such that $\int F^{-n}=\int G^{-n}=1$, and assuming that $\frac{G(x)}{|x|^{\gamma}} \rightarrow_{x \rightarrow \infty} 0$, for some $\gamma>\frac{n}{n-1}$, and that all the integrals exist, we have

$$
\int G^{*}(\nabla F) F^{-n} \geq \frac{1}{n-1} \int G^{1-n}
$$

Question 3.134 (Coredero-Erasquin's proof of Colesanti inequality, 4 points). Prove the following inequality: when $K$ is a $C^{2}$ convex body, II is its second fundamental form and $f \in C^{1}(\partial K)$ is an arbitrary function such that $\int_{\partial K} f=0$, then

$$
\int_{\partial K} \operatorname{tr}(\mathrm{II}) \mathrm{f}^{2}-\left\langle\mathrm{II}^{-1} \nabla_{\partial \mathrm{K}} \mathrm{f}, \nabla_{\partial \mathrm{K}} \mathrm{f}\right\rangle \leq 0 .
$$

Here $\nabla_{\partial K} f$ stands for the intrinsic boundary gradient of $f$. Compare to Question 3.105 part b).

Hint: Use Brascamp-Lieb inequality with $V(x)=\frac{h_{K}^{2}(x)}{2}$ and the "body polar coordinates" formula

$$
\int_{K} F(x) d x=\int_{0}^{\infty} \int_{\partial K} F(t y) t^{n-1}\left\langle y, n_{y}\right\rangle d t d y
$$

where $n_{y}$ is the outer unit normal to $\partial K$ at $y$, and dy stands for the boundary integration.
Question 3.135 (1 point). Show that when $\varphi:[-\pi, \pi]$ is $C^{1}$, even and periodic, then

$$
\int_{-\pi}^{\pi} \varphi^{2}-\frac{1}{2 \pi}\left(\int_{-\pi}^{\pi} \varphi\right)^{2} \leq \frac{1}{4} \int_{-\pi}^{\pi} \dot{\varphi}^{2}
$$

Question 3.136 (1 point). Show that when $\varphi:[-\pi, \pi]$ is $C^{1}$, periodic, and $\varphi(0)=0$, then

$$
\int_{-\pi}^{\pi} \varphi^{2} \leq 4 \int_{-\pi}^{\pi} \dot{\varphi}^{2}
$$

Question 3.137 (1 point). Show that Brascamp-Lieb inequality is "the end of the line" for the linearization method: let $d \mu(x)=e^{-V(x)} d x$ and plug the function $f(x)=\langle\nabla V(x), \theta\rangle+\epsilon \varphi$ into Brascamp-Leib:

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq \int\left\langle\left(\nabla^{2} V\right)^{-1} \nabla f, \nabla f\right\rangle d \mu
$$

and observe that while $\langle\nabla V(x), \theta\rangle$ indeed attains equality in the above inequality, and the terms corresponding to $\epsilon$ cancel out as well, still, the only inequality that we obtain as a result is again the Brascamp-Lieb inequality.

Question 3.138 (1 point). Let $\mu$ be a log-concave probability measure on $\mathbb{R}^{n}$ with the density $e^{-V}$ for some convex function $V$ and the associated Laplacian $L u=\Delta u-\langle\nabla u, \nabla V\rangle$. Let $\lambda_{1}>0$ be the first non-trivial eigenvalue of $L$, that is the smallest number such that there exists a non-zero function $f_{1}$ such that

$$
L f_{1}=-\lambda_{1} f_{1}
$$

Show that

$$
\lambda_{1}=\inf _{f \in W^{1,2}(d \mu)} \frac{\int|\nabla f|^{2} d \mu}{\int f^{2} d \mu}=\inf \frac{\int|\nabla f|^{2} d \mu}{\int f^{2} d \mu-\left(\int f d \mu\right)^{2}} .
$$

Hint: use general convexity/compactness considerations to show that the infimum is attained for some function $f_{1}$. Then consider $f=f_{1}+\epsilon g$ and argue that the derivative in $\epsilon$ of that ratio must be zero. Conclude that $f$ has to be an eigenfunction (use general PDE considerations to argue that it exists).

Question 3.139 (1 point). Show that for a positive definite matrix $A$,

$$
\operatorname{det}(\mathrm{Id}+\mathrm{tA})=1+\mathrm{t} \cdot \operatorname{tr}(\mathrm{~A})+\frac{\mathrm{t}^{2}}{2}\|\mathrm{~A}\|_{\mathrm{HS}}^{2}+\mathrm{o}\left(\mathrm{t}^{2}\right)
$$

where $\|A\|_{H S}^{2}$ is the square of the Hilbert-Schmidt norm (that is, the sum of the squares of all entries).

Question 3.140 (3 points). Show that one can improve the Gaussian Log-Sobolev inequality to the following: suppose $d \mu=e^{-V-\frac{x^{2}}{2}-n \log \sqrt{2 \pi}} d x=e^{-V} d \gamma$ is a probability measure. Then

$$
-\int V d \mu \leq \frac{\int x^{2} d \mu-n}{2}+\frac{n}{2} \log \left(2+\frac{\int|\nabla V|^{2} d \mu-\int x^{2} d \mu}{n}\right)
$$

Question 3.141 (1 point). Prove the following improvement of the Brascamp-Lieb inequality in the unconditional case (recall that a function $f(x)$ is called unconditional if $f\left(\epsilon_{1} x_{1}, \ldots, \epsilon_{n} x_{n}\right)=f(x)$, for every $x \in \mathbb{R}^{n}$ and every choice of signs $\epsilon_{i} \in\{-1,1\}$; that is, $f$ is invariant under coordinate reflections).

Suppose $f, w$ are unconditional and $w$ is convex. Then for the probability measure $d \mu=$ $C e^{-w} d x$ one has

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq \int\left\langle\left(\nabla^{2} w+T\right)^{-1} \nabla f, \nabla f\right\rangle d \mu
$$

where $T=\operatorname{diag}\left[\frac{1}{x_{1}} \frac{\partial w}{\partial x_{1}}, \ldots, \frac{1}{x_{n}} \frac{\partial w}{\partial x_{n}}\right]$.
Hint: use the multiplicative version of Prekopa-Leindler inequality for unconditional functions, as in Question 3.104.

Question 3.142 (2 points, important question). a) Prove the second part of Lemma 7.9 (from the notes) concerning the second derivative of the Legendre of an interpolation: that for a family of convex functions $v_{t}$ such that $v_{t}(x) \in C^{2}(x, t)$, one has

$$
\frac{d^{2}}{d t^{2}} v_{t}^{*}(x)=-\ddot{v}_{t}\left(\nabla v_{t}^{*}\right)+\left\langle\left(\nabla^{2} v_{t}(x)\right)^{-1} \nabla \dot{v}_{t}\left(\nabla v_{t}^{*}\right), \nabla \dot{v}_{t}\left(\nabla v_{t}^{*}\right)\right\rangle .
$$

b) Use it to deduce the Brascamp-Lieb inequality from Prekopa-Leindler directly, without going via the Generalized Log-Sobolev. Namely, note that Prekopa-Leindler ineqaulity implies that

$$
\frac{d^{2}}{d t^{2}} \int e^{-(f+t g)^{*}} \leq 0
$$

and do the computation which confirms that this is equivalent to the Brascamp-Lieb inequality

$$
\int \varphi^{2} d \mu-\left(\int \varphi d \mu\right)^{2} \leq \int\left\langle\left(\nabla^{2} V\right)^{-1} \nabla \varphi, \nabla \varphi\right\rangle d \mu
$$

with $d \mu=e^{-V} d x$, where $V=f^{*}$, and $\varphi(x)=g\left(\nabla f^{*}(x)\right)$, and we assume that $\int d \mu=1$.

## 4 The Blaschke-Santaló inequality and friends

### 4.1 The formulation of the Blaschke-Santaló inequality

Let $K$ be a symmetric convex body, recall

$$
K^{\circ}=\{x: \forall y \in K,\langle x, y\rangle \leq 1\}
$$

is its polar, and let $T$ a linear operator. Recall that $(T K)^{\circ}=\left(T^{-1}\right)^{\top} K^{\circ}$. The volume product $|K| \cdot\left|K^{\circ}\right|$ is affine invariant:

$$
|T K| \cdot\left|(T K)^{\circ}\right|=\operatorname{det} T|K| \operatorname{det} T^{-1}\left|K^{\circ}\right|=1 \cdot|K| \cdot\left|K^{\circ}\right|=|K| \cdot\left|K^{\circ}\right|
$$

In particular, for any ellipsoid $E$,

$$
|E| \cdot\left|E^{\circ}\right|=\left|B_{2}^{n}\right|^{2} \sim\left(\frac{2 \pi e^{2}}{n}\right)^{n}
$$

and for any parallelpiped $P$,

$$
|P| \cdot\left|P^{\circ}\right|=\left|B_{\infty}^{n}\right| \cdot\left|B_{1}^{n}\right|=\frac{4^{n}}{n!} \sim \frac{(4 e)^{n}}{n^{n}}
$$

We formulate the celebrated
Theorem 4.1 (Blaschke-Santalo inequality [161]). For any symmetric convex body $K$,

$$
|K| \cdot\left|K^{\circ}\right| \leq\left|B_{2}^{n}\right|^{2}
$$

And what about the estimate from below?
Conjecture 4.2 (Mahler, 1937 (symmetric version)). For a symmetric convex body $K$ in $\mathbb{R}^{n},|K| \cdot\left|K^{\circ}\right| \geq \frac{4^{n}}{n!}=\left|B_{\infty}^{n}\right| \cdot\left|B_{1}^{n}\right|$.

Mahler proved it in dimension 2, but also see home work. Iryeh, Shibata proved it in dimension 3, and their proof was later simplified by Fradelizi, Hubard, Meyer, RoldanPensado, Zvavitch.

Remark 4.3 (answering Siva's question). Obtaining an isoperimetric inequality in the other direction in a similar fashion is less simple, since it is not invariant under affine transformations. (That is, there are needle-shaped convex bodies with arbitrarily small isoperimetric ratio.) However, we may make it affine invariant by taking the infimum over affine transformations. There is the following theorem by Kieth Ball: Let $K$ be a symmetric convex body with $|K|=\left|B_{\infty}^{n}\right|=2^{n}$, then

$$
\inf _{T: \operatorname{det} T=1}|\partial(T K)|_{n-1} \leq\left|\partial B_{\infty}^{n}\right|_{n-1}
$$

In general, cube often appears as (sometimes conjectured) optimizer in various reverse isoperimetrictype inequalities, thus Mahler conjecture is not the only one of this sort.

### 4.2 Hanner polytopes

The following interesting construction of a class of convex bodies appears in [82].
Definition 4.4 (Hanner polytopes). Hanner polytopes are defined inductively. In dimension 1 , the symmetric interval $[-1,1]$ is the Hanner polyope.

Next, let $H, K$ be Hanner polytopes in $\mathbb{R}^{k}, \mathbb{R}^{n-k}$ respectively. Consider them in orthogonal subspaces in $\mathbb{R}^{n}$. Then $H \times K=\{(x, y): x \in H, y \in K\}$ is a Hanner polytope in $\mathbb{R}^{n}$, and $\operatorname{conv}(K, H)$ - the convex hull of $K \cup H$ - is another Hanner polytope in $\mathbb{R}^{n}$.


The family of Hanner polytopes in $\mathbb{R}^{n}$ includes $B_{1}^{n}$ and $B_{\infty}^{n}$.
In dimension 2, the only Hanner polytope is the square. In dimension 3, the cube $B_{\infty}^{3}$ and the diamond $B_{1}^{3}$ are the only Hanner polytopes (up to rotations and dilations). In dimension 4 , the Hanner polytopes are $B_{\infty}^{4}, B_{1}^{4}$ and $B_{1}^{2} \times B_{1}^{2}$.

It is a homework problem to show that for any Hanner polytope $H \subseteq \mathbb{R}^{n}$ with nonempty interior,

$$
|H| \cdot\left|H^{\circ}\right|=\frac{4^{n}}{n!}=\left|B_{\infty}^{n}\right| \cdot\left|B_{1}^{n}\right| .
$$

Therefore, there is many conjectured minimizers in Mahler's conjecture. Kim [89] showed that all the Hanner polytopes are locally minimal for the volume product.

### 4.3 About Bourgain-Milman's theorem

Bourgain and Milman [38] showed the following "isomorphic version" of the Mahler conjecture:
Theorem 4.5 (Bourgain-Milman 1987). For any symmetric convex body $K \subseteq \mathbb{R}^{n}$,

$$
|K| \cdot\left|K^{\circ}\right| \geq \frac{c^{n}}{n^{n}}=\tilde{c}^{n}\left|B_{\infty}^{n}\right| \cdot\left|B_{1}^{n}\right|
$$

## Corollary 4.6.

$$
\left(n|K| \cdot\left|K^{\circ}\right|\right)^{1 / n} \in\left[c_{1}, c_{2}\right]
$$

Here $c_{2}=2 \pi e^{2}+o(1)$ comes from the Blaschke-Santalo inequality, and $c_{1}$ is the constant from Bourgain-Milman's theorem. If Mahler's conjecture is true, then $c_{1}=4 e+o(1)$. The current best constant is due to Kuperberg [119] who showed $c_{1}=\pi e+o(1)$. See also Nazarov [148], Berndtsson [20], Giannopolous, Paouris, Vritsiou [70].

The Bourgain-Milman theorem has many applications to problems where the dimensional dependence is studied up to an absolute constant.

### 4.4 Volume product of non-symmetric convex bodies and related questions and results

Note that if $0 \notin \operatorname{int}(K)$, then $\left|K^{\circ}\right|=\infty$. Indeed, in this case $K \subset H$ for some half-space $H$ which does not contain the origin, and therefore, $H^{o} \subset K^{o}$. But one may check that $H^{o}$ contains an infinite ray, and therefore is unbounded.

Hence there is no hope to have Blaschke-Santaló inequality for convex bodies without the symmetry assumption, unless we do some trick...

Consider the quantity

$$
\inf _{z \in \mathbb{R}^{n}}|K-z| \cdot\left|(K-z)^{\circ}\right|=|K| \cdot \inf _{z \in \mathbb{R}^{n}}\left|(K-z)^{\circ}\right|
$$

This quantity is now bounded, and in fact, the infimum is attained (by compactness) at a unique point (see home work).

Definition 4.7 (Santaló point). The point $z$ for which the infimum is attained is called the Santaló point of $K, \operatorname{san}(K)$. If $\operatorname{san}(K)=0$, the convex body is said to be in the Santaló position. It is a homework problem to show that the Santaló point exists and is unique.

Theorem 4.8 (the Non-symmetric Blaschke-Santaló inequality). For any convex body K,

$$
|K| \cdot \inf _{z \in \mathbb{R}^{n}}\left|(K-z)^{\circ}\right|=|K| \cdot\left|(K-\operatorname{san}(K))^{\circ}\right| \leq\left|B_{2}^{n}\right|^{2}
$$

Many special centers (besides the Santaló point) can be defined for a non-symmetric convex body. Below we define one of the most commonly used ones:

Definition 4.9 (Center of Mass). The center of mass (or barycenter) of a convex body $K$ is bar $K=\int_{K} x \mathrm{~d} x$.

More generally, for a measure $\mu$ on $\mathbb{R}^{n}$ its barycenter is defined to be the vector $\int_{\mathbb{R}^{n}} x d \mu$. In general, the Santalo point and center of mass do not coincide, and in fact, they may be very far from each other.

One might ask if Blaschke-Santaló inequality holds when the center of $K$ is chosen to be some other point rather than the Santalo point. Naturally, one can claim Blaschke-Santaló inequality when $K^{o}$ is in the Santaló position, rather than $K$. It turns out (home work!) that $K$ is in the Santaló position if and only if $K^{o}$ has the barycenter at the origin! Therefore, the Blaschke-Santalo inequality is still valid if the center is chosen to be the center of mass (of either $K$ or $K^{o}$ ):
Claim 4.10. For any convex body $K$,

$$
|K| \cdot\left|(K-\operatorname{bar} K)^{\circ}\right| \leq\left|B_{2}^{n}\right|^{2}
$$

A notable original proof of the (non-symmetric) Blaschke-Santaló inequality was found by Lehec [129], [130], using explicitly that the body is positioned to have the barycenter at the origin.

The following remarkable recent theorem from [73] interpolates between the two equivalent formulations of the non-symmetric the Blaschke-Santaló inequality, and allows for an estimate for any convex body which contains the origin in its interior!

Theorem 4.11 (Gozlan, Fradelizi, Sadovsky, Zugmeyer). For any convex body $K$ with origin in the interior,

$$
|K| \cdot\left|K^{\circ}\right| \leq\left|B_{2}^{n}\right|^{2}\left(1-\left\langle\operatorname{bar} K, \operatorname{san} K^{\circ}\right\rangle\right)^{n+1} .
$$

Remark 4.12 (An analytic characterization of the Santaló point). If $K$ is a convex body with support function $h_{K}$, then $\rho_{K^{\circ}}=1 / h_{K}$. For a convex body $L$

$$
|L|=\frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_{L}^{n}(\theta) \mathrm{d} \theta
$$

as can be shown using polar coordinates (see home work!) Therefore,

$$
\left|K^{\circ}\right|=\frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K}^{-n}(\theta) \mathrm{d} \theta
$$

Additionally, $h_{K+z}(\theta)=h_{K}(\theta)+\langle\theta, z\rangle$. So,

$$
\left|(K+z)^{\circ}\right|=\frac{1}{n} \int_{\mathbb{S}^{n}-1}\left(h_{K}+\langle\theta, z\rangle\right)^{-n \mathrm{~d} \theta}
$$

Taking derivatives in $z$,

$$
\int_{\mathbb{S}^{n-1}} \frac{\theta_{i}}{h_{K}^{n+1}(\theta)} \mathrm{d} \theta=0
$$

for each $i=1, \ldots, n$.

Finally, we mention the famous
Conjecture 4.13 (Non-symmetric Mahler conjecture). For any convex body K,

$$
|K| \cdot\left|K^{\circ}\right| \geq\left|S_{n-1}\right|^{2}
$$

where $S_{n-1}=\operatorname{conv}\left(a_{1}, \ldots, a_{n+1}\right)$ is the regular simplex.


### 4.5 A connection with the slicing problem

Klartag [93] found a connection between Conjecture 4.13 and the sharp version of the notorious Bourgain's slicing problem [36], [37], [109]:

Conjecture 4.14 (Bourgain, sharp version of the slicing problem). Let $K$ be a convex body, and let $\operatorname{Cov}(K)$ be the covariance matrix of the random vector distributed uniformly on $K$. Then the quantity

$$
\frac{\operatorname{det}(\operatorname{Cov}(K))}{|K|^{2}}
$$

is maximized when $K$ is a regular simplex.
Theorem 4.15 (Klartag, [93]). Conjecture 4.14 implies Conjecture 4.13.
It is worth also noting, that the affirmative answer to Conjecture 4.14 would yield the affirmative answer to the following long-standing and simple-sounding open problem:

Conjecture 4.16 (Bourgain, the slicing problem). Let $K$ be a convex body in $\mathbb{R}^{n},|K|=1$. Then there exists $\theta \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$ such that $\left|K \cap\left(\theta^{\perp}+t \theta\right)\right|_{n-1} \geq c$, where $c>0$ is an absolute constant independent of the dimension.


It should be noted that the (asymptotic) equivalence between Conjectures 4.14 and 4.16 is fairly straightforward, in contrast to the highly non-trivial Theorem 4.15.

Definition 4.17. A measure $\mu$ is called isotropic if $\int x d \mu=0$ and $\operatorname{Cov}(\mu)=I d$, where

$$
\operatorname{Cov}(\mu)=\left(\mathbb{E} X_{i} X_{j}\right)_{i j}
$$

for a random variable $X \sim \mu$.
Remark 4.18. For any (full-dimensional) measures $\mu$ there exists $T$ such that $\mu \circ T$ is isotropic.

One may also show (see e.g. Eldan, Klartag [59]) that Conjecture 4.16 follows from the so-called KLS conjecture:

Conjecture 4.19 (Kannan, Lovasz, Simonovits [88]). Let $\mu$ be an isotropic log-concave measure. Then for any locally-Lipschitz function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$, the Poincare inequality holds with a constant that does not depend on dimension:

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq C \int|\nabla f|^{2} d \mu
$$

where $C>0$ is an absolute constant.


Equivalently, if $K$ is a convex body, and we want to find a cut which splits $K$ into two parts each with volume half of $|K|$, then the cut of least perimeter is the one determined by a hyperplane (up to a constant.) See the home work for the equivalence.

### 4.6 Proof of the symmetric Blaschke-Santalo inequality, and an interesting open problem

Claim 4.20. For any symmetric convex body $K \subseteq \mathbb{R}^{n}$,

$$
|K| \cdot\left|K^{\circ}\right| \leq\left|B_{2}^{n}\right|^{2}
$$

The proof uses Steiner symmetrization. We shall show:
Lemma 4.21. $\left|K^{\circ}\right| \leq\left|S_{u}(K)^{\circ}\right|$ for all $u \in \mathbb{S}^{n-1}$.
Proof. WLOG, suppose $u=e_{n}$. We write

$$
S_{e_{n}}(K)=\left\{\left(x, \frac{s-t}{2}\right):(x, s),(x, t) \in K\right\}
$$

Therefore,

$$
S_{e_{n}}(K)^{\circ}=\left\{(y, z):\langle x, y\rangle+z \frac{s-t}{2} \leq 1 \forall(x, s),(x, t) \in K\right\}
$$

Define the slice $L(r)=\left\{x \in \mathbb{R}^{n-1}:(x, r) \in K\right\}$ for $L \subseteq \mathbb{R}^{n}$. Consider

$$
\frac{K^{\circ}(r)+K^{\circ}(-r)}{2}=\left\{\frac{y+z}{2}:\langle x, y\rangle+s r \leq 1,\langle w, z\rangle-t r \leq 1, \quad \forall(x, s),(w, t) \in K\right\}
$$

By reducing the number of restrictions, we obtain

$$
\begin{aligned}
\frac{K^{\circ}(r)+K^{\circ}(-r)}{2} & \subseteq\left\{\frac{y+z}{2}:\langle x, y\rangle+s r \leq 1,\langle x, z\rangle-t r \leq 1, \forall(x, s),(x, t) \in K\right\} \\
& \subseteq\left\{\frac{y+z}{2}:\left\langle x, \frac{y+z}{2}\right\rangle+\frac{s-t}{2} r \leq 1 \forall(x, s),(x, t) \in K\right\} \\
& =\left\{v:\langle x, v\rangle+\frac{s-t}{2} r \leq 1 \forall(x, s),(x, t) \in K\right\} \\
& =S_{e_{n}}(K)^{\circ}(r)
\end{aligned}
$$



Next, $|K(r)|$ is an even function of $r$ because $K$ is symmetric,. By the Brunn-Minkowski inequality,

$$
\left|\frac{K^{\circ}(r)+K^{\circ}(-r)}{2}\right| \geq \sqrt{\left|K^{\circ}(r) \cdot\right| K^{\circ}(-r) \mid}=\left|K^{\circ}(r)\right|
$$

which implies $\left|S_{e_{n}}(K)^{\circ}(r)\right| \geq\left|K^{\circ}(r)\right|$. Therefore, using Fubini's theorem, we get

$$
\left|K^{\circ}\right|=\int_{-\infty}^{\infty}\left|K^{\circ}(r)\right| \mathrm{d} r \leq \int_{-\infty}^{\infty}\left|S_{e_{n}}(K)^{\circ}(r)\right| \mathrm{d} r=\left|S_{e_{n}}(K)^{\circ}\right|
$$

In order to derive the Blaschke-Santalo inequality from Lemma 4.21, we select a sequence of directions such that the successive symmetrizations of $K$ approach a ball, and note that the polar volume increases along this sequence, while the volume remains preseved. Namely, choose a sequence $u_{1}, u_{2}, \ldots$ such that $S_{u_{k}, \ldots, u_{1}} K \rightarrow R B_{2}^{n}$, where $R=\frac{|K|^{1 / n}}{\left|B_{2}^{n}\right|^{1 / n}}$. Then

$$
|K| \cdot\left|K^{\circ}\right| \leq\left|R B_{2}^{n}\right|\left|\left(R B_{2}^{n}\right)^{\circ}\right|=\left|B_{2}^{n}\right|^{2} . \square
$$

We mention another open question:
Conjecture 4.22 (Cordero-Erasquin). Let $\mu$ be an even log-concave measure and $K$ a symmetric convex body. Then

$$
\mu(K) \mu\left(K^{o}\right) \leq \mu\left(B_{2}^{n}\right)^{2} .
$$

In particular, for any symmetric convex body $L$,

$$
|K \cap L| \cdot\left|K^{o} \cap L\right| \leq\left|B_{2}^{n} \cap L\right|^{2} .
$$

Klartag proved this holds for some rotation invariant measures [96].
Question: Is it even true that $\mu\left(R B_{2}^{n}\right) \mu\left(\left(R B_{2}^{n}\right)^{\circ}\right) \leq \mu\left(B_{2}^{n}\right)^{2}$ ?
Answer: yes! and this fact is equivalent to the recent deep result of Cordero-Erasquin and Rotem [49] (which we will discuss in more detail in Remark 7.35).

### 4.7 A fun inequality on the circle

Recall that (see Remark 4.12): if $K$ is a symmetric convex body in $\mathbb{R}^{n}$ with support function $h_{K}$, then $\rho_{K^{\circ}}=1 / h_{K}$, and we have

$$
\left|K^{\circ}\right|=\frac{1}{n} \int_{\mathbb{S}^{n}-1} h_{K}^{-n}(\theta) \mathrm{d} \theta
$$

We now want to deduce the volume of $K$ (on the plane) in terms of the support function. To start, suppose that $K$ is a polygon with $h_{i}$ denoting the distance to side $i$ with length $a_{i}$ (see Figure 4.7). Then, $|K|=\frac{1}{2} \sum_{i} a_{i} h_{i}$


What happens as $K$ becomes smooth? We need to understand what the heights $h_{i}$ and side lengths $a_{i}$ converge to. Indeed, the geometric interpretation of the support function tells us that $h_{i}$ converges to $h_{K}$ (see Remark 2.12), and the side lengths converge to the surface area measure, the inverse of which is called the curvature, denoted $k_{K}(\theta)$. Also, we note that $k_{K}=\frac{1}{h_{K}+\check{h}_{K}}($ see HW). This implies

$$
\begin{equation*}
|K|=\frac{1}{2} \int_{\mathbb{S}^{1}} h_{K}(\theta) \frac{1}{k_{K}(\theta)} \mathrm{d} \theta=\frac{1}{2} \int_{\mathbb{S}^{1}} h_{K}(\theta) \cdot\left(h_{K}(\theta)+\ddot{h}_{K}(\theta)\right) \mathrm{d} \theta . \tag{8}
\end{equation*}
$$

Note that $h$ is a periodic function, and on integrating by parts we obtain

$$
\int_{-\pi}^{\pi} h \ddot{h}=-\int_{-\pi}^{\pi} \dot{h}^{2} .
$$

Combined with (8), we conclude

$$
\begin{equation*}
|K|=\frac{1}{2} \int_{\mathbb{S}^{1}}\left(h_{K}^{2}(\theta)-\dot{h}_{K}^{2}(\theta)\right) \mathrm{d} \theta \tag{9}
\end{equation*}
$$

Equation (9) is called Santalo's formula. Recall that the Blaschke-Santalo inequality in $\mathbb{R}^{2}$ implies $|K| \cdot\left|K^{\circ}\right| \leq \pi^{2}$, for a symmetric convex body $K$. Thus,

$$
\left(\frac{1}{2} \int_{-\pi}^{\pi}\left(h_{K}^{2}(\theta)-\dot{h}_{K}^{2}(\theta)\right) \mathrm{d} \theta\right) \cdot\left(\frac{1}{2} \int_{-\pi}^{\pi} h_{K}^{-2}(\theta) \mathrm{d} \theta\right) \leq \pi^{2}
$$

which gives

$$
\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi}\left(h_{K}^{2}(\theta)-\dot{h}_{K}^{2}(\theta)\right) \mathrm{d} \theta\right) \cdot\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} h_{K}^{-2}(\theta) \mathrm{d} \theta\right) \leq 1 .
$$

In fact, we do not need to assume that $h$ is a support function. We have the following general result.

Proposition 4.23. Suppose $f:[-\pi, \pi] \rightarrow \mathbb{R}$ such that $f \in C^{1}$, $\pi$-periodic. Then,

$$
\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi}\left(\dot{f}^{2}(\theta)-f^{2}(\theta)\right) \mathrm{d} \theta\right)+\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} f^{-2}(\theta) \mathrm{d} \theta\right)^{-1} \geq 0
$$

Proof. Without loss of generality we may assume that $f \geq 0$ and $f \in C^{2}(-\pi, \pi)$. If $f$ is the support function of a symmetric convex body in $\mathbb{R}^{2}$, then result holds by the Blaschke-Santalo inequality, as we explained above. Note that $f$ is a support function if the 1-homogeneous extension of $f$ into $\mathbb{R}^{2}$ is convex. This means that $f+f \geq 0$.

So, we consider any function $f \geq 0$, and consider $F: \mathbb{R}^{2} \rightarrow \mathbb{R}$ such that $F(x)=$ $|x| \cdot f\left(\frac{x}{|x|}\right)$, and $G(x)=F^{\star \star}(x)$, the double-Legendre transform of $F$ (or, the convexification of $F$ ). Furthermore, define $g(\theta)=G(\theta)$ where $g$ is a support function such that $g \leq f$ and $g$ is the largest such function. Then, we have the following properties:

1. $0 \leq g \leq f$,
2. $g+\ddot{g} \geq 0$, and
3. if $f+\ddot{f} \geq 0$, then $g=f$.

Using these properties, we can show the following result.

$$
\int\left(\dot{f}^{2}-f^{2}\right)+\left(\int f^{-2}\right)^{-1} \geq \int\left(\dot{g}^{2}-g^{2}\right)+\left(\int g^{-2}\right)^{-1}
$$

First, by property 1, we have

$$
\begin{equation*}
\left(\int f^{-2}\right)^{-1} \geq\left(\int g^{-2}\right)^{-1} \tag{10}
\end{equation*}
$$

Next, consider

$$
\begin{aligned}
\int\left(\dot{f}^{2}-f^{2}\right) & =-\int f(f+\ddot{f}) \\
& =\underbrace{-\int_{f+\ddot{f}<0} f \cdot(f+\ddot{f})}_{\geq 0}-\int_{f+\ddot{f} \geq 0} f \cdot(f+\ddot{f}) \\
& \geq-\int_{f+\ddot{f} \geq 0} f \cdot(f+\ddot{f}) \\
& =-\int_{f+\ddot{f} \geq 0} g \cdot(g+\ddot{g}) \\
& \geq-\int g \cdot(g+\ddot{g}) \\
& =\int\left(\dot{g}^{2}-g^{2}\right)
\end{aligned}
$$

where the first equality uses integration by parts, the third equality uses property 3 , the second inequality uses $-\int_{f+\ddot{f}<0} g \cdot(g+\ddot{g})<0$ (which follows from properties 1 and 2), and the final equality again uses integration by parts. Combined with (10), we get the desired inequality, which concludes the proof.

In fact, the following more general result holds (from the non-symmetric Blaschke-Santalo inequality).

Theorem 4.24. For all functions $f \in C^{1}(-\pi, \pi)$ and periodic, there exists $t, s \in \mathbb{R}$ such that $f_{t, s}(\theta)=f(\theta)+t \cos \theta+s \sin \theta$ such that

$$
\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \dot{f}_{t, s}^{2}-f_{t, s}^{2}\right)+\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} f_{t, s}^{-2}\right)^{-1} \geq 0
$$

Remark 4.25. This was obtained via an independent proof by Lutwak, Young, Zhang [45], who in fact proved a more general result, and in particular found a new proof of the BlaschkeSantaló inequality on the plane.

Now, consider the result in Proposition 4.23, and reparametrize it such that $\phi(\theta)=f\left(\frac{\theta}{2}\right)$. Since $f$ was $\pi$-periodic, $\phi$ is $2 \pi$-periodic, and we obtain the following result.

Theorem 4.26. For all functions $\phi \in C^{1}(-\pi, \pi)$ and circle-periodic, we have

$$
4 \cdot\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \dot{\phi}^{2}\right) \geq\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi^{2}\right)-\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi^{-2}\right)^{-1}
$$

Remark 4.27. This result looks similar to the p-Beckner inequality (see Theorem 3.66). We recall it here for completeness. For $p \in[1,2)$, we have

$$
\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi^{2}\right)-\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \phi^{p}\right)^{\frac{2}{p}} \leq(2-p)\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \dot{\phi}^{2}\right) .
$$

We have shown that the Blaschke-Santalo inequality in $\mathbb{R}^{2}$ is equivalent to the $p$-Beckner inequality on the circle for $p=-2$.

A HW question. Is there an interval of negative $p$ where the $p$-Beckner inequality holds on the circle?
Remark 4.28. If $g=\phi^{-2}$, then

$$
\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{1}{g}\right)-\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} g\right)^{-1} \leq\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{\dot{g}^{2}}{g^{3}}\right)
$$

and if $f=\phi^{2}$, we obtain

$$
\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} f\right)-\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} f^{-1}\right)^{-1} \leq\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{\dot{f}^{2}}{f}\right)
$$

### 4.8 Functional version of the Blaschke-Santalo inequality

Below we present a functional version of the Blaschke-Santalo inequality. introduced by Ball [10]. We shall see that it implies the usual (geometric) Blaschke-Santalo inequality; our proof will also be based on the geometric version, following the work of Arstein-Avidan, Klartag, Milman [5]. For simplicity, we focus on the symmetric version, but a non-symmetric functional Blaschke-Santalo is available too [5]. We recommend also the proof by Lehec [129] which did not rely on the geometric Blaschke-Santalo inequality.

Theorem 4.29 (Ball [10]; Arstein-Avidan, Klartag, Milman [5]; Lehec [129]). If $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is an even function such that $\int e^{-\psi}<\infty$, then

$$
\int e^{-\psi} \cdot \int e^{-\psi^{\star}} \leq\left(e^{-x^{2} / 2}\right)^{2}=(2 \pi)^{n}
$$

Recall that given a function $\psi$, the function $\psi^{*}$ is the smallest of the functions $\varphi$ which satisfy for every $x, y \in \mathbb{R}^{n}$ the inequality $\psi(x)+\varphi(y) \geq\langle x, y\rangle$. Therefore, we get:
Corollary 4.30. Suppose that $f, g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ are such that $f(x) \cdot g(y) \leq e^{-\langle x, y\rangle}$, then $\int f \cdot \int g \leq(2 \pi)^{n}$.

Proof of theorem 4.29. For any constant $c$, we have $(\psi+c)^{\star}=\psi^{\star}-c$, and so we can assume that $\psi \geq 0$. Moreover, we assume that $\psi(0)=0$ which implies that $\psi^{\star}(0)=0$ and $\psi^{\star} \geq 0$.

Furthermore, we can assume without loss of generality that $\psi$ is convex. Indeed, otherwise we can replace the left hand side with $\psi^{* *}$ and it only increases.

We have

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} e^{-\psi} \mathrm{d} x & =\int_{0}^{\infty}\left|\left\{e^{-\psi}>t\right\}\right| \mathrm{d} t \\
& =\int_{0}^{\infty} e^{-s}|\{\psi<s\}| \mathrm{d} s
\end{aligned}
$$

where the first equality follows by the Fubini theorem, the second equality applies the change of variable $t=e^{-s}$. Similarly, we obtain

$$
\int_{\mathbb{R}^{n}} e^{-\psi^{\star}} \mathrm{d} x=\int_{0}^{\infty} e^{-s}\left|\left\{\psi^{\star}<s\right\}\right| \mathrm{d} s
$$

We make the following claim.
Claim 4.31. For any $s, t \geq 0$,

$$
\left\{\psi^{\star}<t\right\} \subset(s+t) \cdot\{\psi<s\}^{\circ} .
$$

Proof. Consider $x \in\{\psi<s\}$ and $y \in\left\{\psi^{\star}<t\right\}$ and note that it suffices to show that $\langle x, y\rangle \leq(s+t)$. By the property of the Legendre transform, we know that

$$
\langle x, y\rangle \leq \psi(x)+\psi^{\star}(y) \leq s+t
$$

Consider the following three functions on $\mathbb{R}_{+}$:

- $f(s)=e^{-s} \cdot|\{\psi<s\}|$
- $g(t)=e^{-t} \cdot\left|\left\{\psi^{\star}<t\right\}\right|$
- $h(x)=\left|B_{2}^{n}\right| \cdot 2^{n / 2} \cdot e^{-x} \cdot x^{n / 2}$

We will apply the Prekopa-Leindler inequality on these functions. First, we claim that

$$
h\left(\frac{s+t}{2}\right) \geq \sqrt{f(s) \cdot g(t)}
$$

Indeed, we can write

$$
h^{2}\left(\frac{s+t}{2}\right)=\left|B_{2}^{n}\right|^{2} \cdot 2^{n} \cdot e^{-(s+t))} \cdot\left(\frac{s+t}{2}\right)^{n}=\left|B_{2}^{n}\right|^{2} \cdot e^{-s} \cdot e^{-t} \cdot(s+t)^{n}
$$

We have,

$$
\begin{aligned}
f(s) \cdot g(t) & =e^{-s} \cdot|\{\psi<s\}| \cdot e^{-t} \cdot\left|\left\{\psi^{\star}<t\right\}\right| \\
& \leq e^{-s} \cdot|\{\psi<s\}| \cdot e^{-t} \cdot(s+t)^{n} \cdot\left|\{\psi<s\}^{\circ}\right| \\
& \leq e^{-s} \cdot e^{-t} \cdot(s+t)^{n} \cdot\left|B_{2}^{n}\right|^{2}=h^{2}\left(\frac{s+t}{2}\right),
\end{aligned}
$$

where the first inequality uses Claim 4.31, and the second inequality uses the BlaschkeSantalo inequality. Thus, the three functions satisfy the conditions of the Prekopa-Leindler inequality, and we get

$$
\int e^{-\psi} \cdot \int e^{-\psi^{\star}}=\int e^{-s}|\psi<s| \cdot \int e^{-t}\left|\psi^{\star}<t\right| \leq\left(\int e^{-x} \cdot x^{n / 2}\right)^{2} \cdot 2^{n} \cdot\left|B_{2}^{n}\right|^{2}=(2 \pi)^{n}
$$

Remark 4.32. Setting $\psi(x)=\frac{\|x\|_{K}^{2}}{2}$ and $\psi^{\star}(x)=\frac{\|x\|_{K^{\circ}}^{2}}{2}$, and integrating in polar coordinates recovers the usual Blaschke-Santalo inequality (see home work).

Remark 4.33. The equality case occurs if and only if $\psi(x)=\frac{|x|^{2}}{2}$, as was shown in [5].
We also mention the following theorem (see HW).
Theorem 4.34 (Fradelizi-Meyer [67]). Consider even functions $f, g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that for all $x, y, \in \mathbb{R}^{n}, f(x) \cdot g(y) \leq \rho(\langle x, y\rangle)$ whenever $\langle x, y\rangle \geq 0$. Then,

$$
\left(\int f\right) \cdot\left(\int g\right) \leq\left(\int \rho\left(|x|^{2}\right)\right)^{2}
$$

This holds for any $\rho: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$.

### 4.9 Linearizing Theorem 4.29.

Take $\psi=\frac{|x|^{2}}{2}+\epsilon f$ for some function $f$. Then, we have

$$
\int e^{-\left(\frac{|x|^{2}}{2}+\epsilon f\right)} \cdot \int e^{-\left(\frac{|x|^{2}}{2}+\epsilon f\right)^{\star}} \leq(2 \pi)^{n} .
$$

Recall that

$$
v_{t}^{\star}=v_{t}-t \dot{v}_{t}\left(\nabla v_{t}\right)-\frac{t^{2}}{2} \ddot{v}_{t}\left(\nabla v_{t}\right)+\frac{t^{2}}{2}\left\langle\left(\nabla^{2} v_{t}\right)^{-1} \nabla\left[\left.\dot{v}_{t}\right|_{\nabla v_{t}}\right], \nabla\left[\left.\dot{v}_{t}\right|_{\nabla v_{t}^{*}(x)}\right]\right\rangle .
$$

Then

$$
\left(\frac{|x|^{2}}{2}+\epsilon f\right)^{\star}=\frac{|x|^{2}}{2}-\epsilon f+\frac{\epsilon^{2}}{2}|\nabla f|^{2}+o\left(\epsilon^{2}\right)
$$

since $\nabla v_{0}=x$ and $\nabla^{2} v_{0}=$ Id. So, we have (up to the terms of order $o\left(\epsilon^{2}\right)$ ):

$$
\int e^{-\frac{|x|^{2}}{2}-\epsilon f} \cdot \int e^{-\frac{|x|^{2}}{2}+\epsilon f-\frac{\epsilon^{2}}{2}|\nabla f|^{2}} \leq(2 \pi)^{n} .
$$

Using $e^{-\delta}=1-\delta+\frac{\delta^{2}}{2}$ (up to lower order terms), we get

$$
(2 \pi)^{n} \geq\left(\int e^{-\frac{|x|^{2}}{2}} \cdot\left(1-\epsilon f+\frac{\epsilon^{2}}{2} f^{2}\right)\right) \cdot\left(\int e^{-\frac{|x|^{2}}{2}} \cdot\left(1-\epsilon f+\frac{\epsilon^{2}}{2}|\nabla f|^{2}-\frac{\epsilon^{2} f^{2}}{2}\right)\right)+o\left(\epsilon^{2}\right) .
$$

Dividing both sides by $(2 \pi)^{n}$ gives

$$
1 \geq\left(\int\left(1-\epsilon f+\frac{\epsilon^{2}}{2} f^{2}\right) \mathrm{d} \gamma\right) \cdot\left(\int\left(1-\epsilon f+\frac{\epsilon^{2}}{2}|\nabla f|^{2}-\frac{\epsilon^{2} f^{2}}{2}\right) \mathrm{d} \gamma\right)+o\left(\epsilon^{2}\right)
$$

where $\mathrm{d} \gamma$ is the Gaussian measure. We note that the constant terms cancel out, and so do the terms which are multiplied by $\epsilon$. Collecting the terms multiplied by $\epsilon^{2}$ gives the following inequality:
Theorem 4.35. For all even functions $f$, we have

$$
\int_{\mathbb{R}^{n}} f^{2} \mathrm{~d} \gamma-\left(\int_{\mathbb{R}^{n}} f \mathrm{~d} \gamma\right)^{2} \leq \frac{1}{2} \int_{\mathbb{R}^{n}}|\nabla f|^{2} \mathrm{~d} \gamma
$$

Remark 4.36. Note the improved the constant as compared to the Gaussian Poincare's inequality which we obtained from Theorem 3.66.
Remark 4.37. We note that using the non-symmetric version of the Blaschke-Santalo inequality, it suffices to assume $\int \nabla f \mathrm{~d} \gamma=0$, as integrating by parts, we obtain

$$
\int \frac{\partial f}{\partial x_{i}} \mathrm{~d} \gamma=\int\left\langle\nabla f, \nabla x_{i}\right\rangle \mathrm{d} \gamma=-\int f \cdot L_{\gamma} x_{i} \mathrm{~d} \gamma=\int f \cdot x_{i} \mathrm{~d} \gamma
$$

So, $\int \nabla f \mathrm{~d} \gamma=0$ is equivalent to showing that for all linear functions $\langle x, \theta\rangle$, we have $\int f \cdot\langle x, \theta\rangle \mathrm{d} \gamma=0$. This implies that the second eigenvalue of the Ornstein-Uhlenbeck operator is 2 .

Remark 4.38. The same result could be obtained using methods of Fourier Analysis and the decomposition into Hermite polynomials.

Recall that the Brascamp-Lieb inequality allowed us to restrict the Gaussian Poincare's inequality to any convex set $K$; that is, $\int_{K} f^{2} \frac{\mathrm{~d} \gamma}{\gamma(K)}-\left(\int_{K} f \frac{\mathrm{~d} \gamma}{\gamma(K)}\right)^{2} \leq \int_{K}|\nabla f|^{2} \frac{\mathrm{~d} \gamma}{\gamma(K)}$. Can we also do this for Theorem 4.35?

### 4.10 A brief excursion into mass transport

Below we present a brief (one leg here one leg there!) excursion into mass transport. For a more detailed (but still brief) introduction we recommend Klartag [105]. Another insightful survey was done by Ball [11]. See also books by Villani [170], Figalli, Glaudo [64] and Bogachev, Kolesnikov, Shaposhnikov [28].

Definition 4.39. Let $\mu, \nu$ be Borel finite probability measures on $\mathbb{R}^{n}$. We say that a map $T: \operatorname{supp}(\mu) \rightarrow \mathbb{R}^{n}$ transports $\mu$ into $\nu$, denoted $T_{\star} \mu=\nu$ if for all $A \subset \mathbb{R}^{n}$ (measurable), we have

$$
\nu(A)=\mu\left(T^{-1}(A)\right)
$$

or equivalently, for all $\phi \in L^{1}(\nu)$,

$$
\int_{\mathbb{R}^{n}} \phi \mathrm{~d} \nu=\int_{\mathbb{R}^{n}} \phi \circ T \mathrm{~d} \mu .
$$

We consider some examples.
Example 4.40. Consider a discrete measure $\mu$ such that $\mu=\sum_{i} \lambda_{i} \delta_{x_{i}}$. Suppose $T$ is any bijection from $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, and $\nu=\sum_{i} \lambda_{i} \delta_{T x_{i}}=T_{\star} \mu$. In other words, fully atomic measures can be transported into other fully atomic measures via bijections. See the picture below.


Example 4.41. Let $\gamma$ be the Gaussian measure on $\mathbb{R}$, and let $\Phi$ be the corresponding cumulative distribution function; that is, $\Phi(t)=\int_{-\infty}^{t} \frac{1}{\sqrt{2 \pi}} e^{-s^{2} / 2} \mathrm{~d} s$. Then, $\Phi_{\star} \gamma=\operatorname{Unif}[0,1]$. To see this, consider $[x, y] \subset[0,1]$. We have

$$
|[x, y]|=y-x=\int_{\Phi^{-1}(x)}^{\Phi^{-1}(y)} \Phi^{\prime}(t) \mathrm{d} t=\int_{\Phi^{-1}(x)}^{\Phi^{-1}(y)} \mathrm{d} \gamma
$$

which verifies the claim.
Remark 4.42. Suppose $T_{\star} \mu=\nu$, and $T x=\left(T_{1} x, \ldots, T_{n} x\right)$. If the map $T$ has a Jacobian, where $\operatorname{Jac}(T)=\left(\frac{\partial T_{i} x}{\partial x_{j}}\right)_{i, j}$ then, by change of variables, we have $\int \phi \mathrm{d} \nu=\int \phi \circ T J a c(T) \mathrm{d} \nu$. Combined with $\int_{\mathbb{R}^{n}} \phi \mathrm{~d} \nu=\int_{\mathbb{R}^{n}} \phi \circ T \mathrm{~d} \mu$ we get, $\operatorname{Jac}(T) \mathrm{d} \nu=\mathrm{d} \mu$.

Definition 4.43 (Optimal Transport). A map $\widehat{T}: \mu \rightarrow \nu$ such that $\widehat{T}_{\star} \mu=\nu$ is called optimal with respect to quadratic cost if for all $T$ such that $T_{\star} \mu=\nu$ :

$$
\int_{\mathbb{R}^{n}}|\widehat{T} x-x|^{2} \mathrm{~d} \mu \leq \int_{\mathbb{R}^{n}}|T x-x|^{2} \mathrm{~d} \mu .
$$

Theorem 4.44 (Brenier). Suppose $\mu, \nu$ are absolutely continuous Borel measures and $T_{\star} \mu=$ $\nu$. Then $T$ is optimal with respect to quadratic cost if, and only if, there exists a convex function $F: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{\infty\}, F<\infty$ a.e. on $\operatorname{supp}(\mu)$ such that $T=\nabla F$.

By a standard compactness argument, we get
Corollary 4.45. Suppose $\mu$ and $\nu$ are Borell finite absolutely continuous probability measures.

There exists a convex function $F: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that $\nabla F_{\star} \mu \rightarrow \nu$ for all absolutely continuous probability measures (this is unique upto constant addition).

Remark 4.46. We know that $J a c(\nabla F)=\nabla^{2} F$ (if $F \in C^{2}\left(\mathbb{R}^{n}\right)$ ). Then, for any probability measure $\mu, \mathrm{d} \nu=\operatorname{det}\left(\nabla^{2} F\right) \mathrm{d} \mu$, and $\nabla F_{\star} \mu=\nu$.

We consider some examples.
Example 4.47. Suppose that we want to transport the Lesbesgue measure into the following measure: $\mathrm{d} \nu=\left(\delta_{e_{1}}+\delta_{-e_{1}}\right)$ where $\delta_{x}(y)=\infty$ if $y=x$, and is 0 otherwise. What is the optimal way to transport all the points in space into the two points $e_{1}$ and $-e_{1}$ ? It is easy to observe that, given any point $x$, we transport its measure to the "closer" point. We want $f(x)=g(T x)$, so it suffices to set

$$
T x= \begin{cases}e_{1} & \text { if } x_{1}>0 \\ -e_{1} & \text { if } x_{1} \leq 0\end{cases}
$$



Observe that $T x=\nabla F(x)$ where $F(x)=\left|\left\langle x, e_{1}\right\rangle\right|$. The pictures above and below are (c) Ball.


Note that this examples is not in the setting of the aforementioned theorem since the measures considered are not finite.

Example 4.48. More generally, if we want to transport the Lesbesgue measure into a measure with density which is the sum of some points, say $\sum_{i} \delta_{\theta_{i}}$. Then, one can check that $T_{\star} \lambda=\sum_{i} \delta_{\theta_{i}}$ with $T x=\nabla F(x)$ for $F=\max \left(\left\langle x, \theta_{i}\right\rangle\right)$. Here $\lambda$ denotes the Lesbesgue measure.

We now consider a discrete version of the easy direction of Brenier's theorem.
Lemma 4.49. Suppose we have points $x_{1}, \ldots, x_{N}, y_{1}, \ldots, y_{N} \in \mathbb{R}^{n}$. Furthermore, suppose there exists a convex function $F: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that, $\nabla F\left(x_{i}\right)=y_{i}$ for all $i \in[N]$. Then, for any permutation $\sigma$ on $N$ elements,

$$
\sum_{i=1}^{N}\left|x_{i}-y_{\sigma(i)}\right|^{2} \geq \sum_{i=1}^{N}\left|x_{i}-y_{i}\right|^{2}
$$

Proof. Since $F$ is convex, we have $F(y)-F(x) \geq\langle\nabla F(x), y-x\rangle$.


By summing up this inequality over pairs $\left(x_{i}, x_{\sigma(i)}\right)_{i \in[N]}$, we get

$$
\sum_{i=1}^{N}\left\langle\nabla F\left(x_{i}\right), x_{\sigma(i)}-x_{i}\right\rangle \leq \sum_{i=1}^{N}\left(F\left(x_{\sigma(i)}\right)-F\left(x_{i}\right)\right)=0
$$

where the final equality follows from the fact that $\sigma$ is a permutation of $x_{1}, \ldots, x_{N}$. So, we get

$$
\sum_{i=1}^{N}\left\langle\nabla F\left(x_{i}\right), x_{\sigma(i)}\right\rangle \leq \sum_{i=1}^{N}\left\langle\nabla F\left(x_{i}\right), x_{i}\right\rangle,
$$

which implies $\sum_{i=1}^{N}\left|F\left(x_{i}\right)-x_{i}\right|^{2} \leq \sum_{i=1}^{N}\left|F\left(x_{i}\right)-x_{\sigma(i)}\right|^{2}$, where we once again used the commutativity of addition.

Lastly, we state the following important result by Cafarelli [41] (see also Kolesnikov [111]):
Theorem 4.50 (Cafarelli's contraction theorem). Let $\gamma$ be the Gaussian measure, and let $\mu$ be a "strongly log-concave" measure; that is, $\mathrm{d} \mu=e^{-v} \mathrm{~d} x$ such that $\nabla^{2} v \geq \mathrm{Id}$. In other words, $\left(\frac{\mathrm{d} \mu}{\mathrm{d} x}\right) /\left(\frac{\mathrm{d} \gamma}{\mathrm{d} x}\right)$ is log-concave. Then the Brenier map $T: T_{\star} \gamma=\mu$ is 1-Lipschitz; that is, for all $x, y \in \mathbb{R}^{n},|T x-T y| \leq|x-y|$.

Remark 4.51. Recall that when $K$ is a convex set, the restriction of the Gaussian measure onto $K$, given by $\mathrm{d} \gamma_{K}=\mathbb{I}_{K} \cdot \mathrm{~d} \gamma \cdot \frac{1}{\gamma(K)}$, is strongly log-concave. By Theorem 4.50, the Brenier map between $\gamma$ and $\gamma_{K}$ is a contraction.

### 4.11 Restricting the symmetric Gaussian Poincaré inequality onto a symmetric convex set

Using Theorem 4.35 (the corollary of the Blaschke-Santalo inequality) and Cafarelli's contraction theorem, we prove the following result.

Theorem 4.52. For any symmetric convex set $K \subset \mathbb{R}^{n}$, and any even function $f: K \rightarrow \mathbb{R}$, we have

$$
\begin{equation*}
\int_{K} f^{2} \frac{\mathrm{~d} \gamma}{\gamma(K)}-\left(\int_{K} f \frac{\mathrm{~d} \gamma}{\gamma(K)}\right)^{2} \leq \frac{1}{2} \int_{K}|\nabla f|^{2} \frac{\mathrm{~d} \gamma}{\gamma(K)} \tag{11}
\end{equation*}
$$

Proof. Let $\mathrm{d} \gamma_{K}=\frac{1}{\gamma(K)} \cdot \mathbb{I}_{K} \mathrm{~d} \gamma$ be the restriction of the Gaussian measure on $K$. Consider the Brenier map $T: T_{\star} \gamma=\gamma_{K}$, which by Theorem 4.50 is a contraction. Consider the L.H.S. of (11), and using the definition of mass transport, we have

$$
\int_{K} f^{2} \frac{\mathrm{~d} \gamma}{\gamma(K)}-\left(\int_{K} f \frac{\mathrm{~d} \gamma}{\gamma(K)}\right)^{2}=\int_{\mathbb{R}^{n}}(f \circ T)^{2} \mathrm{~d} \gamma-\left(\int_{\mathbb{R}^{n}} f \circ T \mathrm{~d} \gamma\right)^{2}
$$

Additionally, by the Lipschitz property, we have

$$
\int_{\mathbb{R}^{n}}|\nabla(f \circ T)|^{2} \mathrm{~d} \gamma \leq \int_{\mathbb{R}^{n}}|\nabla(f) \circ T|^{2} \mathrm{~d} \gamma=\int_{K}|\nabla f|^{2} \frac{\mathrm{~d} \gamma}{\gamma(K)} .
$$

Applying Theorem 4.35, we get

$$
\frac{1}{2} \int_{\mathbb{R}^{n}}|\nabla(f \circ T)|^{2} \mathrm{~d} \gamma \geq \int_{\mathbb{R}^{n}}(f \circ T)^{2} \mathrm{~d} \gamma-\left(\int_{\mathbb{R}^{n}} f \circ T \mathrm{~d} \gamma\right)^{2},
$$

which on combining with the prior two (in)equalities completes the proof. Note that we could apply Theorem 4.35 because $f \circ T$ is an even function, which follows since $f$ is even (by assumption), and $T$ is even as its a Brenier map from $\gamma \rightarrow \gamma_{K}$ for symmetric K.

Remark 4.53. In fact, Klartag [96] used Cafarelli's theorem to deduce the following extension of the Blaschke-Santaloinequality: for any even log-concave measure $\mu$ and any even function $\psi$ on $\mathbb{R}^{n}$ one has

$$
\int e^{-\psi} d \mu \cdot \int e^{-\psi^{*}} d \mu\left(\int e^{-\frac{x^{2}}{2}} d \mu\right)^{2}
$$

Specializing to $d \mu=1_{K}(x) d x$ and using the computation similar to the one in subsection 4.9 one gets from this Theorems 4.52 and thus 4.56.

### 4.12 The B-conjecture

Conjecture 4.54 (B-conjecture). Let $\mu$ be an even log-concave measure on $\mathbb{R}^{n}$ and $K a$ symmetric convex set. Then $\mu\left(e^{t} K\right)$ is log-concave in $t \in \mathbb{R}$, i.e.

$$
\mu(\sqrt{a b} K) \geq \sqrt{\mu(a K) \mu(b K)} \quad \forall a, b \geq 0
$$

Remark 4.55. By Prekopa-Leindler, we know that, for any convex set $K$,

$$
\mu\left(\frac{a+b}{2} K\right) \geq \sqrt{\mu(a K) \mu(b K)}
$$

Note that, by the AM-GM inequality, this is weaker than the B-conjecture.

### 4.13 The B-theorem for the Gaussian measure due to CorderoErasquin, Fradelizi, Maurey

In the case of the standard Gaussian measure, the Conjecture 4.54 was verified:
Theorem 4.56 (B-theorem, Cordero-Erasquin, Fradelizi, Maurey [50]). Let $\gamma$ be the standard Gaussian measure and $K$ a symmetric convex set. Then $\gamma\left(e^{t} K\right)$ is log-concave in $t \in \mathbb{R}$.
Proof of theorem 4.56. It is enough to show

$$
\left.\frac{d^{2}}{d t^{2}} \log \gamma\left(e^{t} K\right)\right|_{t=0} \leq 0
$$

Let us introduce the auxiliary function

$$
F(s):=\sqrt{2 \pi}^{n} \gamma(s K)=\int_{s K} e^{-\frac{x^{2}}{2}} d x=\int_{K} s^{n} e^{-\frac{(s y)^{2}}{2}} d y, \quad s \geq 0 .
$$

We have

$$
F^{\prime}(s)=n s^{n-1} \int_{K} e^{-\frac{(s y)^{2}}{2}} d \gamma-s^{n+1} \int_{K} y^{2} e^{-\frac{(s y)^{2}}{2}} d \gamma
$$

and therefore

$$
F^{\prime}(1)=\sqrt{2 \pi}^{n}\left(n \gamma(K)-\int_{K} y^{2} d \gamma\right)
$$

Further differentiating yields

$$
\begin{aligned}
F^{\prime \prime}(1) & =\sqrt{2 \pi}^{n}\left(n(n-1) \gamma(K)-n \int_{K} y^{2} d \gamma+(n+1) \int_{K} y^{2} d \gamma+\int y^{4} d \gamma\right) \\
& =\sqrt{2 \pi}^{n} \gamma(K)\left(n^{2}-n-(2 n+1) f_{K} y^{2} d \gamma+f_{K} y^{4} d \gamma\right)
\end{aligned}
$$

We use this to calculate

$$
\begin{aligned}
\left.\frac{d^{2}}{d t^{2}} \log \gamma\left(e^{t} K\right)\right|_{t=0}=\left.\frac{d^{2}}{d t^{2}} F\left(e^{t}\right)\right|_{t=0} & =\left.\frac{d}{d t} \frac{e^{t} F^{\prime}\left(e^{t}\right)}{F\left(e^{t}\right)}\right|_{t=0} \\
& =\left.\frac{\left(e^{t} F^{\prime}\left(e^{t}\right)+e^{2 t} F^{\prime \prime}\left(e^{t}\right)\right) F\left(e^{t}\right)-e^{2 t} F^{\prime}\left(e^{t}\right)}{F\left(e^{t}\right)^{2}}\right|_{t=0} \\
& =\frac{F^{\prime}(1) F(1)+F^{\prime \prime}(1) F(1)-F^{\prime}(1)^{2}}{F(1)^{2}}
\end{aligned}
$$

Denoting $\mathbb{E} Y^{2}=f_{K} y^{2} d \gamma$ and $\mathbb{E} Y^{4}=f_{K} Y^{2} d \gamma$, we continue

$$
\begin{align*}
& =n-\mathbb{E} Y^{2}+n(n-1)-(2 n+1) \mathbb{E} Y^{2}+\mathbb{E} Y^{4}-\left(n-\mathbb{E} Y^{2}\right)^{2} \\
& =-2 \mathbb{E} Y^{2}+\mathbb{E}\left(n-Y^{2}\right)^{2}-\left(\mathbb{E}\left(n-Y^{2}\right)\right)^{2} \\
& =-2 \mathbb{E} Y^{2}+\mathbb{E}\left(Y^{2}\right)^{2}-\left(\mathbb{E} Y^{2}\right)^{2} \\
& =-2 \mathbb{E} Y^{2}+\operatorname{Var}\left(Y^{2}\right) \\
& =-2 f_{K} y^{2} d \gamma+f_{K} y^{4} d \gamma-\left(f_{K} y^{2} d \gamma\right)^{2} . \tag{12}
\end{align*}
$$

Recall that we proved for all symmetric convex $K$ and locally-Lipschitz and even function $f$

$$
\begin{equation*}
f_{K} f^{2} d \gamma-\left(f_{K} f d \gamma\right)^{2} \leq \frac{1}{2} f_{K}|\nabla f|^{2} d \gamma \tag{13}
\end{equation*}
$$

If we apply this with $f(y):=y^{2}$ and $\nabla f=2 y$, we get

$$
f_{K} y^{4} d \gamma-\left(f_{K} y^{2} d \gamma\right)^{2} \leq 2 f_{K} y^{2} d \gamma
$$

which together with eq. (12) implies that $\left.\frac{d^{2}}{d t^{2}} \log \gamma\left(e^{t} K\right)\right|_{t=0} \geq 0$ which yields the claim.
Remark 4.57. In general, Cafarelli's theorem is a universal tool in transporting Gaussian inequalities to measures $d \mu=e^{-v} d x$ with $\nabla^{2} v \geq I d$ (more generally $\nabla^{2} v \geq k \cdot I d$ ), including $\gamma_{\mid K}$.

There are several applications of the B-theorem:

- Klartag, Vershynin [110]: to small-ball estimates (which we will discuss later in subsection ??).
- Bobkov [23]: Let $T$ be any volume preserving linear transformation (i.e. $|\operatorname{det} T|=1$ ) and $K$ a symmetric convex body with $|K|=1$. Then $\max _{T} \int_{T K} d \gamma$ is attained for $T=I d$ if and only if

$$
\left.\gamma\right|_{K}=e^{-\frac{x^{2}}{2}} \frac{2}{\sqrt{2 \pi}^{n} \gamma(K)} \mathbb{1}_{K} d x
$$

is isotropic (recall that the measure is called isotropic when its barycenter is zero and its covariance matrix is identity).

### 4.14 Some more history on the B-conjecture

The B-conjecture is known in the following cases:

- Gaussian (see theorem 4.56 by Cordero-Erasquin, Fradelizi, Maurey [50]);
- $d \mu=e^{-\|x\|_{1}} c_{n} d x$ (Eskenazis, Nayar, Tkocz [60]) - this result directly relies on the Theorem 4.56;
- in dimension 2 (Livne Bar-On [132], Böröczky, Lutwak, Yang, Zhang [35] and Saraglou [163])
- Rotation-invariant measures (Cordero-Erasquin, Rotem [54]) - we shall discuss this result later in the semester;
- Unconditional case (Saraglou [164], Cordero-Erasquin, Fradelizi, Maurey [50]; see below section 4.15).

In fact, we state below a more general version of the B-conjecture, which is allegedly the "Original conjecture" made by Banazchyk. One may see (home work) that the affirmative answer to this implies Theorem 4.56.

Conjecture 4.58 (Banazchyk). Let $K$ be a symmetric convex body and $\gamma$ the standard Gaussian measure. Then for any $z \in \mathbb{R}^{n}$

$$
\frac{\gamma(z+t K)}{\gamma(t K)}
$$

is monotonic increasing for $t>0$.

### 4.15 B-conjecture in the unconditional case

Definition 4.59. A set $A \subset \mathbb{R}^{n}$ is called unconditional if for all $x=\left(x_{1}, \ldots, x_{n}\right) \in A$ the point $\left(\varepsilon_{1} x_{1}, \ldots, \varepsilon_{n} x_{n}\right) \in A$ for any choice of signs $\varepsilon_{i} \in\{-1,1\}$. In other words, $A$ is invariant under coordinate symmetries. A function is unconditional if

$$
f(x)=f\left(\left(\varepsilon_{1} x_{1}, \ldots, \varepsilon_{n} x_{n}\right)\right), \quad \forall x \in \mathbb{R}^{n}, \varepsilon_{i} \in\{-1,1\} .
$$



Theorem 4.60 (Multiplicative Prekopa on $\mathbb{R}_{+}^{n}$ ). Let $f, g, h: \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$, where

$$
\mathbb{R}_{+}^{n}:=\left\{x: x_{i} \geq 0 \forall i\right\}
$$

such that

$$
\sqrt{f(x) g(y)} \leq h\left(\sqrt{x_{1}, y_{1}}, \ldots, \sqrt{x_{n} y_{n}}\right)
$$

Then,

$$
\int_{\mathbb{R}_{+}^{n}} f \int_{\mathbb{R}_{+}^{n}} g \leq\left(\int_{\mathbb{R}_{+}^{n}} h\right)^{2} .
$$

Proof. The proof is done by a change of variables in Prekopa-Leindler inequality. Given $f$, consider $F: \mathbb{R}^{n} \rightarrow \mathbb{R}$,

$$
F\left(x_{1}, \ldots, x_{n}\right):=f\left(e^{x_{1}}, \ldots, e^{x_{n}}\right) e^{\sum_{i=1}^{n} x_{i}}
$$

and define $G$ and $H$ analogously. The assumption is then equivalent to

$$
H\left(\frac{x+y}{2}\right) \geq F(x)^{1 / 2} G(y)^{1 / 2}, \quad \forall x, y \in \mathbb{R}^{n}
$$

Furthermore, we have

$$
\int_{\mathbb{R}_{+}^{n}} f=\int_{\mathbb{R}^{n}} F
$$

and the analogous statement for $G$ and $F$. Now, Prekopa-Leindler inequality Theorem 3.20 implies the conclusion.

Corollary 4.61 (Cordero-Erasquin, Fradelizi, Maurey). Let $\mu$ be a log-concave unconditional measure and $K$ a convex unconditional set. Then, $\mu\left(e^{t} K\right)$ is log-concave in $t \in \mathbb{R}$.

Proof. For a set $A$ in $\mathbb{R}^{n}$ denote by $A^{+}=A \cap \mathbb{R}_{+}^{n}$. Consider $f=\mathbb{1}_{(a K)+} e^{-v}, g=\mathbb{1}_{(b K)^{+}} e^{-v}$ and $h=\mathbb{1}_{(\sqrt{a b} K)^{+}} e^{-v}$, where $d \mu=e^{-v} d x$. Checking the conditions of theorem 4.60 yields $\sqrt{\mu(a K) \mu(b K)} \leq \mu(\sqrt{a b} K)$. See home work for the details and the case $\lambda \neq 1 / 2$.

### 4.16 About Log-Brunn-Minkowski conjecture

Definition 4.62 (Log-addition). Let $K, L$ be symmetric convex sets. We define

$$
\frac{K+{ }_{0} L}{2}:=\bigcap_{u \in S^{n-1}}\left\{x \in \mathbb{R}^{n}:\langle x, u\rangle \leq \sqrt{h_{k}(x) h_{L}(x)}\right\}
$$

where $h_{K}$ and $h_{L}$ are the support functions of $K$ and $L$, respectively.


Note that we can write the usual Minkowski sum as

$$
\frac{K+L}{2}:=\bigcap_{u \in S^{n-1}}\left\{x \in \mathbb{R}^{n}:\langle x, u\rangle \leq \frac{h_{k}(x)+h_{L}(x)}{2}\right\} .
$$

Therefore, by AM-GM, we have

$$
\begin{equation*}
\frac{K+{ }_{0} L}{2} \subset \frac{K+L}{2} . \tag{14}
\end{equation*}
$$

Therefore, the following conjecture is a strengthening of the Brunn-Minkowski inequality in the case of symmetric convex bodies:

Conjecture 4.63 (Böröczky, Lutwak, Young, Zhang 2013 [35]). Let $K, L$ be symmetric and convex sets. Then

$$
\left|\frac{K+{ }_{0} L}{2}\right| \geq \sqrt{|K||L|} .
$$

It turns out, the validity of this inequality for Lebesgue measure is equivalent to the validity of the same statement for any given even log-concave measure:

Theorem 4.64 (Saraglou [163]). Equation (14) is equivalent to the following: For all logconcave even measures $\mu$ on $\mathbb{R}^{n}$ and all symmetric convex sets $K, L$, it holds

$$
\begin{equation*}
\mu\left(\frac{K+{ }_{0} L}{2}\right) \geq \sqrt{\mu(K) \mu(L)} \tag{15}
\end{equation*}
$$

Proof. For " $\Longrightarrow$ ", use Prekopa-Leindler inequality (homework). For " $\Longleftarrow "$ : any logconcave measure behaves like the Lebesgue measure near the origin.

We conclude that the Log-Brunn-Minkowski conjecture 4.63 implies the B-conjecture conjecture 4.54. Indeed, for any log-concave even measure eq. (15) holds. We set $K \leftarrow a K$ and $L \leftarrow b K$ and have $\frac{a K+{ }_{0} b K}{2} \geq \sqrt{a b} K$ Therefore, the validity of Conjecture 4.63 implies

$$
\mu(\sqrt{a b} K) \geq \sqrt{\mu(a K) \mu(b K)}
$$

In $\mathbb{R}^{2}$, Böröczky, Lutwak, Young, Zhang 2013 [35] showed that Log-Brunn-Minkowski is true, and therefore the B-conjecture conjecture 4.54 is known to be true on the plane.

Let us get back to the Blaschke-Santaló inequality (see theorems 4.1 and 4.29).

### 4.17 Reverse Log-Sobolev inequality

Recall the notation

$$
\operatorname{Ent}(f)=\int_{\mathbb{R}^{n}} f \log f d x
$$

In this subsection we shall discuss another nice corollary of the Blaschke-Santaló inequality:

Theorem 4.65 (Artstein-Avidan, Klartag, Schutt, Werner [6]). Let $v: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a convex even function with $\int e^{-v} d x=1$ (i.e. $d \mu:=e^{-v} d x$ is a probability measure). Then

$$
\begin{equation*}
-\int v d \mu=\operatorname{Ent}\left(e^{-v}\right) \geq \int \log \sqrt{\operatorname{det} \nabla^{2} v} d \mu-n \log \sqrt{2 \pi e} \tag{16}
\end{equation*}
$$

Remark 4.66. Equality holds if and only if $v(x)=\frac{x^{2}}{2}+n \log \sqrt{2 \pi}$. Equation (16) is also equivalent to

$$
\operatorname{Ent}\left(e^{-v}\right)-\operatorname{Ent}\left(\frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}}\right) \geq \int \log \sqrt{\operatorname{det} \nabla^{2} v} d \mu
$$

Indeed, we have

$$
\int \frac{x^{2}}{2}+n \log \sqrt{2 \pi} d \gamma=\frac{n}{2}+n \log \sqrt{2 \pi}
$$

where we used $\operatorname{det}(I d)=1$ and

$$
\int x^{2} d \gamma=\sum_{i=1}^{n} \int_{\mathbb{R}^{n}} x_{i}^{2} d \gamma=\sum_{i=1}^{n} \int_{-\infty}^{\infty} t^{2} d \gamma(t)=n
$$

Remark 4.67. Recall the generalized Log-Sobolev inequality (cf. theorem 3.45):

$$
\operatorname{Ent}\left(e^{-v}\right) \leq \int G^{*}(\nabla v) d \mu-n
$$

whenever $\int e^{-v}=\int e^{-G}=1$. In the Gaussian case,

$$
G^{*}=\frac{x^{2}}{2}-n \log \sqrt{2 \pi}
$$

we get

$$
\operatorname{Ent}\left(e^{-v}\right) \leq \int \frac{|\nabla v|^{2}}{2} d \mu-n \log (\sqrt{2 \pi} e)=\frac{1}{2} \int \Delta\left(v-\frac{x^{2}}{2}\right)-n \log (\sqrt{2 \pi} e)
$$

For the last inequality, recall $L u=\Delta u-\langle\nabla v, \nabla u\rangle$. Then for all functions $u$, we have $\int L u d \mu=0$. In particular, $\int L v d \mu=0$, so

$$
\int \Delta v d \mu=\int|\nabla v|^{2} d \mu
$$

We have

$$
\frac{1}{2} \int \log \operatorname{det}\left(\nabla^{2} v\right) d \mu \leq \operatorname{Ent}\left(e^{-v}\right)+n \log \sqrt{2 \pi e} \leq \frac{1}{2} \int \Delta\left(v-\frac{x^{2}}{2}\right) d \mu
$$

If $d \mu=d \gamma$, the inequalities become $0 \leq 0 \leq 0$. Note that both inequalities are sharp in the Gaussian case. In total, it is all about measuring the distance to the Gaussian. The second inequality can be made even stronger (homework):

$$
\operatorname{Ent}\left(e^{-v}\right) \leq \frac{n}{2} \log \frac{\int \Delta v d \mu}{n}-n \log \sqrt{2 \pi e}
$$

Proof of theorem 4.65. Without loss of generality, let $v \in C^{2}\left(\mathbb{R}^{n}\right), v \neq \infty$ and $V$ strictly convex. By Blaschke-Santaló, since $\int e^{-v}=1$, we have

$$
\int e^{v^{*}}=\int e^{-v} \int e^{v^{*}} \leq(2 \pi)^{n}
$$

A change of variables with $x=\nabla v(y)$ yields

$$
\int e^{-v^{*}(\nabla v(y))} \operatorname{det} \nabla^{2} v(y) d y \leq(2 \pi)^{n}
$$

Since $v^{*}(\nabla v)=\langle\nabla v, y\rangle-v(y)$, we get

$$
\int e^{-\langle\nabla v, y\rangle+v(y)} \operatorname{det} \nabla^{2} v(y) d y \leq(2 \pi)^{n}
$$

Recall Jensen's inequality (Theorem 2.3), which states that for any convex function $F$ and probability measure $\mu$, we have

$$
F\left(\int f d \mu\right) \leq \int F(f) d \mu
$$

We take the convex $F=e^{t}$, and further write

$$
\begin{aligned}
(2 \pi)^{n} \geq & \int e^{-\langle\nabla v, y\rangle+v(y)} \operatorname{det} \nabla^{2} v(y) d y=\int e^{-\langle\nabla v, y\rangle+2 v(y)} \operatorname{det} \nabla^{2} v(y) d \mu \\
& \geq \exp \left(\int-\langle\nabla v, y\rangle+2 v(y)+\log \operatorname{det} \nabla^{2} v(y) d \mu\right)
\end{aligned}
$$

This implies

$$
n \log (2 \pi) \geq \int-\langle\nabla v, y\rangle+2 v(y)+\log \operatorname{det} \nabla^{2} v(y) d \mu
$$

We claim that $\int-\langle\nabla v, y\rangle d \mu=n$. Indeed, $\int L u d \mu=0$ for all $u$. Taking $u:=\frac{x^{2}}{2}$ we calculate

$$
L \frac{x^{2}}{2}=\Delta \frac{x^{2}}{2}-\left\langle\nabla \frac{x^{2}}{2}, \nabla v\right\rangle=n-\langle x, \nabla v\rangle .
$$

With the claim, we get

$$
n \log (2 \pi e) \geq \int 2 v+\log \operatorname{det} \nabla^{2} v d \mu
$$

which finishes the proof.
Remark 4.68 (On the Reverse Log-Sobolev Inequality). In the proof of Theorem 4.65 we assumed that $v \in C^{2}\left(\mathbb{R}^{n}\right)$. However the inequality holds for a more general class of $v$ by approximation and a direct proof was outlined in a paper of Caglar, Fradelizi, Gozlan, Lehec, Schutt, Werner [43], leading to the equality case characterization.

Remark 4.69 (Entropy Power Inequality). Let $X$ be a random vector whose density with respect to lebesgue measure is $f$. Define the entropy function $h: \mathbb{R}^{n} \rightarrow \mathbb{R}$ as

$$
h(X):=-\int f \log (f) d x=-\operatorname{Ent}(f)
$$

Let $X, Y$ be random vectors and let $X^{\prime}, Y^{\prime}$ be independent Gaussian random vectors that satisfy $h(X)=h\left(X^{\prime}\right), h(Y)=h\left(Y^{\prime}\right)$. Then the entropy function satisfies

$$
h(X+Y) \geq h\left(X^{\prime}+Y^{\prime}\right)
$$

Since the equality cases of the Log-Sobolev inequality and Reverse Log-Sobolev inequality are achieved by gaussians, one can ask if the Log-Sobolev inequality combined with the Reverse Log-Sobolev are sufficient to imply the Entropy Power Inequality. If so, it would give an interesting connection to information theory.
Remark 4.70. Suppose $F, G$ are 2-homogeneous convex functions on $\mathbb{R}^{n}$ (i.e. $F(t x)=$ $t^{2} F(x)$ ). An ODE exercise (home work) tells you that

$$
F(t x)=t^{2} F(x) \Longleftrightarrow 2 F(x)=\langle\nabla F(x), x\rangle
$$

From Proposition 7.6 we know that the Legendre transform satisfies $F(x)+F^{*}(\nabla F)=$ $\langle\nabla F, x\rangle$. Therefore by the previous equivalence we get

$$
F^{*}(\nabla F)=F
$$

This leads to the following claim (see home work):

$$
\int e^{-\frac{F+G}{2}} \operatorname{det}\left(\frac{\nabla^{2}(F)+\nabla^{2}(G)}{2}\right) \geq \sqrt{\int e^{-F} \operatorname{det}\left(\nabla^{2} F\right) \int e^{-G} \operatorname{det}\left(\nabla^{2} G\right)}
$$

This inequality follows from applying the Prekopa Leindler inequality and verifying the Prekopa Leindler condition, which involves the same trick we used to prove that $\int e^{-F^{*}}$ is log-concave.

### 4.18 Fathi's symmetrized Transport-Entropy Inequality

Let $\mu, \nu$ be probability measures so that $\mu$ is absolutely continuous with respect to $\nu$, and write $d \mu=f d \nu$ where $f$ is the density of $\mu$ with respect to $\nu$. We write

$$
\operatorname{Ent}_{d \nu}(\mu)=\int f \log f d \nu
$$

Remark 4.71. Take $d \mu=e^{-v} d \gamma=e^{-v-x^{2} / 2-n \log (\sqrt{2 \pi})} d x$. Suppose $\int d \mu=1$. Then we have

$$
\begin{array}{r}
\operatorname{Ent}_{\gamma}(\mu)=\int-v e^{-v} d \gamma \\
\operatorname{Ent}_{d x}(\mu)=\int-\left(v+x^{2} / 2+n \log (\sqrt{2 \pi})\right) e^{-v} d \gamma \\
=\operatorname{Ent}_{\gamma}(\mu)-\frac{1}{2} \int \frac{x^{2}}{2} d \mu-n \log (\sqrt{2 \pi})
\end{array}
$$

Remark 4.72. For any measure $\mu$ we have $\operatorname{Ent}_{\mu}(\mu)=\int 1 \log (1) d \mu=0$. In this way one can think of $\operatorname{Ent}_{\nu}(\mu)$ as measuring the distance between $\nu$ and $\mu$ (known as the entropy distance).

Definition 4.73. Suppose $\nu$ and $\mu$ are absolutely continuous Borel finite measures. Recall that $T_{*} \mu=\nu(T$ transports $\mu$ into $\nu)$ if

$$
\forall \operatorname{Lip} \varphi, \int \varphi(x) d \mu(x)=\int \varphi(T x) d \nu(x)
$$

We define the Wasserstein 2-distance between $\nu$ and $\mu$ as

$$
W_{2}(\mu, \nu)=\inf _{T: T_{*} \mu=\nu} \sqrt{\int|x-T x|^{2} d \mu}
$$

By Brenier's theorem 4.44, if $\mu$ and $\nu$ are Borel finite absolutely continuous measures, the infimum is attained when $T=\nabla F$ for some convex function $F$.

Remark 4.74. In general one can define a "coupling" between measures $\mu$ and $\nu$, both on $\mathbb{R}^{n}$, to be a measure $\pi$ on $\mathbb{R}^{n} \times \mathbb{R}^{n}$ such that the marginals of $\pi$ are $\mu$ and $\nu$ :

$$
d \pi(x, y)=F(x, y) d x d y
$$

where $d \mu=f d x, d \nu=g d y$ and $\int F(x, y) d x=g(y), \int F(x, y) d y=f(x)$.
In particular one can formulate the definition of Wasserstein distance as

$$
W_{2}(\mu, \nu)=\inf _{\pi-\text { coupling of } \mu, \nu} \sqrt{\int_{\mathbb{R}^{2 n}}|x-y|^{2} d \pi(x, y)} .
$$

When $\mu$ and $\nu$ are absolutely continuous then $W_{2}$ is attained on the coupling "supported" on the surface $\{T x=y\}$, with $T$ being the Brenier map.


Finally, we formulate the main result of this subsection, which we will derive as a corollary of the Blaschke-Santaló inequality.

Theorem 4.75 (Fathi [63]). Consider a pair of even probability measures $\mu, \nu$. Then

$$
W_{2}(\mu, \nu)^{2} \leq 2 \operatorname{Ent}_{\gamma}(\mu)+2 \operatorname{Ent}_{\gamma}(\nu)
$$

Remark 4.76. In fact, instead of assuming that both measures are even, it suffices to assume that one of the measures is mean zero.

Fathi's Theorem yields as a Corollary the following important fact (which was historically proven much earlier and via different means):

Corollary 4.77 (Talagrand [166]).

$$
W_{2}(\mu, \gamma)^{2} \leq 2 \operatorname{Ent}_{\gamma}(\mu)
$$

Remark 4.78. Since $W_{2}$ is a metric, we have by the triangle inequality that the result of Talagrand implies the result of Fathi with an additional multiplicative factor of 2. This improved factor will play crucial role for applications to the concentration of measure, for instance.

As an immediate application of the Talagrand inequality and the Log-Sobolev inequality, we get:
Corollary 4.79 (Talgrand + Gaussian Log-Sobolev). For $d \mu=e^{-\left(v+x^{2} / 2+n \log (\sqrt{2 \pi})\right)} d x=$ $e^{-v} d \gamma$ one has

$$
W_{2}(\mu, \gamma)^{2} \leq \int|\nabla v-x|^{2}
$$

In order to prove Fathi's result, we shall need a few Lemmas. First, we point out
Lemma 4.80. Fathi's theorem is equivalent to the fact that

$$
\int_{\pi-c o u p l i n g ~ o f ~}^{\mu, \nu}, ~ \int-\langle x, y\rangle d \pi(x, y) \leq \operatorname{Ent}_{d x}(\mu)+\operatorname{Ent}_{d x}(\nu)+n \log (2 \pi) .
$$

Proof. This follows right away from the definition of Wasserstein distance and Remark 4.71.

Without loss of generality we will assume that $\nu, \mu$ are Borel finite absolutely continuous measures with bounded densities. We outline the following variational formula for the entropy (see also e.g. Artstein-Avidan, Giannopolous, Milman [4]):

## Lemma 4.81.

$$
\operatorname{Ent}_{d x}(\mu)=\sup _{f}\left(\int f d \mu-\log \int e^{f} d x\right)
$$

Proof. Write $d \mu(x)=\int g(x) d x$ with $|g| \leq C$ (follows from $g$ being a bounded density). Next consider the class of functions $\mathcal{F}$ for which $e^{f} \leq C$ for all $f \in \mathcal{F}$. Then

$$
\sup _{f \in \mathcal{F}}\left(\int f d \mu-\log \int e^{f} d x\right) .
$$

is bounded from above and so by compactness the supremum is obtained by some $f \in \mathcal{F}$. Suppose now $f_{o}$ is a maximizer. Then $f_{o}$ is a local maximizer in the following sense: Define $f_{\varepsilon}=f_{0}+\varepsilon \varphi$ and define

$$
F(\varepsilon)=\int\left(f_{o}+\varepsilon \varphi\right) g d x-\log \int e^{f_{o}+\varepsilon \varphi} d x
$$

Then $F$ satisfies

$$
F^{\prime}(0)=0, \quad F^{\prime \prime}(0) \leq 0
$$

We now determine $f_{0}$. By differentiating with respect to $\varepsilon$ we get

$$
F^{\prime}(\varepsilon)=\int \varphi g d x-\frac{\int e^{f_{o}} \varphi e^{\varepsilon \varphi}}{\int g e^{\varepsilon \varphi}}
$$

Setting $\varepsilon=0$ we get

$$
F^{\prime}(0)=\int \varphi g d x-\int \varphi e^{f_{o}} \text { for reasonable } \varphi
$$

Now take $\varphi_{z}$ to be a sufficiently nice function who support is essentially the point $z$. By the definition of $F^{\prime}(0)$ we conclude that a local maximizer must satisfy $g=e^{f_{o}}$ at $z$. By varying $z$ over $\mathbb{R}^{n}$ we conclude that $g=e^{f_{o}}$ almost everwhere. Thus $f_{o}=\log g$ almost everywhere. This means $f_{0}=\log g$ is a local maximizer. To show that it's a global maximizer one needs to argue that $F^{\prime \prime}(0) \leq 0$. But by the choice of $f_{o}$ we can show that

$$
F^{\prime \prime}(0)=-\int g \varphi^{2}+\left(\int g \varphi\right)^{2}=-\operatorname{Var}_{\mu}(\phi) \leq 0
$$

This solves the case where $f$ is bounded. Observe now that without loss of generality we can assume that $f$ is bounded since we can always approximate $f$ by a bounded function and then appeal to the bounded case. The result follows.

## Lemma 4.82.

$$
\int_{\pi} \int-\langle x, y\rangle d \pi(x, y)=\sup _{f, g: f(x)+g(y) \leq-\langle x, y\rangle}\left(\int f d \mu+\int g d \nu\right) .
$$

Proof. This is an application of the Kantorovich duality theorem. Suppose $\mu, \nu$ are absolutely continuous finite. Then from the definition of a coupling one can write

$$
\int f(x) d \mu(x)+\int g(y) d \nu(y)=\iint(f(x)+g(y)) d \pi(x, y)
$$

This immediately gives $L H S \leq R H S$ of the desired inequality since the supremum is over all $f, g$ satisfying $f(x)+g(y) \leq-\langle x, y\rangle$. To get $L H S \geq R H S$ we first define $\varphi=-f, \psi=-g$. Then the RHS becomes

$$
-\inf _{\varphi(x)+\psi(y) \geq\langle x, y\rangle} \iint \varphi(x)+\psi(y) d \pi(x, y) .
$$

By the definition of the legendre transform this is at least

$$
-\inf _{\varphi} \iint \varphi(x)+\varphi^{*}(y) d \pi(x, y) .
$$

Taking the difference of the above and the LHS we see

$$
R H S-L H S \geq \inf _{\varphi} \inf _{\pi} \iint-\varphi(x)-\varphi^{*}(y)+\langle x, y\rangle d \pi(x, y)
$$

It remains to recall by Brenier's theorem that there exists a function $\phi$ for which $y=$ $\nabla \phi(x)$ on the entire support of $T$. Using the identity $\varphi(x)+\varphi^{*}(\nabla \varphi)=\langle x, \nabla \varphi\rangle$, we conclude that the above is non-negative, and the lemma follows.

Proof of Theorem 4.75. Recall that the functional version of Blaschke-Santalo says that if $\varphi$ is an even function then

$$
\int e^{-\varphi} \int e^{-\varphi^{*}} \leq(2 \pi)^{n}
$$

and equivalently, for all non-negative even functions $f, g$ satisfying $f(x)+g(y) \geq\langle x, y\rangle$ for all $x, y \in \mathbb{R}^{n}$ one has

$$
\int e^{-f} \int e^{-g} \leq(2 \pi)^{n}
$$

To prove Theorem 4.75 we will use Lemma 4.80, Lemma 4.81, Lemma 4.82 and the equivalent version of Blaschke-Santalo Inequality. To that end recall that, by Lemma 4.80, Theorem 4.75 is equivalent to

$$
\int_{\pi \text {-coupling of } \mu, \nu} \int-\langle x, y\rangle d \pi(x, y) \leq \operatorname{Ent}_{d x}(\mu)+\operatorname{Ent}_{d x}(\nu)+n \log (2 \pi)
$$

To prove this inequality we apply Lemma 4.81 to the left hand side to get

$$
\begin{array}{r}
\operatorname{Ent}_{d x}(\mu)+\operatorname{Ent}_{d x}(\nu)+n \log (2 \pi) \\
=\sup _{f, g}\left(\int f d \mu-\log \int e^{f} d x+\int g d \nu-\log \int e^{g} d x+n \log 2 \pi\right) . \\
\geq \sup _{f, g: f(x)+g(y) \leq-\langle x, y\rangle \forall x, y}\left(\int f d \mu-\log \int e^{f} d x+\int g d \nu-\log \int e^{g} d x+n \log 2 \pi\right) .
\end{array}
$$

Then we apply Blaschke-Santalo inequality to get

$$
\begin{aligned}
\sup _{f, g: f(x)+g(y) \leq-\langle x, y\rangle \forall x, y}\left(\int f d \mu-\log \right. & \left.\int e^{f} d x+\int g d \nu-\log \int e^{g} d x+n \log 2 \pi\right) \\
& \geq \sup _{f, g: f(x)+g(y) \leq-\langle x, y\rangle \forall x, y}\left(\int f d \mu+\int g d \nu\right) .
\end{aligned}
$$

Thus

$$
\sup _{f, g: f(x)+g(y) \leq-\langle x, y\rangle \forall x, y}\left(\int f d \mu+\int g d \nu\right) \leq \operatorname{Ent}_{d x}(\mu)+\operatorname{Ent}_{d x}(\nu)+n \log (2 \pi)
$$

Finally by Lemma 4.82 this is equivalent to

$$
\int_{\pi \text {-coupling of } \mu, \nu} \int-\langle x, y\rangle d \pi(x, y) \leq \operatorname{Ent}_{d x}(\mu)+\operatorname{Ent}_{d x}(\nu)+n \log (2 \pi)
$$

Remark 4.83. We note that Fathi's inequality not only follows from, but is equivalent to the Blaschke Santalo Inequality (Home work)

### 4.19 Home work

Question 4.84 (1 point). Let $P$ be a polytope given by

$$
P=\left\{x \in \mathbb{R}^{n}:\left\langle x, u_{i}\right\rangle \leq a_{i}, \forall i=1, \ldots, N\right\}
$$

for some unit vectors $u_{1}, \ldots, u_{N}$ and positive numbers $a_{1}, \ldots, a_{N}$, and suppose that $P$ is bounded. Show that

$$
P^{o}=\operatorname{con} v\left\{\frac{u_{1}}{a_{1}}, \ldots, \frac{u_{N}}{a_{N}}\right\} .
$$

Conclude that $\left(B_{1}^{n}\right)^{o}=B_{\infty}^{n}$.
Question 4.85 (1 point). In this question, $K$ and $L$ stand for convex bodies in $\mathbb{R}^{n}$ with non-empty interior, containing the origin.
a) Prove that $K^{o o}=K$.
b) Prove that for a linear operator $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ with $\operatorname{det} T \neq 0$,

$$
\left(T^{t} K\right)^{o}=T^{-1} K^{o} .
$$

Conclude that a polar of an ellipsoid is an ellipsoid.
c) Prove that

$$
\left(B_{p}^{n}\right)^{o}=B_{q}^{n},
$$

where $\frac{1}{p}+\frac{1}{q}=1$, for all $p, q>1$.
d) Prove that

$$
(K \cap L)^{o}=\operatorname{conv}\left(K^{o} \cup L^{o}\right) .
$$

e) Prove that for every subspace $H$ of $\mathbb{R}^{n}$

$$
(K \mid H)^{o} \cap H=K^{o} \cap H
$$

f) Prove that if $K \subset L$, one has $L^{o} \subset K^{o}$.
g) Prove that if $K$ is symmetric then $K^{o}$ is symmetric.
h) Show that for any (possibly non-convex) set $A$, we have $A^{o}=(\operatorname{conv}(A))^{o}$. Conclude that the polar is always a convex set.

Question 4.86 (1 point). Let $K$ be a symmetric convex body. Show that if $K=K^{o}$ then $K=B_{2}^{n}$.

Question 4.87 (1 point). Show that for any symmetric convex body $K$, we have $h_{K}(\theta) \rho_{K^{o}}(\theta)=$ 1 for all $\theta \in \mathbb{R}^{n}$.

Question 4.88 (3 points). Verify Mahler's conjecture in $\mathbb{R}^{2}$ for symmetric polygons: show that for any symmetric polygon $P$ in $\mathbb{R}^{2}$,

$$
|P| \cdot\left|P^{o}\right| \geq 8
$$

Question 4.89 (1 point). Given a Borel measurable set $A$ in $\mathbb{R}^{n}$, a function $\alpha: A \rightarrow \mathbb{R}$ and $a$ vector $v \in \mathbb{R}^{n} \backslash 0$, consider the shadow system

$$
K_{t}=\operatorname{conv}\{x+\alpha(x) v: x \in A\},
$$

and define the convex body

$$
\tilde{K}=\operatorname{conv}\left\{x+t \alpha(x) e_{n+1}\right\} \subset \mathbb{R}^{n+1}
$$

Show that for $u \in e_{n+1}^{\perp}$,

$$
h_{K_{t}}(u)=h_{\tilde{K}}\left(u+t\langle u, v\rangle e_{n+1}\right) .
$$

Question 4.90 (2 points). Prove the Blaschke-Santalo inequality using shadow systems. Hint 1. Express $\left|K_{t}^{o}\right|$ combining the formulas from Questions 2.26, 4.87 and 4.89.
Hint 2. pass the integration on $\mathbb{S}^{n-1}$ to the integration on $B_{2}^{n-1}=\left\{x \in \mathbb{R}^{n}:\langle x, v\rangle=0\right\}$ with the map $x=\theta-\langle\theta, v\rangle v$.
Hint 3: now extend the integration to $\mathbb{R}^{n-1}$.
Hint 4. Conclude that $\left|K_{t}^{o}\right|$ is -1 -concave in $t$ for any shadow system, using Question 3.103.
Hint 5. Notice that Steiner symmetrization can be realized as a shadow system, and, using the fact that $\left|K^{o}\right|=\left|\bar{K}^{o}\right|$ for any reflection $\bar{K}$ of $K$, and the -1 - concavity of $\left|K_{t}^{o}\right|$ along any shadow system, conclude that Steiner symmetrization increases $\left|K^{\circ}\right|$. Conclude the BlaschkeSantalo inequality.
(this proof was discovered by Campi and Gronchi).

Question 4.91 (1 point). a) For any $\varphi: \mathbb{R} \rightarrow \bar{R}$ one has $\varphi^{*}$ is a convex function.
b) If $\varphi$ is convex then $\varphi^{* *}=\varphi$.
c) If $f \geq g$ then $f^{*} \leq g^{*}$.
d) Find $\left|x_{1}\right|^{*}$.
e) Find $\left(\frac{\|x\|_{p}^{q}}{q}\right)^{*}$.
f) For a convex body $K$, one has $\left(-\log 1_{K}\right)^{*}=h_{K}$.
g) For an $a \in \mathbb{R}$, find $(a \varphi)^{*}$ in terms of $\varphi^{*}$.
h) Letting $\varphi_{a}(x)=\varphi($ ax $)$ for some $a \in \mathbb{R}$, find $\varphi_{a}^{*}$.
i) Show that $(\varphi+a)^{*}=\varphi^{*}-a$, for any $a \in \mathbb{R}$.
j) Show that

$$
\left(f^{*}+g^{*}\right)^{*}(z)=\inf _{x, y \in \mathbb{R}^{n}: x+y=z}(f(x)+g(y)) .
$$

k) Fix $\alpha>1$. Show that is $f$ is $\alpha$-homogeneous (i.e. $f(t x)=t^{\alpha} f(x)$ for all $t \in \mathbb{R}$ ) then $f^{*}(\nabla f)=(\alpha-1) f$.
Hint: use one of the properties we proved in class, combined with the fact that for an $\alpha$-homogeneous function one has $\langle\nabla f, x\rangle=\alpha f$ (verify this).

Question 4.92 (1 point). Find an alternative short proof of the functional Blaschke-Santalo inequality for unconditional functions by passing the integration from $\mathbb{R}^{n}$ to the set

$$
\left\{x \in \mathbb{R}^{n}: \forall i=1, \ldots, n, x_{i} \geq 0\right\}
$$

and making a change of variables in the Prekopa-Leindler inequality given by $\left(x_{1}, \ldots, x_{n}\right)=$ $\left(e^{t_{1}}, \ldots, e^{t_{n}}\right)$. (see also a similar Question 3.104).

Question 4.93 (1 point). Show that the Santaló point of a convex body exists and is unique.
Question 4.94 (4 points). Find a statement and a proof for the Blaschke-Santalo inequality and the functional Blaschke-Santalo inequality for non-symmetric convex sets and non-even functions (as per our discussion in class).

Question 4.95 (3 points). a) Note that the Blaschke-Santalo inequality on the plane is equivalent to showing that for any even periodic function $h \in C^{2}([-\pi, \pi])$, such that $h \geq 0$ and $h+\ddot{h} \geq 0$,

$$
F(h)=\int_{-\pi}^{\pi} h^{-2} d t \cdot \int_{-\pi}^{\pi} h^{2}-\dot{h}^{2} d t \leq 4 \pi^{2}
$$

(or equivalently, one may drop the even assumption and restrict to $[0, \pi]$ ).
Hint: use Questions 4.87 and 2.26 to conclude that

$$
\left|K^{o}\right|=\frac{1}{2} \int_{-\pi}^{\pi} h^{-2} d t
$$

Also use Question 3.117.
b) Observe that the equality is attained when $h$ is the support function of an ellipse.
c) Find some way to show that this inequality is true.

Option 1: maybe use basic Harmonic Analysis (I don't know if it is possible and would love to see it if it works)?

Option 2: maybe use variational approach? That is, suppose that a given function $h$ maximizes the functional $F(h)$, argue* that it suffices to assume that $h \in C^{1}([-\pi, \pi])$ and $h>$ 0 and $h+\dot{h}>0$, then argue that for any $\epsilon>0$ and any even smooth $\psi>0, \frac{d}{d \epsilon} F(h+\epsilon \psi)=0$, and conclude some ODE that $h$ must satisfy (in view of the arbitrarity of $\psi$ ). Then conclude that the support function of an ellipsoid is the only type of function that satisfies this ODE.

* This "argue" may not be easy and you are encouraged to pursue other steps in this hint even if this step is not clear at first.

Option 3: try whatever you like! :)
Question 4.96 (1 point). Let $K$ be a smooth convex body with II $>0$. For $x \in \partial K$ let $x^{*} \in \partial K^{*}$ be given by $x^{*}=\nabla\|x\|_{K}$. Show that the Gauss curvature at $x$ of $\partial K$ is inverse to the Gauss curvature at $x^{*}$ of $K^{o}$.
Hint: use the properties of Legendre transform of $h_{K}(x)$.
Question 4.97 (5 points). a) Find an example of a non-symmetric convex body for which the Santaló point and the center of mass do not coincide.
b) How far could they be?
c) For a convex body $K$ in $\mathbb{R}^{n}$, let $d(K)$ be the distance between the center of mass and the Santaló point. How large could $\frac{d(K)}{\operatorname{diam(K)}}$ be?

Question 4.98 (1 point). Let $H$ be a Hanner polytope (as defined inductively in class). Show that indeed

$$
|H|\left|H^{o}\right|=\frac{4^{n}}{n!}
$$

Question 4.99 (2 points, Saint-Raimond's theorem via Meyer's proof). Prove the (symmetric) Mahler conjecture in the case when the body $K$ is unconditional (that is, it is symmetric with respect to every coordinate hyperplane).
Hint 1: Note that the result is true in dimension 1 and proceed by induction.
Hint 2: Consider $K^{+}=\left\{x \in K: x_{i} \geq 0 \forall i=1, \ldots, n\right\}$. Given a point $x \in K^{+}$consider $n$ cones

$$
K_{i}=\operatorname{conv}\left\{x, K^{+} \cap e_{i}^{\perp}\right\} .
$$

Note that

$$
|K| \geq 2^{n} \sum_{i=1}^{n}\left|K_{i}\right|
$$

recall Question 1.20 and write the above out to deduce that the vector with coordinates (..., $\frac{2\left|K \cap e_{i}^{+}\right|}{n|K|}, \ldots$ ) belongs to $K^{o}$ (use the unconditionality in the process).

Hint 3: Do the same argument for $K^{o}$, and then use properties of polarity along with the
fact that $K \cap e_{i}^{\perp}=K \mid e_{i}^{\perp}$ (which is another place where the fact that $K$ is unconditional is used!!!), to conclude that

$$
|K|\left|K^{o}\right| \geq \frac{4}{n^{2}} \sum_{i=1}^{n}\left|K \cap e_{i}^{\perp}\right| \cdot\left|\left(K \cap e_{i}^{\perp}\right)^{o}\right|
$$

and use induction.
Question 4.100 (5 points). Iryeh and Shibata's proof of Mahler's conjecture in $\mathbb{R}^{3}$ followed the same idea as in Question 4.99, and hinged on the fact that it is possible to bring a symmetric convex body in $\mathbb{R}^{3}$ into a position where it is possible to split it into 8 parts with coordinate hyperplanes so that each part has the same volume, and each of the three coordinate hyperplane sections of $K$ is split into four equal parts, and also each projection of $K$ onto coordinate hyperplane coincides with a section.
a) verify that this fact ensures the validity of Mahler conjecture (in the same way as above);
b) prove this challenging fact.

Question 4.101 (3 points). Verify the non-symmetric Mahler conjecture in dimension 2.
Question 4.102 (3 points). Using the ideas from Question 4.99, prove the result of Barthe, Fradelizi: if a convex body $K$ in $\mathbb{R}^{n}$ has all the symmetries of the regular simplex then it verifies the non-symmetric Mahler conjecture, that is, $|K|\left|K^{o}\right| \geq\left|S_{n}\right|^{2}$ where $S_{n}$ is the selfdual regular simplex.

Question 4.103 (10 points). Is it possible to use the ideas from Question 4.102 to prove the non-symmetric Mahler conjecture in $\mathbb{R}^{3}$, that is, to show that for any convex body $K$ in $\mathbb{R}^{3}$ one has $|K|\left|K^{o}\right| \geq\left|S_{3}\right|^{2}$ where $S_{3}$ is the self-dual regular simplex? Maybe one could prove the appropriate non-symmetric version of the fact proved by Iryeh and Shibata about bringing $K$ into a certain position?

Question 4.104 (2 points). Prove the following result of Fradelizi and Meyer: Mahler's conjecture is equivalent to the following functional version. For any convex function $\varphi$ on $\mathbb{R}^{n}$ one has

$$
\int e^{-\varphi} \cdot \int e^{-\varphi^{*}} \geq 4^{n}
$$

Question 4.105 (2 points). Prove the following result of Fradelizi and Meyer which extends the functional Blaschke-Santalo: let $\rho:[0, \infty) \rightarrow[0, \infty)$ be a measurable function and suppose $f$ and $g$ are even log-concave functions such that $f(x) g(y) \leq \rho^{2}(\langle x, y\rangle)$ whenever $\langle x, y\rangle \geq 0$. Then

$$
\int f \cdot \int g \leq\left(\int \rho\left(|x|^{2}\right)\right)^{2}
$$

Question 4.106 (5 points). We saw in class that the $p$-Beckner inequality on the circle for periodic functions

$$
\frac{1}{2 \pi} \int_{-\pi}^{\pi} f^{2}-\left(\frac{1}{2 \pi} \int_{-\pi}^{\pi} f^{p}\right)^{\frac{2}{p}} \leq(2-p) \frac{1}{2 \pi} \int_{-\pi}^{\pi} \dot{f}^{2}
$$

holds not only for $p \in[1,2)$ but also for $p=-2$. By any chance, is it possible to argue that there is a range of negative $p$ for which this holds (rather than just one value $p=-2$ )? Maybe argue similarly to Question 3.127?

Question 4.107 (2 points). Show that Talagrand's transport-entropy inequality implies the Gaussian Poincare inequality.
Hint: linearize.
Question 4.108 (10 points). Try and make some progress on the question we discussed in class: for any even log-concave measure $\mu$ and any symmetric convex body $K$ one has

$$
\mu(K) \mu\left(K^{o}\right) \leq \mu\left(B_{2}^{n}\right)^{2} .
$$

Maybe you can find a proof in some partial case - for some class of measures, for unconditional measures/bodies, in dimension 2, etc?

Question 4.109 (1 point). Prove the symmetric Gaussian Poincare inequality

$$
\operatorname{Var}_{\gamma}(f) \leq \frac{1}{2} \mathbb{E}_{\gamma}|\nabla f|^{2}
$$

for all even locally-Lipschitz functions $f$ on $\mathbb{R}^{n}$ by using the decomposition into Hermite polynomials (rather than by linearizing Blaschke-Santalo inequality like we did in class).

Question 4.110 (1 point). Show that the Blaschke-Santalo inequality and Fathi's inequality are in fact equivalent (in class we only deduced the latter from the former).

Question 4.111 (2 points). Prove the result of Saraglou.
a) See the lecture notes for the definition of the log-addition. Show that the Log-BrunnMinkowski inequality for Lebesgue measure

$$
\left|\frac{K+{ }_{0} L}{2}\right| \geq \sqrt{|K| \cdot|L|}
$$

(for any symmetric convex bodies $K$ and $L$ in $\mathbb{R}^{n}$ ) implies the Log-Brunn-Minkowski inequality for any even log-concave measure $\mu$ on $\mathbb{R}^{n}$ with full support:

$$
\mu\left(\frac{K+{ }_{0} L}{2}\right) \geq \sqrt{\mu(K) \mu(L)}
$$

(for any symmetric convex bodies $K$ and $L$ in $\mathbb{R}^{n}$ ). Conclude that the Log-Brunn-Minkowski conjecture implies the B-conjecture.
Hint: use the Prekopa-Leindler inequality.
b) Show the converse implication.

Hint: consider the situation near the origin and use the scale-invariance of the inequality in the Lebesgue case.

Question 4.112 (2 points). Confirm that the validity of the B-conjecture for all rotationinvariant log-concave measures is equivalent to the fact that for any even log-concave measure $\mu$,

$$
\mu\left(R B_{2}^{n}\right) \mu\left(\frac{1}{R} B_{2}^{n}\right) \leq \mu\left(B_{2}^{n}\right)^{2}
$$

(recall that this corresponds to a very partial case and a sanity check in the Conjecture from Question 4.108.)

Question 4.113 (2 points). Show Klartag's theorem generalizing the functional BrunnMinkowski inequality: for any even log-concave measure $\mu$,

$$
\int e^{-\phi} d \mu \cdot \int e^{-\phi^{*}} d \mu \leq\left(\int e^{-\frac{x^{2}}{2}} d \mu\right)^{2}
$$

Hint: use Cafarelli's contraction theorem.
Question 4.114 (10 points). Attempt to make any progress on the "original B-conjecture": let $z \in \mathbb{R}^{n}$ and let $K$ be a symmetric convex set in $\mathbb{R}^{n}$. Then the function

$$
\frac{\gamma(t K+z)}{\gamma(t K)}
$$

is non-decreasing in $t \geq 1$. Here $\gamma$ is the standard Gaussian measure.
Question 4.115 (2 points). Show that the B-theorem of Cordero-Erasquin, Fradelizi and Maurey would follow from the confirmation of the conjecture from Question 4.114.

Hint: write the conclusion in terms of a non-negative derivative at $t=1$; then note that the arising inequality implies that certain function which depends on $z \in \mathbb{R}^{n}$ is increasing along each ray, and therefore it is convex at the point $z=0$. Consider the Laplacian in $z$.

Question 4.116 (2 points). Prove the result of Bobkov: the following are equivalent:

- For a symmetric convex body $K$ of volume 1 , the measure with the density

$$
\frac{1}{\sqrt{2 \pi}^{n} \gamma(K)} e^{-\frac{x^{2}}{2}} 1_{K}(x) d x
$$

is isotropic.

- For a symmetric convex body $K$ of volume 1 and for any volume-preserving linear transformation $T$ on $\mathbb{R}^{n}, \gamma(K) \geq \gamma(T K)$.

Hint: use the $B$-theorem.
Question 4.117 (3 points). Prove the improved Log-Sobolev inequality: for any convex function $V$ on $\mathbb{R}^{n}$ such that $\int e^{-V}=1$,

$$
-\int V e^{-V} \leq \frac{n}{2} \log \frac{\int \Delta V e^{-V}}{n}-n \log \sqrt{2 \pi e}
$$

Question 4.118 (10 points). Is it possible to deduce from the Reverse Log-Sobolev inequality and/or the (generalized) Log-Sobolev inequality the following corollary of the Entropy Power Inequality?

Let $X$ and $Y$ be any two centered random vectors in $\mathbb{R}^{n}$ and $X^{\prime}$ and $Y^{\prime}$ are independent centered Gaussians (whose covariance matrices are scalar), such that $h(X)=h\left(X^{\prime}\right)$ and $h(Y)=h\left(Y^{\prime}\right)$. Then

$$
h(X+Y) \geq h\left(X^{\prime}+Y^{\prime}\right)
$$

where

$$
h(X)=-\int f \log f
$$

where $f$ is the density according to which $X$ is distributed.
Question 4.119 (2 points). Find Fathi's original proof for his inequality, which relies on the Reverse Log-Sobolev inequality (which we discussed) as well as the following fact (following from works of Cordero-Erasquin, Klartag and Santambrogio).

Let $\mu$ be a centered probability measure whose support has non-empty interior. Then there exists an essentially continuous convex function $\varphi$, unique up to translations, such that $\rho=e^{-\varphi} d x$ is a probability measure on $\mathbb{R}^{n}$ whose push-forward by the map $\nabla \varphi$ is $\mu$. Moreover, it satisfies

$$
\rho=\operatorname{argmin}\left\{-\frac{1}{2} W_{2}(\mu, \nu)^{2}+\operatorname{Ent}_{\gamma}(\nu)\right\} .
$$

Clarification: do not aim to prove this fact, only aim for the implication of Fathi's theorem from this fact combined with the Reverse Log-Sobolev.

Question 4.120 (1 point). Suppose $u, v$ on $\mathbb{R}^{n}$ are 2-homogeneous convex functions. Prove that

$$
\int e^{-\frac{u+v}{2}} \operatorname{det}\left(\frac{\nabla^{2} u+\nabla^{2} v}{2}\right) \geq \sqrt{\int e^{-u} \operatorname{det}\left(\nabla^{2} u\right) \cdot \int e^{-v} \operatorname{det}\left(\nabla^{2} v\right)}
$$

Hint: use the fact that for a 2-homogeneous function, $2 u=\langle\nabla u, x\rangle$ and the change of variables that we used when proving the Reverse Log-Sobolev inequality, together with the Prekopa-Leindler inequality.

Question 4.121 (1 point). Prove the conclusion of Question 4.108 under the assumption that both $K$ and $\mu$ are unconditional.

## 5 Concentration of Measure Phenomenon: the soft approach

We shall now use all the acquired knowledge in order to study concentration of measure phenomenon stemming from convexity. Often times we will be happy with approximate estimates. The phenomena we shall encounter will often grow out of the Brunn-Minkowski inequality (and its many consequences which we learned).

### 5.1 Introduction to the basic concepts related to concentration of measure

Let $\mu$ be a measure on a metric space $(X, d)$. For every point $x \in X$, and subset $A \subseteq X$ we define the distance between $x$ and $A$ as

$$
\operatorname{dist}(A, x):=\inf _{y \in A} d(x, y)
$$

We define the thickening of $A$ by $t$ as

$$
A_{t}:=\{x \in X: d(x, A)<t\}
$$



We will study lower bounds for the quantity

$$
\inf _{A: \mu(A) \geq \alpha} \mu\left(A_{t} \backslash A\right)
$$

A related question is the isoperimetry. Define, as before,

$$
\mu^{+}(\partial A)=\liminf _{\epsilon \rightarrow 0} \frac{\mu\left(A+\epsilon B_{2}^{n}\right)-\mu(A)}{\epsilon}
$$

What are the lower bounds for the quantity

$$
\inf _{A: \mu(A) \geq \alpha} \mu^{+}(\partial A) ?
$$

A complete answer to the first question gives an answer to the second question. In many instances an answer to the second question can give the complete answer to the first question.

Example 5.1 (Concentration of Measure on the Sphere). Let $\sigma$ denote the Haar measure over the sphere $\mathbb{S}^{n-1}$ (the uniform probability distribution on $\mathbb{S}^{n-1}$ ). Take $d$ to be the metric on the sphere defined by $d(x, y):=\angle(x, y)$ for all $x, y \in \mathbb{S}^{n-1}$. Then

$$
\sigma\left(A_{t}\right) \geq 1-\sqrt{\frac{\pi}{8}} e^{-t^{2} / 2} \quad \forall A \subseteq \mathbb{S}^{n-1}, \sigma(A) \geq 1 / 2
$$

Example 5.2 (Concentration of Measure on in Gauss space). Let $\gamma$ denote the $n$ dimensional Gaussian measure. Then taking $d$ to be euclidean metric, we get

$$
\gamma(A) \geq 1-\frac{1}{2} e^{-t^{2} / 2} \quad \forall A \subseteq \mathbb{R}^{n}, \gamma(A) \geq 1 / 2
$$

Example 5.3 (Concentration of Measure on the Hamming Cube). Let $Q=\{-1,1\}^{n}$ denote the hamming cube. Define $d$ to be the hamming distance on $Q$ :

$$
d(x, y):=\frac{1}{n} \cdot \#\left\{i: x_{i} \neq y_{i}\right\}
$$

We define $\mu$ to be the normalized counting measure on $Q$ (i.e. $\mu(A):=2^{-n}|A|$ ). Then

$$
\mu(A) \geq 1-\frac{1}{2} e^{-2 t^{2} / n} \quad \forall A \subseteq Q, \mu(A) \geq 1 / 2
$$

We now state and prove a soft version of the concentration of measure theorem on the sphere:

Theorem 5.4 (non-sharp concentration on $\mathbb{S}^{n-1}$ ). Let $A \subseteq \mathbb{S}^{n-1}$ satisfy $\sigma(A)=1 / 2$. Let $t \in[0, \pi / 2]$. Then the following inequality is true:

$$
\sigma\left(A_{t}\right) \geq 1-c_{1} e^{-c_{2} t^{2} n}
$$

where $c_{1}, c_{2}$ are absolute constants.
The proof of 5.4 will require the following lemma.
Lemma 5.5. Let $\mu$ be a Borel measure on $B_{2}^{n}$ defined by $\mu(A)=|A| /\left|B_{2}^{n}\right|$, where $|\cdot|$ denotes volume. Let $A, B$ be disjoint Borel measurable sets and define

$$
\rho(A, B):=\inf \{|a-b|: a \in A, b \in B\}=: \rho \geq 0
$$

Fix an $\alpha>0$. Then whenever $\mu(A) \geq \alpha$ and $\mu(B) \geq \alpha$, we have:

$$
\alpha \leq e^{-\rho^{2} n / 8}
$$

In other words, if two sets on the sphere are far away, then at least one of them must have a small measure.

Proof. We first note that the Brunn-Minkowski inequality implies that

$$
\begin{aligned}
\frac{1}{2}\left|A \cap B_{2}^{n}\right|^{1 / n}+ & \frac{1}{2}\left|B \cap B_{2}^{n}\right|^{1 / n} \leq\left|\frac{A+B}{2} \cap B_{2}^{n}\right|^{1 / n} \\
& \Longrightarrow \mu\left(\frac{A+B}{2}\right) \geq \frac{\alpha}{2}+\frac{\alpha}{2}=\alpha
\end{aligned}
$$

Next for every $a \in A, b \in B$ we use the parallelogram rule, the definition of $\rho$, and the fact that $A$ and $B$ are sets are on the sphere to get

$$
|a+b|^{2}=2|a|^{2}+2|b|^{2}-|a-b|^{2} \leq 2|a|^{2}+2|b|^{2}-\rho^{2} \leq 4-\rho^{2} .
$$

Since this holds for every $a \in A, b \in B$ we conclude that

$$
\frac{A+B}{2} \subseteq \sqrt{1-\frac{\rho^{2}}{4}} B_{2}^{n} .
$$

Hence

$$
\mu\left(\frac{A+B}{2}\right) \leq \mu\left(\sqrt{1-\rho^{2} / 4} B_{2}^{n}\right)=\left(1-\rho^{2} / 4\right)^{n / 2} \leq \exp \left(-n \rho^{2} / 8\right)
$$

We now prove the theorem.
Proof of Theorem 5.4. Recall that $A \subseteq \mathbb{S}^{n-1}$ satisfies $\sigma(A)=1 / 2$ and $t \in[0, \pi / 2]$. Let $\lambda \in(0,1)$ be a parameter that we'll specify later. Let $B=\mathbb{S}^{n-1} \backslash A_{t}$. We define the following relevant sets

$$
\tilde{A}=\cup\{s A: s \in[\lambda, 1]\}, \quad \tilde{B}=\cup\{s B: s \in[\lambda, 1]\} .
$$

Observe that $\rho(\tilde{A}, \tilde{B})=2 s \sin (t / 2)$ (exercise). Furthermore $2 s \sin (t / 2) \geq 2 s t / \pi$ for $t \leq \pi$. Applying Lemma 5.5 to sets $\tilde{A}, \tilde{B}$ we get

$$
\min (\mu(\tilde{A}), \mu(\tilde{B})) \leq \exp \left(-\rho^{2}(\tilde{A}, \tilde{B}) n / 8\right)=\exp \left(-n(2 s t / \pi)^{2} / 8\right)=\exp \left(-n s^{2} t^{2} /\left(2 \pi^{2}\right)\right)
$$

Since $\sigma(A)=1 / 2$ by assumption one has $\mu(\tilde{A})=\left(1-\lambda^{n}\right) \sigma(A) \geq\left(1-\lambda^{n}\right) / 2$. On the other hand $\mu(\tilde{B})=\left(1-\lambda^{n}\right) \sigma(B) \leq\left(1-\lambda^{n}\right) / 2$. Therefore the minimum is achieved by $\mu(\tilde{B})$.


In particular Lemma 5.5 implies that

$$
\mu(B) \leq \frac{1}{1-\lambda^{n}} e^{-\lambda^{2} t^{2} n /\left(2 \pi^{2}\right)}
$$

Since $B$ is the complement of $A_{t}$ in $\mathbb{S}^{n-1}$ the above inequality is equivalent to

$$
1-\sigma\left(A_{t}\right) \leq \frac{1}{1-\lambda^{n}} e^{-\lambda^{2} t^{2} n /\left(2 \pi^{2}\right)}
$$

Taking $\lambda=1 / 2$ we conclude that

$$
\sigma\left(A_{t}\right) \geq 1-c_{1} e^{-c_{2} t^{2} n}
$$

where $c_{1}=2, c_{2}=1 /\left(8 \pi^{2}\right)$.
Proposition 5.6. Let $A$ be a non-empty Borel measurable set in $\mathbb{R}^{n}$. Then

$$
\int_{\mathbb{R}^{n}} e^{d(x, A)^{2} / 4} d \gamma \leq \frac{1}{\gamma(A)}
$$

Proof. We will prove this inequality via Prekopa-Leindler. To that end we define the following functions:

$$
\begin{array}{r}
\varphi(x)=\frac{1}{(\sqrt{2 \pi})^{n}} e^{-\|x\|^{2} / 2} \\
f(x)=\exp \left(d(x, A)^{2} / 4\right) \cdot \varphi(x) \\
g(x)=\mathbf{1}_{A}(x) \cdot \varphi(x) \\
h(x)=\varphi(x) .
\end{array}
$$

We now verify that $h\left(\frac{x+y}{2}\right) \geq \sqrt{f(x) g(y)}$ on $\mathbb{R}^{n}$ so as to apply Prekopa-Leindler. To do this we will case on $y$. If $y$ is not in $A$ then $g(y)=0$, hence:

$$
h\left(\frac{x+y}{2}\right) \geq 0=\sqrt{f(x) g(y)}
$$

If $y \in A$ we first note that $\|x-y\| \geq d(x, A)$. Next by the parallelogram rule and the inequality we just mentioned we have

$$
\frac{d(x, A)^{2}}{4}-\frac{\|x\|^{2}}{2}-\frac{\|y\|^{2}}{2} \leq-\frac{\|x+y\|^{2}}{4}
$$

Thus we have

$$
\begin{array}{r}
f(x) g(y)=(2 \pi)^{-n} e^{d(x, A)^{2} / 4-\|x\|^{2} / 2-\|y\|^{2} / 2} \\
\leq(2 \pi)^{-n} e^{-\|x+y\|^{2} / 4}=(\sqrt{2 \pi})^{-2 n} e^{-\|(x+y) / 2\|^{2}} \\
\Longrightarrow \sqrt{f(x) g(y)} \leq(\sqrt{2 \pi})^{-n} e^{-\frac{\|(x+y) / 2\|^{2}}{2}} \\
=\varphi\left(\frac{x+y}{2}\right)=h\left(\frac{x+y}{2}\right) .
\end{array}
$$

Therefore for every choice of $y$ the Prekopa-Leindler condition holds. Applying PrekopaLeindler to $f, g, h$ we conclude that

$$
\begin{array}{r}
1=\int h d x \\
\geq \sqrt{\int f d x} \sqrt{\int g d x} \\
=\sqrt{\int e^{d(x, A)^{2} / 4} d \gamma} \sqrt{\gamma(A)} \\
\Longrightarrow \int e^{d(x, A)^{2} / 4} d \gamma \leq \frac{1}{\gamma(A)}
\end{array}
$$

Theorem 5.7 (Gaussian concentration with weaker constraints). Let $A$ be a Borel measurable set satisfying $\gamma(A) \geq 1 / 2$. Then

$$
\gamma\left(A_{t}\right) \geq 1-2 e^{-t^{2} / 4}
$$

Proof. First we decompose the integral into two parts

$$
\int_{\mathbb{R}^{n}} e^{d(x, A)^{2} / 4} d \gamma=\int_{d(x, A)<t} e^{d(x, A)^{2} / 4} d \gamma+\int_{d(x, A) \geq t} e^{d(x, A)^{2} / 4} d \gamma
$$

Since $e^{d(x, A)^{2} / 4}$ is non-negative we may lowerbound the first integral by 0 . Since $d(x, A) \geq$ $t$ on the domain of the second integral we may lowerbound the integrand of the second integral by $t$. These two facts imply that

$$
\int_{\mathbb{R}^{n}} e^{d(x, A)^{2} / 4} d \gamma \geq e^{t^{2} / 4} \gamma(\{x: d(x, A) \geq t\})
$$

By 5.6 the left hand side of this inequality is at most $1 / \gamma(A)$. Therefore

$$
\frac{1}{\gamma(A)} \geq e^{t^{2} / 4} \gamma(\{x: d(x, A) \geq t\})
$$

Since $A_{t}^{c}=\{x: d(x, A) \geq t\}$ and $\gamma(A) \geq 1 / 2$ we may rearrange the inequality to conclude

$$
e^{t^{2} / 4} \gamma\left(A_{t}^{c}\right) \leq 2 \Longrightarrow \gamma\left(A_{t}^{c}\right) \leq 2 e^{-t^{2} / 4} \Longrightarrow \gamma\left(A_{t}\right) \geq 1-2 e^{-t^{2} / 4}
$$

### 5.2 Levy's concentration function and types of estimates

Recall that we consider a Borel subset $A$ of a metric probability measure space $(X, d, \mu)$. For $t>0$, we have been studying

$$
A_{t}=\left\{x \in \mathbb{R}^{n}: \operatorname{dist}(x, A) \leq t\right\}
$$

We have been interested in obtaining estimates of the type $\mu\left(A_{t}\right) \geq 1-g(t)$ where $g(t) \rightarrow 0$. We now define a function that encodes the concentration phenomenon for a given metric probability space.
Definition 5.8 (Levy's Concentration Function). Given a metric measure space ( $X, d, \mu$ ) where $\mu$ is a probability measure, we define $\alpha_{\mu}: \mathbb{R}^{+} \rightarrow \mathbb{R}$ by

$$
\alpha_{\mu}(t):=\sup \left\{1-\mu\left(A_{t}\right): \mu(A) \geq \frac{1}{2}\right\}=1-\inf _{\mu(A) \geq \frac{1}{2}} \mu\left(A_{t}\right)
$$

The concentration function is the best (i.e., smallest) function that satisfies $\mu\left(A_{t}\right) \geq$ $1-\alpha_{\mu}(t)$ for all $A \subset X$ with $\mu(A) \geq \frac{1}{2}$.
Claim 5.9 (Home work). For any metric probability space, $\alpha_{\mu}(t) \xrightarrow{t \rightarrow \infty} 0$.
Indeed, one may notice that as $t \rightarrow \infty$, one has $\mu\left(A_{t}\right) \rightarrow 1$ and $A_{t} \rightarrow X$.
There are two types of concentration phenomena that people especially care about:
Definition 5.10. A measure $\mu$ on $(X, d, \mu)$ has normal or sub-Gaussian concentration if $\alpha_{\mu}(t) \leq C e^{-c t^{2}}$ (where $c, C$ depend on $X$.)

For the Gaussian measure, we saw $\gamma\left(A_{t}\right) \geq 1-c_{1} e^{-c_{2} t^{2}}$ for all Borel measurable sets $A$ satisfying $\gamma(A) \geq \frac{1}{2}$, thus Gaussian measure satisfies the sub-Gaussian concentration. Same is true about the Haar measure on the sphere and the uniform distribution on the Hamming cube.

Definition 5.11. A measure $\mu$ on $(X, d, \mu)$ has exponential concentration if $\alpha_{\mu}(t) \leq C e^{-c t}$ (where $c, C$ depend on $X$.)


Below we shall see many examples of both phenomena.

### 5.3 Concentration of measure for Lipschitz functions

Recall that a function $f: X \rightarrow \mathbb{R}$ is called $p$-Lipschitz if for all $x, y \in X$,

$$
|f(x)-f(y)| \leq p d(x, y)
$$

Recall that if $f: \mathbb{R}^{n} \rightarrow \mathbb{R}, f \in C^{1}\left(\mathbb{R}^{n}\right)$, then $f$ is $p$-Lipschitz if and only if $|\nabla f| \leq p$.
Definition 5.12. Let $f:(X, d, \mu) \rightarrow \mathbb{R}$ be a function where $\mu$ is a probability measure. Then, the median of $f$, denoted $\operatorname{med}(f)$, is the real number such that

$$
\mu(\{x \in X: f \geq \operatorname{med}(f)\}) \geq \frac{1}{2}
$$

and

$$
\mu(\{x \in X: f \leq \operatorname{med}(f)\}) \geq \frac{1}{2}
$$

The next classical result implies that when a metric measure space enjoys some concentration phenomenon, then one can say that Lipschitz functions on this space are almost constant, in the appropriate sense.

Theorem 5.13. Let $(X, d, \mu)$ be a metric probability space, and $f: X \rightarrow \mathbb{R}$ be a 1-Lipschitz function. Then,

$$
\mu(\{x \in X:|f(x)-\operatorname{med}(f)|>t\}) \leq 2 \alpha_{\mu}(t)
$$

Proof. Let $A:=\{x \in X: f(x) \geq \operatorname{med}(f)\}$ and $B:=\{x \in X: f(x) \leq \operatorname{med}(f)\}$. Consider $y \in A_{t}$. Then, there exists $x \in A$ such that $d(x, y) \leq t$. We have

$$
f(y)=f(y)-f(x)+f(x) \geq-d(x, y)+f(x) \geq-d(x, y)+\operatorname{med}(f) \geq-t+\operatorname{med}(f)
$$

Similarly, for any $y \in B_{t}$, we have

$$
f(y) \leq d(x, y)+\operatorname{med}(f) \leq t+\operatorname{med}(f)
$$

As a result,

$$
\{x \in X:|f(x)-\operatorname{med}(f)|>t\} \subset\left(A_{t} \cap B_{t}\right)^{c}=A_{t}^{c} \cup B_{t}^{c} .
$$

We know that $\mu\left(A_{t}\right) \geq 1-\alpha_{\mu}(t)$ and $\mu\left(B_{t}\right) \geq 1-\alpha_{\mu}(t)$. Therefore,

$$
\mu(\{x \in X:|f(x)-\operatorname{med}(f)|>t\}) \leq \mu\left(A_{t}^{c}\right)+\mu\left(B_{t}^{c}\right) \leq 2 \alpha_{\mu}(t)
$$

This completes the proof.
Corollary 5.14. If $f$ is L-Lipschitz, then

$$
\mu(\{x \in X:|f(x)-\operatorname{med}(f)| \geq t\}) \leq 2 \alpha_{\mu}\left(\frac{t}{L}\right)
$$

Proof. Consider $g=f / L$.
Next, the following proposition establishes the converse of Theorem 5.13.
Proposition 5.15. Let $(X, d, \mu)$ be a metric probability space. For given $t$, if there exists $\eta>0$ such that

$$
\mu(\{x \in X:|f(x)-\operatorname{med}(f)|>t\}) \leq \eta .
$$

for all 1-Lipschitz functions $f$, then $\alpha_{\mu}(t) \leq \eta$.
That is, Theorem 5.13 is sharp up to a factor of 2 .
Proof. Consider $f(x)=d(x, A)$ for some Borel set $A$ with $\mu(A) \geq \frac{1}{2}$. Then, $f: X \rightarrow \mathbb{R}$ is 1-Lipschitz. We have

$$
|f(x)-f(y)|=|d(x, A)-d(y, A)| \leq d(x, y)
$$

by the triangle inequality. By assumption,

$$
\mu(\{x \in X:|d(x, A)-\operatorname{med}(f)| \geq t\}) \leq \eta
$$

We also have $\operatorname{med}(f)=\operatorname{med}(d(\cdot, A))=0$. Taking supremum over all $A$ with $\mu(A) \geq \frac{1}{2}$, we get

$$
\alpha_{\mu}(t)=\sup \left\{1-\mu\left(A_{t}\right): \mu(A) \geq \frac{1}{2}\right\} \leq \eta
$$

## Corollary 5.16.

1. If $f: S^{n-1} \rightarrow \mathbb{R}$ is 1 -Lipschitz, then for all $t>0, \sigma(\{|f-\operatorname{med}(f)| \geq t\}) \leq 2 e^{-c t^{2} n}$.
2. If $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is 1-Lipschitz, then for all $t>0, \gamma(\{|f-\operatorname{med}(f)| \geq t\}) \leq 2 e^{-c t^{2}}$.

Now, we shall consider estimates on deviation from the mean. More often than not, it is easier to compute the mean, as compared to the median.

Theorem 5.17. Let $f: S^{n-1} \rightarrow \mathbb{R}$ be Lipschitz continuous with $\operatorname{Lip}(f)=b>0$. Then, for all $t>0$,

$$
\sigma(\{|f(x)-\mathbb{E} f| \geq b t\}) \leq 4 e^{-c t^{2} n}
$$

Here, $\mathbb{E} f=\int_{S^{n-1}} f d \sigma$ and $c$ is an absolute constant independent of $n$.
Proof. Without loss of generality, assume $b=1$. Consider $\widetilde{f}$ - an independent copy of $f$ on $S^{n-1}$. Consider the product space $S^{n-1} \times S^{n-1}$. We have

$$
\begin{gathered}
\sigma \otimes \sigma\left(\left\{(x, y) \in S^{n-1} \times S^{n-1}:|f(x)-\widetilde{f}(y)| \geq t\right\}\right) \leq \\
\sigma\left(|f(x)-\operatorname{med}(f)| \geq \frac{t}{2}\right)+\sigma\left(|\widetilde{f}(y)-\operatorname{med}(f)| \geq \frac{t}{2}\right) \leq 4 e^{-c_{1} t^{2} n / 4}
\end{gathered}
$$

Also,

$$
\mathbb{E}_{\sigma \otimes \sigma} e^{\Lambda^{2}|f-\widetilde{f}|^{2}}=2 \int_{0}^{\infty} \Lambda^{2} t e^{\Lambda^{2} t^{2}} \sigma \otimes \sigma(\{|f(x)-\widetilde{f}(y)| \geq t\}) d t
$$

by the layer-cake formula. Therefore, by the inequality shown above,

$$
\mathbb{E}_{\sigma \otimes \sigma} e^{\Lambda^{2}|f-\widetilde{f}|^{2}} \leq 8 \Lambda^{2} \int_{0}^{\infty} t e^{\Lambda^{2} t^{2}-\widetilde{c} t^{2} n} d t
$$

Choosing $\Lambda$ appropriately, we see that

$$
\mathbb{E}_{\sigma} \mathbb{E}_{\sigma} e^{c_{3} n|f-\tilde{f}|^{2}}=\mathbb{E}_{\sigma \otimes \sigma} e^{c_{3} n|f-\tilde{f}|^{2}} \leq 4
$$

By Jensen's inequality (Theorem 2.3) applied to one of the expectations,

$$
\begin{equation*}
\mathbb{E}_{\sigma} e^{c_{3} n|f-\mathbb{E} f|^{2}} \leq 4 \tag{17}
\end{equation*}
$$

Finally, by Markov's inequality, we have

$$
\begin{aligned}
\sigma(\{x \in X:|f-\mathbb{E} f| \geq t\}) & =\sigma\left(\left\{x \in X: e^{c_{3} n|f-\mathbb{E} f|^{2}} \geq e^{c_{3} n t^{2}}\right\}\right) \\
& \leq e^{-c_{3} n t^{2}} \mathbb{E}_{\sigma} e^{c_{3} n|f-\mathbb{E} f|^{2}} \\
& \leq 4 e^{-c_{3} n t^{2}}
\end{aligned}
$$

where in the last step we used (17). This finished the proof.

### 5.4 Log-Concave Measure, Borell's Lemma and the Reverse Hölder inequality

Lemma 5.18 (Borell's Lemma). Let $\mu$ be a log-concave probability measure on $\mathbb{R}^{n}$. Then, for any symmetric convex set $A \subset \mathbb{R}^{n}$ with $\mu(A)=\alpha \in\left[\frac{1}{2}, 1\right)$, we have

$$
\mu(t A) \geq 1-\alpha\left(\frac{1-\alpha}{\alpha}\right)^{\frac{t+1}{2}}=1-e^{-c t}
$$

for all $t>1$.
Proof. First of all, observe that $\frac{2}{t+1}(t A)^{c}+\frac{t-1}{t+1} A \subset A^{c}$. Indeed, let $x \in(t A)^{c}, y \in A$. Then, $\frac{2 x}{t+1}+\frac{(t-1) y}{t+1} \notin A$ since $A$ is convex and symmetric.

As $\mu$ is log-concave,

$$
1-\mu(A)=\mu\left(A^{c}\right) \geq(1-\mu(t A))^{\frac{2}{t+1}} \mu(A)^{\frac{t-1}{t+1}}
$$

so

$$
1-\alpha \geq(1-\mu(t A))^{\frac{2}{t+1}} \alpha^{\frac{t-1}{t+1}}
$$

which implies

$$
1-\mu(t A) \leq \alpha\left(\frac{1-\alpha}{\alpha}\right)^{\frac{t+1}{2}}
$$

Theorem 5.19 (Reverse Hölder's Inequality for Semi-Norms of Log-Concave Measures). Let $\mu$ be a non-degenerate log-concave probability measure on $\mathbb{R}^{n}$. Consider a semi-norm (convex, 1-homogeneous, even function) $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$. Then, for all $q>p \geq 1$, we have

$$
\left(\int|f|^{p} d \mu\right)^{1 / p} \leq\left(\int|f|^{q} d \mu\right)^{1 / q} \leq \frac{c q}{p}\left(\int|f|^{p} d \mu\right)^{1 / p}
$$

where $c>0$ is an absolute constant.
Proof. Note that the left inequality is a straightforward consequence of Hölder's inequality. The actual content of the Theorem is the right inequality, which we will now prove.

Consider $A=\left\{x \in \mathbb{R}^{n}:|f(x)| \leq 3\|f\|_{p}\right\}$ where $\|f\|_{p}=\left(|f|^{p} d \mu\right)^{1 / p} . A$ is symmetric and convex as $f$ is a semi-norm. Note that

$$
\begin{aligned}
\mu(A) & =1-\mu\left(\left\{x \in X:|f(x)|>3\|f\|_{p}\right\}\right) \\
& =1-\mu\left(\left\{x \in X:|f(x)|^{p}>3^{p}\|f\|_{p}^{p}\right\}\right) \\
& \geq 1-3^{-p}\|f\|_{p}^{-p} \mathbb{E}_{\mu}|f|^{p} \\
& =1-3^{-p}
\end{aligned}
$$

If $\mu(A)=\alpha$, then $\frac{1-\alpha}{\alpha} \leq e^{-p / 2}$ as $\alpha \geq 1-3^{-p}$.
Consider $t A=\left\{x:|f(x)| \leq 3 t\|f\|_{p}\right\}$. We have $\mu(t A) \geq 1-e^{-c_{1} p t}$ for all $t>1$ using Borell's lemma together with 1-homogeneity of $f$.

Fix $\infty>q \geq p>1$. It is easy to see that

$$
\int|f|^{q} d \mu=\int_{0}^{\infty} q s^{q-1} \mu(\{x:|f(x)| \geq s\}) d s
$$

Splitting the integral in two parts,

$$
\begin{aligned}
\int|f|^{q} d \mu & =\int_{0}^{3\|f\|_{p}} q s^{q-1} \mu(\{|f| \geq s\}) d s+\int_{3\|f\|_{p}}^{\infty} q s^{q-1} \mu(\{|f| \geq s\}) d s \\
& \leq \int_{0}^{3\|f\|_{p}} q s^{q-1} d s+\left(3\|f\|_{p}\right)^{q} \int_{1}^{\infty} q t^{q-1} e^{-c_{1} p t} d t \\
& =\left(3\|f\|_{p}\right)^{q}+q\left(3\|f\|_{p}\right)^{q} \int_{1}^{\infty} t^{q-1} e^{-c_{1} p t} d t \\
& =c^{q}\|f\|_{p}^{q}\left(1+q^{q}\right)
\end{aligned}
$$

and the result follows. In the last step, we use

$$
\begin{aligned}
\int_{0}^{\infty} t^{q-1} e^{-c_{1} p t} d t & =\left(c_{1} p\right)^{q} \int_{0}^{\infty} t^{q-1} e^{-t} d t \\
& =\left(c_{1} p\right)^{q} \Gamma(q) \\
& =\left(c_{1} p\right)^{q} q^{q} c_{2}^{q}
\end{aligned}
$$

Let us outline an application of the above theorem.
Claim 5.20. Let $X$ be a log-concave, isotropic ${ }^{1}$ random vector. Then for all $t \geq 1$,

$$
\mathbb{P}(|X| \geq c t \sqrt{n}) \leq e^{-c_{1} t}
$$

Proof. Using Chernoff's trick, we have for $\Lambda>0$,

$$
\mathbb{P}(|X| \geq c t \sqrt{n})=\mathbb{P}\left(e^{\Lambda|X|} \geq e^{c t \Lambda \sqrt{n}}\right) \leq e^{-c t \Lambda \sqrt{n}} \mathbb{E} e^{\Lambda|X|}
$$

Furthermore,

$$
\mathbb{E} e^{\Lambda|X|}=\mathbb{E} \sum_{k=0}^{\infty} \frac{(\Lambda|X|)^{k}}{k!}=\sum_{k=0}^{\infty} \frac{\Lambda^{k} \mathbb{E}|X|^{k}}{k!} \leq \sum_{k=0}^{\infty} \frac{\Lambda^{k}(c k \mathbb{E}|X|)^{k}}{k!}
$$

where the last step follows from the preceding theorem applied to $p=1$ and $q=k$.
We also have $\mathbb{E}|X| \leq \sqrt{\mathbb{E}|X|^{2}}$. As $X$ is isotropic, we have $\mathbb{E}|X|^{2}=\sum \mathbb{E}\left|X_{i}\right|^{2}=n$. Therefore, $\mathbb{E}|X| \leq \sqrt{n}$. It follows that

$$
\mathbb{E} e^{\Lambda|X|} \leq c_{0}
$$

for an absolute constant $c_{0}$, if $\Lambda$ in the calculations above is chosen to be $c_{1} / \sqrt{n}$ for an appropriate constant $c_{1}$. Finally, we get the bound

$$
\mathbb{P}(|X| \geq c t \sqrt{n}) \leq e^{-c t \Lambda \sqrt{n}} \mathbb{E} e^{\Lambda|X|} \leq c_{0} e^{-\widetilde{c} t}
$$

### 5.5 Paouris's inequality (statement)

In fact, the statement of the Claim 5.20 misses the correct inequality by a factor of $\sqrt{n}$ :
Theorem 5.21 (Paouris [152]). Let $X$ be a log-concave, isotropic random vector. Then, for any $t \geq 1$,


This result was further re-proved (and in some sense strengthened) by Guedon, E. Milman [79].

[^0]
### 5.6 Sub-exponential concentration of linear functions for Log-Concave measures, and of other semi-norms

In fact, one may repeat the argument of Claim 5.20 to show:
Claim 5.22. Let $\mu$ be a log-concave probability measure and $X$ be a random vector such that $X \sim \mu$. Let $f$ be any semi-norm. Then for any $t>0$,

$$
P\left(f(X) \geq \mathbb{E}_{\mu} f(X)+t\right) \leq C e^{-\frac{c t}{\mathbb{E}_{\mu} f(X)}} .
$$

In the particular case when $f(x)=|\langle x, \theta\rangle|$ (which is, in some sense, the simplest possible semi-norm) one gets a result which is tight:

Corollary 5.23. Let $\mu$ be a log-concave probability measure and $X$ be a random vector such that $X \sim \mu$. Then, for all $\theta \in S^{n-1}$,

$$
\mu\left(\left\{x:|\langle x, \theta\rangle| \geq t \mathbb{E}_{\mu}|\langle X, \theta\rangle|\right\}\right) \leq 2 e^{-c t}
$$

for every $t>0$.
Proof. We have

$$
\begin{aligned}
\mu\left(\left\{|\langle X, \theta\rangle| \geq t \mathbb{E}_{\mu}|\langle X, \theta\rangle|\right\}\right) & =\mu\left(\left\{e^{\frac{c|\langle X, \theta\rangle|}{\mathbb{E}_{\mu}|\langle X, \theta\rangle|}} \geq e^{c t}\right\}\right) \\
& \leq \mathbb{E}_{\mu}\left(e^{\frac{c|\langle X, \theta\rangle\rangle}{\mathbb{E}_{\mu}|\langle X, \theta\rangle|}}\right) e^{-c t} \\
& \leq c_{1} e^{-c t}
\end{aligned}
$$

where

$$
\mathbb{E}_{\mu} e^{\boldsymbol{\uparrow}|\langle X, \theta\rangle|}=\mathbb{E}_{\mu} \sum_{k=0}^{\infty} \frac{\boldsymbol{\phi}^{k}|\langle X, \theta\rangle|^{k}}{k!}=\sum_{k=0}^{\infty} \frac{\boldsymbol{\phi}^{k}\left(E_{\mu}|\langle X, \theta\rangle|\right)^{k} k^{k}}{k!} \leq c_{1},
$$

provided that $\boldsymbol{\uparrow}$ is chosen to be of order $\left(c \mathbb{E}_{\mu}|\langle X, \theta\rangle|\right)^{-1}$.
One may check the tightness of the above result by considering for instance the measure $d \mu=e^{-\|x\|_{1}} d x$.

In fact, similar results could be obtained on the sphere, not just with respect to logconcave measures on $\mathbb{R}^{n}$ : indeed, considering $\mu$ to be a rotation-invariant log-concave measure (say, the Gaussian), writing the integrals in polar coordinates, and using the Reverse Hölder inequality, one gets:

Corollary 5.24 (Home work). Let $f: S^{n-1} \rightarrow \mathbb{R}$ be a semi-norm. Then,

$$
\left(\int_{S^{n-1}}|f(\theta)|^{q} d \sigma(\theta)\right)^{1 / q} \leq \frac{c q}{p} \sqrt{\frac{n+p}{n+q}}\left(\int_{S^{n-1}}|f(\theta)|^{p} d \sigma(\theta)\right)^{1 / p} .
$$

Therefore, one has sub-exponential concentration for norms on the sphere:

$$
P(f(\theta) \geq \mathbb{E} f+t) \leq C e^{-\frac{c t}{\mathbb{E} f}}
$$

Here $\mathbb{E}$ stands for the expectation with respect to the Haar measure on the sphere.

### 5.7 The Thin Shell Conjecture and the KLS conjecture

Conjecture 5.25 (Antilla, Ball, Perissinaki [2]; Bobkov, Koldobsky [24]). Let X be a logconcave, isotropic random vector (that is, $\mathbb{E} X=0$ and $\mathbb{E} X_{i} X_{j}=\delta_{i j}$ ). Then

$$
\operatorname{var}(|X|) \leq C
$$

Remark 5.26. Let $T$ be a volume-preserving linear transformation and $\mu$ be a log-concave measure with $\operatorname{bar}(\mu)=0$. Define $\mu_{T}:=T_{\#} \mu$. Then, for every such $\mu$, there exists $T$ such that $\mu_{T}$ is isotropic. Also, if $X_{T} \sim \mu_{T}$, then $\inf _{T} \operatorname{var}\left(\left|X_{T}\right|\right)$ is attained for the isotropic position (the infimum is taken over all volume-preserving linear transformations.)

Remark 5.27. The thin shell conjecture implies the slicing Conjecture 4.16 of Bourgain, as was shown by Eldan and Klartag [59].

Recall the Kannan-Lovász-Simonovits Conjecture [88] (1995) which was stated above as Conjecture 4.19: for log-concave isotropic probability measures $\mu$ on $\mathbb{R}^{n}$ and all locally Lipschitz functions $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$, we have

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq C \int|\nabla f|^{2} d \mu
$$

where $C>0$ is an absolute constant.
Remark 5.28. The Kannan-Lovász-Simonovits Conjecture implies the Thin Shell Conjecture. To see this, take $f(x)=|x|$. Then,

$$
\operatorname{var}(|x|)=\int|x|^{2} d \mu-\left(\int|x| d \mu\right)^{2} \leq c \int 1 d \mu=c
$$

The current best bound for the KLS conjecture, as well the Thin Shell conjecture and Bourgain's sclicing problem is the following:

Theorem 5.29 (Klartag 2022 [103]). For a log-concave isotropic probability measure $\mu$ on $\mathbb{R}^{n}$ and all locally Lipschitz functions $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$, we have

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq C \sqrt{\log n} \int|\nabla f|^{2} d \mu
$$

where $C$ is an absolute constant.

### 5.8 Klartag's bound on the thin shell in the unconditional case

Below we shall outline a relatively uncomplicated proof, due to Klartag [101], of the Thin Shell conjecture in the partial case when the measure is unconditional, that is, symmetric with respect to all coordinate reflections. The reader is also referred to Klartag [99].

Theorem 5.30 (Klartag 2008 [101]). If $X$ is an unconditional log-concave isotropic random vector on $\mathbb{R}^{n}$, then $\operatorname{var}(|X|) \leq C$ where $C$ is a constant independent of the dimension.
Lemma 5.31. Let $\mu$ be a probability measure on $\mathbb{R}_{+}^{n}:=\left\{x \in \mathbb{R}^{n}: x_{i} \geq 0\right\}$. Consider $d \mu=e^{-\psi} d x-a \log$-concave, unconditional measure. Then, for any $f \in W^{1,2}\left(\mathbb{R}_{+}^{n}, \mu\right)$, we have $\operatorname{var}_{\mu}(f) \leq 4 \int_{\mathbb{R}_{+}^{n}} \sum_{i=1}^{n} x_{i}^{2}\left|\partial_{i} f\right|^{2} d \mu$.

Proof. Consider a map $\pi\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}^{2}, \ldots, x_{n}^{2}\right)$. We note that $\psi \circ \pi$ is convex, as $\psi$ is convex and unconditional (see home work). Consider $\varphi(x):=\psi(\pi(x))-\sum_{i=1}^{n} \log \left(2 x_{i}\right)$ and the measure $\nu$ with density $e^{-\varphi}$. Then $\pi$ transports $\mu$ into $\nu$, that is, for all appropriate $f$ we have:

$$
\int f\left(\pi^{-1}(x)\right) e^{-\psi(x)} d x=\int f(y) e^{-\varphi(y)} d y
$$

Note that

$$
\nabla^{2} \varphi=\nabla^{2}(\psi \circ \pi)-\nabla^{2}\left(\sum_{i=1}^{n} \log \left(2 x_{i}\right)\right) \geq-\nabla^{2}\left(\sum_{i=1}^{n} \log \left(2 x_{i}\right)\right)=\left[\begin{array}{cccc}
\frac{1}{x_{1}^{2}} & 0 & \cdots & 0 \\
0 & \frac{1}{x_{2}^{2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{x_{n}^{2}}
\end{array}\right]
$$

This gives

$$
\left(\nabla^{2} \varphi\right)^{-1} \leq\left[\begin{array}{cccc}
x_{1}^{2} & 0 & \cdots & 0 \\
0 & x_{2}^{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_{n}^{2}
\end{array}\right]
$$

By the Brascamp-Lieb inequality (Theorem 3.75]),

$$
\begin{equation*}
\operatorname{var}_{e^{-\varphi}}(g) \leq \int_{\mathbb{R}_{+}^{n}}\left\langle\left(\nabla^{2} \varphi\right)^{-1} \nabla g, \nabla g\right\rangle e^{-\varphi} \leq \int_{\mathbb{R}_{+}^{n}} \sum_{i=1}^{n} x_{i}^{2}\left|\partial_{i} g\right|^{2} e^{-\varphi} \tag{18}
\end{equation*}
$$

We have

$$
\begin{equation*}
\operatorname{var}_{e^{-\varphi}}(g)=\operatorname{var}_{e^{-\psi}}(f) \tag{19}
\end{equation*}
$$

where $g=f \circ \pi$. Also,

$$
\begin{equation*}
\int_{\mathbb{R}_{+}^{n}} \sum_{i=1}^{n} x_{i}^{2}\left|\partial_{i} g\right|^{2} e^{-\varphi}=\int_{\mathbb{R}_{+}^{n}} \sum_{i=1}^{n} x_{i}^{2}\left|\partial_{i} f\right|^{2} e^{-\psi} \tag{20}
\end{equation*}
$$

Combining (18), (19) and (20) finishes the proof.
Proof of Theorem 5.30. We write

$$
\operatorname{var}(|X|)=\mathbb{E}(|X|-E|X|)^{2} \leq \mathbb{E}(|X|-\sqrt{n})^{2}=\mathbb{E} \frac{\left(X^{2}-n\right)^{2}}{(X+\sqrt{n})^{2}} \leq
$$

$$
\frac{1}{n} \mathbb{E}\left(|X|^{2}-n\right)^{2}=\frac{1}{n} \operatorname{var}\left(|X|^{2}\right) \leq \frac{4}{n} \sum_{i=1}^{n} \mathbb{E} X_{i}^{2}\left(2 X_{i}\right)^{2}=\frac{16}{n} \sum_{i=1}^{n} \mathbb{E} X_{i}^{4} \leq c,
$$

where for the last inequality we use $\mathbb{E} X_{i}^{4} \leq c\left(\mathbb{E} X_{i}^{2}\right)^{2} \leq c$ (by the Reverse Hölder inequality), while the second to last inequality follows from Lemma 5.31 applied with $f(x)=|x|^{2}$ with $\partial_{i} f=2 x_{i}$.

### 5.9 Exponential concentration via Poincare Inequality

Recall: Suppose $\mu$ is a probability measure on a metric space $M=(X, d)$. $\mu$ is said to satisfy the Poincare inequality with constant $\beta$ if for any locally-Lipschitz function $f$ one has

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq \beta \cdot \int|\nabla f|^{2} d \mu
$$

We have the following examples:

- Gaussian measure, $\beta=1$;
- Gaussian measure, even functions (for which $\beta=\frac{1}{2}$ );
- KLS conjecture / Klartag's result: if $\mu$ is isotropic and log-concave then we can choose $\beta=c \sqrt{\log n}$;
- Payne-Weinberger Theorem 3.85: If $\mu$ is uniform on a convex set $K$ then $\beta \leq \frac{\operatorname{diam}(K)}{\pi}$;
- Other examples: $\mu$ uniform on the sphere, on the Hamming cube, etc (see the home work);
- Fix $k>0$. If $\mu$ is "more log-concave than the Gaussian", that is $d \mu=e^{-v} d x$ with $\nabla^{2} v \geq k \cdot$ Id then one has $\beta \leq k^{-1}$ (as a corollary of the Brascamp-Lieb inequality Theorem 3.75).

We shall see that establishing a Poincare inequality directly allows to establish a subexponential concentration result (see also the presentation in [4]). Recall that

$$
\alpha_{\mu}(t)=\sup _{\mu(A) \geq \frac{1}{2}}\left(1-\mu\left(A_{t}\right)\right)
$$

is the concentration function that governs how quickly sets accumulate measure.
Theorem 5.32 (Gromov-Milman [75]). Suppose ( $X, d, \mu$ ) is a metric probability space. Suppose $\mu$ satisfies on $X$ the Poincare inequality with constant $\beta$. Then

$$
\alpha_{\mu}(t) \leq e^{-\frac{t}{3 \sqrt{\beta}}} .
$$

We immediately notice some corollaries.

Corollary 5.33. If $\nabla^{2} v \geq k \cdot I d, d \mu=e^{-v} d x$ on $\mathbb{R}^{n}$ then $\alpha_{\mu}(t) \leq e^{-\frac{\sqrt{k} t}{3}}$.
Corollary 5.34. If $\mu$ is isotropic log-concave then $\alpha_{\mu}(t) \leq e^{-\frac{c t}{(\log n)^{1 / 4}}}$.
Remark 5.35. Note also that Payne-Weinberger one has that if $\mu$ is uniform over $K$ then $\alpha_{\mu}(t) \leq e^{-\frac{c t}{\sqrt{\text { diam K }}}}$.

Before we outline the proof, recall that earlier we related the bounds on concentration function to the large deviation bounds on Lipschitz functions from its median. It turns out, one could also express a bound in terms of large deviation from the mean.

Proposition 5.36 (Home Work). If for all Lipschitz functions $f$ one has

$$
\mu\left(f \geq \int f d \mu+t\right) \leq \alpha(t)
$$

then one has the implication that for all Borell sets $A \subset X$ with $\mu(A)>0$, for all $t>0$ one has

$$
1-\mu\left(A_{t}\right) \leq \alpha(\mu(A) t)
$$

In particular, $\alpha_{\mu}(t) \leq \alpha(t / 2)$ for each $t>0$.
Proof of Theorem 5.32. Consider $f: X \rightarrow \mathbb{R}$ to be some bounded 1-Lipschitz function with mean zero (note WLOG we can take this for all such functions, up to subtracting a constant, and both functions are invariant under shift operations). Consider $\lambda>0$ and $g_{\lambda}=e^{\frac{\lambda f}{2}}$. One has, by a Poincare inequality assumption, that

$$
\int g_{\lambda}^{2}-\left(\int g_{\lambda} d \mu\right)^{2} \leq \beta \int\left|\nabla g_{\lambda}\right|^{2} d \mu
$$

holds for some $\beta$. We can compute $\nabla g_{\lambda}=\frac{\lambda}{2} \nabla f e^{\frac{\lambda f}{2}}$, so that the above is then

$$
\int e^{\lambda f} d \mu-\left(\int e^{\frac{\lambda f}{2}} d \mu\right)^{2} \leq \frac{\beta \lambda^{2}}{4} \int e^{\lambda f}|\nabla f|^{2} d \mu .
$$

By Lipschitz criterion, one has $|\nabla f| \leq 1$ and hence the RHS is $\frac{\beta \lambda^{2}}{4} \int e^{\lambda f} d \mu$. Set $S(\lambda)=$ $\int e^{\lambda f} d \mu$, then $S(\lambda)-S\left(\frac{\lambda}{2}\right)^{2} \leq \frac{\beta \lambda^{2}}{4} S(\lambda)$. Equivalently, one has $S(\lambda) \leq \frac{1}{1-\frac{\beta \lambda^{2}}{4}} \cdot S(\lambda / 2)$. Iterating this with $\lambda=\frac{1}{2 \sqrt{\beta}}$, we have

$$
S(\lambda) \leq \prod_{k=0}^{n-1}\left(\frac{1}{1-\frac{1}{4^{k+1}}}\right)^{2^{k}} \cdot S^{2^{n}}\left(\lambda / 2^{n}\right)
$$

We claim that $\left(S\left(\lambda / 2^{n}\right)\right)^{2^{n}} \rightarrow 1$ as $n \rightarrow \infty$, then $S(\lambda)=1-\lambda \mathbb{E}_{\mu} f+O\left(\lambda^{2}\right)=1+O\left(\lambda^{2}\right)$. Using the estimate $\left(1+c \lambda^{2} / 2^{2 n}\right)^{2^{n}} \approx e^{-c 2^{-n}}$, which goes to 1 as $n \rightarrow \infty$, and using algebra/calculus one has $S(1 / 2 \sqrt{\beta}) \leq \prod_{k=0}^{\infty}\left(\frac{1}{1-4^{k+2}}\right)^{2^{k}} \leq 3$.

To proceed, we use the so-called Chernoff's trick. We have

$$
\begin{aligned}
\mu(f \geq t)=\mu\left(f \geq \mathbb{E}_{\mu} f+t\right. & =\mu\left(e^{\frac{f}{2 \sqrt{\beta}}} \geq e^{\frac{t}{2 \sqrt{\beta}}}\right) \\
& \leq S(1 / 2 \sqrt{\beta}) e^{-\frac{t}{2 \sqrt{\beta}}} \leq 3 e^{-\frac{5}{2 \sqrt{\beta}}} .
\end{aligned}
$$

Recall that it's enough to have Poincare for $e^{c \cdot d(x, A)}$ in order for $\mu\left(A_{t}\right) \geq 1-e^{-\tilde{c} t}$, and for this setting, this distance function $d(x, A)$ is a Lipschitz function.

Remark 5.37. Examples of spaces that do not satisfy the Poincaré inequality are unions of almost-disconnected convex domains connected by a thin neck; Poincare inequality worsens as neck gets thinner. See the picture below (c) Klartg.


Picture (c) Klartag

### 5.10 Marton's argument: almost-optimal concentration for the Gaussian via Fathi

Recall that Talagrand inequality says that

$$
W_{2}(\mu, \gamma)^{2} \leq 2 \operatorname{Ent}_{\gamma} \mu
$$

recall Fathi's Theorem 4.75 which states that

$$
W_{2}(\mu, \nu)^{2} \leq 2 \operatorname{Ent}_{\gamma} \mu+2 \operatorname{Ent}_{\gamma} \nu
$$

In particular, it is enough to assume that either $\mu$ or $\nu$ is bary-centered, by the BlaschkeSantaló inequality. As a corollary, we get the following:

Corollary 5.38 (Fathi). Suppose $A$ is a Borel set in $\mathbb{R}^{n}$ with barycenter zero. Then

$$
1-\gamma\left(A_{r}\right) \leq \frac{1}{\gamma(A)} e^{-r^{2} / 2}
$$

Here $A_{r}$, as usual, is the r-thickening of the set $A$.

Proof. Consider $d \mu$ to be the restriction of the Gaussian measure to $A$, given by $d \mu=$ $\frac{1}{\gamma(A)} \mathbb{1}_{A} d \gamma$. Then we set $d \nu=\frac{1}{1-\gamma\left(A_{r}\right)} \mathbb{1}_{A_{r}^{c}} d \gamma$. Note that

$$
\operatorname{Ent}_{\gamma}(\mu)=\int \frac{d \mu}{d \gamma} \log \frac{d \mu}{d \gamma}-0=-\log \gamma(A)
$$

and

$$
\operatorname{Ent}_{\gamma}(\nu)=-\log \left(1-\gamma\left(A_{r}\right)\right)
$$

The Wasserstein distance between $\mu$ and $\nu$ satisfies

$$
\begin{aligned}
W_{2}(\mu, \nu)^{2}=\inf _{\pi \text { couplings }} \int|x-y|^{2} d \mu(x, y) & =\inf _{\pi} \int_{\left\{x \in A, y \in A_{r}^{c}\right\}}|x-y|^{2} d \mu(x, y) \\
& \geq r^{2} \inf _{\pi} \int_{\{\ldots\}} d \pi(x, y) \\
& =r^{2} .
\end{aligned}
$$

By Fathi's result (Theorem 4.75), we know that $r^{2} \leq-2 \log \gamma(A)-2 \log \left(1-\gamma\left(A_{r}\right)\right.$ ), which implies that $e^{-r^{2} / 2} \geq \gamma(A)\left(1-\gamma\left(A_{r}\right)\right)$, finishing the proof.

Remark 5.39. Note that this is not an optimal bound, but it is a useful technique that in general leads to other useful estimates. The type of argument is referred to as Marton's argument [142].

### 5.11 Concentration and Laplace Functional

We follow the presentation in [4]. Assume $(X, d, \mu)$ is a metric probability space, and set $\lambda \geq 0$. Define a function $E_{\mu}(\lambda)$ to be

$$
E_{\mu}(\lambda):=\sup \left\{\int e^{\lambda f} d \mu \mid f: X \rightarrow \mathbb{R} \text { is } 1 \text {-Lipschitz, } \mathbb{E}_{\mu} f=0\right\}
$$

This is called the Laplace functional. One has the following relation between the Laplace functional and concentration:

Proposition 5.40. Suppose $(X, d, \mu)$ is a metric probability space. Then for each $t>0$, one has

$$
\alpha_{\mu}(t) \leq \inf _{\lambda \geq 0}\left(e^{-\lambda t / 2} E_{\mu}(\lambda)\right) .
$$

Proof. Let $f$ be a 1-Lipschitz function, and set $g=f-\mathbb{E}_{\mu} f$. One has $\int e^{\lambda g} d \mu \leq E_{\mu}(\lambda)$ by definition of the Laplace functional $E_{\mu}(\lambda)$. Therefore,

$$
\begin{aligned}
\mu(g \geq t) & =\mu\left(e^{\lambda g} \geq e^{\lambda t}\right) \\
& \leq \int e^{\lambda g} d \mu \cdot e^{-\lambda t} \\
& \leq E_{\mu}(\lambda) \cdot e^{-\lambda t} .
\end{aligned}
$$

Thus

$$
\mu\left(\left(f \geq \mathbb{E}_{\mu} f+t\right) \leq E_{\mu}(\lambda) e^{-\lambda t} \quad \forall \lambda \geq 0\right.
$$

and hence $\alpha_{\mu}(t) \leq E_{\mu}(t) e^{-\lambda t / 2}$. The result then follows after taking infimum.
In some particular cases, bounding the Laplace functional might lead to sub-Gaussian concentration.

Corollary 5.41. If $E_{\mu}(\lambda) \leq e^{C_{0} \lambda^{2}}$ then one has $\alpha_{\mu}(t) \leq e^{-c_{1} t^{2}}$.
Proof. Indeed, $\inf _{\lambda} e^{-\lambda t / 2+C_{0} \lambda^{2}}=e^{\inf \left\{c_{0} \lambda^{2}-\lambda t / 2\right\}}=e^{-c_{1} t^{2}}$.
For example, if the diameter of a space $X$ is uniformly bounded by some $D$ then $E_{\mu}(\lambda) \leq$ $e^{c D^{2} \lambda^{2}}$. As a corollary, we get an almost-optimal concentration on the Hamming cube. See home work for more details.

### 5.12 The Herbst Argument

This classical argument due to Herbst [78] allows to obtain sub-Gaussian concentration via the Log-Sobolev inequality.

Theorem 5.42 (Herbst). Let $(X, d, \mu)$ be a probability metric space such that for any locallyLipschitz functions one has the Log-Sobolev inequality with constant $\beta>0$, that is,

$$
\operatorname{Ent}_{\mu}\left(f^{2}\right) \leq 2 \beta \int|\nabla f|^{2} d \mu
$$

Then for all $t>0$, one has $\alpha_{\mu}(t) \leq e^{-t^{2} / 8 \beta}$.
As an example, if $d \mu=e^{-v} d x$ and $\nabla^{2} v \geq k \cdot$ Id then $\beta \leq k^{-1}$. This follows from the Gaussian log-Sobolev and Cafarelli's Theorem (see home work). Therefore, we get

Corollary 5.43. Suppose $d \mu=e^{-v} d x$ and $\nabla^{2} v \geq k \cdot I d$. Then for all Borel $A \subset \mathbb{R}^{n}$ with $\mu(A) \geq \frac{1}{2}$ we have

$$
\mu\left(A_{t}\right) \geq 1-e^{-k t^{2} / 8} \quad(\forall t \geq 0)
$$

Remark 5.44. Recall that log-Sobolev with constant $\beta$ immediately implies Poincare with constant $\beta$, because if $E n t_{\mu}\left(f^{2}\right) \leq 2 \beta \int|\nabla f|^{2} d \mu$ and one sets $f=1+\epsilon g$ then one gets Poincare's inequality in the $\epsilon^{2}$ term (home work). Therefore, Herbst's Theorem 5.12 assumes more than Gromov-Milman [75], but the conclusion is stronger.

To prove this theorem, we need some machinery.
Definition 5.45 (log-moment generating function). Given $f: X \rightarrow \mathbb{R}$ consider the function

$$
\psi(\lambda)=\log \mathbb{E}_{\mu} e^{\lambda\left(f-\mathbb{E}_{\mu} f\right)}
$$

This function is called the log-moment generating function of $f$.
We have the following Lemma:
Lemma 5.46. Suppose for $f: X \rightarrow \mathbb{R}$ one has

$$
\operatorname{Ent}_{\mu}\left(e^{\lambda f}\right) \leq \frac{\lambda^{2} \sigma^{2}}{2} \mathbb{E}_{\mu} e^{\lambda f}
$$

Then $\psi(\lambda) \leq \frac{\sigma^{2} \lambda^{2}}{2}$.
Proof. One has

$$
\psi(\lambda)=\log \mathbb{E}_{\mu} e^{\lambda f}-\lambda \mathbb{E}_{\mu} f+O\left(\lambda^{2}\right)
$$

so that then

$$
\begin{aligned}
\frac{d}{d \lambda} \frac{\psi(\lambda)}{\lambda} & =\frac{1}{\lambda} \cdot \frac{\mathbb{E}_{\mu}\left(f e^{\lambda f}\right)}{\mathbb{E}_{\mu}\left(e^{\lambda f}\right.}-\frac{1}{\lambda^{2}} \log \mathbb{E}_{\mu} e^{\lambda f} \\
& =\frac{1}{\lambda^{2}} \cdot \frac{\operatorname{Ent}_{\mu}\left(e^{\lambda f}\right)}{\mathbb{E}_{\mu}\left(e^{\lambda f}\right)}
\end{aligned}
$$

Then by Newton-Leibniz one has

$$
\begin{aligned}
\frac{\psi(\lambda)}{\lambda)}-\lim _{\lambda \rightarrow 0} \frac{\psi(\lambda)}{\lambda)} & =\int_{0}^{\lambda} \frac{1}{s^{2}} \frac{\operatorname{Ent}_{\mu}\left(e^{s f)}\right.}{\mathbb{E}_{\mu}\left(e^{s f}\right)} d s \\
& \leq \int_{0}^{\lambda} \frac{1}{s^{2}} \frac{\sigma^{2} s^{2}}{2} d s \\
& =\lambda \sigma^{2} / 2 .
\end{aligned}
$$

One can check that $\lim _{\lambda \rightarrow 0} \frac{\psi(\lambda)}{\lambda}=0$, then

$$
\psi(\lambda) \leq \frac{\lambda^{2} \sigma^{2}}{2}
$$

When $f=e^{-x^{2} / 2}(2 \pi)^{-n / 2}$, then $\psi(\lambda) \approx c \lambda^{2}$. This motivates the following definition:
Definition 5.47. A function $f$ is said to be sub-Gaussian if for this $f, \psi(\lambda) \leq \frac{\lambda^{2} \sigma^{2}}{2}$ (here $\sigma>0$ is the sub-Gaussian constant).

There is a relation between concentration and $\psi^{*}(t):=\sup _{\lambda}(t \lambda-\psi(\lambda))$. Namely:
Lemma 5.48. Suppose $f \in L^{1}(\mu)$. Then

$$
\mu\left(\left|f-\mathbb{E}_{\mu} f\right| \geq t\right) \leq 2 e^{-\psi^{*}(t)}
$$

Proof. Recall that $\psi(\lambda)=\log \mathbb{E}_{\mu} e^{\lambda f-\lambda \mathbb{E}_{\mu} f}$. Then

$$
\begin{aligned}
\mu\left(f-\mathbb{E}_{\mu} f \geq t\right) & =\mu\left(e^{\lambda\left(f-\mathbb{E}_{\mu} f\right)} \geq e^{\lambda t}\right) \\
& \leq \mathbb{E} e^{\lambda\left(f-\mathbb{E}_{\mu} f\right)} e^{-\lambda t} \\
& \leq e^{-(\lambda t-\psi(\lambda)}
\end{aligned}
$$

The LHS does not depend on $\lambda$, so we can take infimum in $\lambda$ to conclude that

$$
\begin{aligned}
\mu\left(\left|f-\mathbb{E}_{\mu} f\right| \geq t\right) & \leq \inf _{\lambda} e^{-(\lambda t-\psi(\lambda))} \\
& =e^{-\sup _{\lambda}(\lambda t-\psi(\lambda))} \\
& =e^{-\psi^{*}(t)}
\end{aligned}
$$

The lower tail of the absolute value can be computed similarly.
As a corollary of Lemma 5.48, we get
Corollary 5.49. For all sub-Gaussian $f$ one has

$$
\mu\left(\left|f-\mathbb{E}_{\mu} f\right| \geq t\right) \leq 2 e^{-t^{2} / 2 \sigma^{2}}
$$

Proof of Herbst Theorem . As usual, in order to bound the concentration function $\alpha_{\mu}(t)$, we bound $\mu\left(f-\mathbb{E}_{\mu} f \geq t\right)$ for all Lipschitz functions. Consider any Lipschitz function $g$, with Lipschitz constant bounded by 1 , and WLOG take $\mathbb{E}_{\mu} g=0$. As before, consider $f=e^{\lambda g / 2}$. One then has

$$
\nabla f=\frac{\lambda}{2} \nabla g e^{\lambda g / 2}
$$

Under the assumption that $\mu$ satisfies the Log-Sobolev inequality, we conclude that

$$
\begin{aligned}
\operatorname{Ent}_{\mu}\left(f^{2}\right) & \leq 2 \beta \int|\nabla f|^{2} d \mu \\
& =\frac{\lambda \beta}{2} \int|\nabla g|^{2} e^{\lambda g} d \mu \\
& \leq \frac{\lambda^{2} \beta}{2} \int e^{\lambda g} d \mu .
\end{aligned}
$$

By Lemma 5.46, we have that if $\operatorname{Ent}_{\mu}\left(e^{\lambda g}\right) \leq \frac{\lambda^{2} \beta}{2} \mathbb{E}_{\mu} e^{\lambda g}$, then $\psi(\lambda) \leq \frac{\beta \lambda^{2}}{2}$, and hence $\psi^{*}(t) \geq$ $\frac{t^{2}}{2 \beta}$. By Corollary 5.49 , we show that

$$
\mu\left(g-\mathbb{E}_{\mu} g \geq t\right) \leq e^{-t^{2} / 2 \beta}=: \alpha(t)
$$

It remains to recall (by Proposition 5.36) that $\alpha_{\mu}(t) \leq \alpha(t / 2)$, and to conclude that $\alpha_{\mu}(t) \leq e^{-t^{2} / 8 \beta}$ as desired.

### 5.13 Small ball estimates for norms on the sphere by Klartag and Vershynin

Recall that for a non-negative random variable $\xi$, a "small-ball" type estimate is an inequality similar to

$$
P(\xi \leq \epsilon) \leq \delta(\epsilon)
$$

where ideally we would like $\delta(\epsilon) \rightarrow 0$, the faster the better.


More generally, for a random vector $X$ in $\mathbb{R}^{n}$ and $z \in \mathbb{R}^{n}$, one could try and get an upper estimate for $P(|X-z| \leq \epsilon)$ independent of $z$. This would mean that the random vector $X$ rarely falls into the small ball of radius $\epsilon$ centered at any given $z$ (hence the name).

We learned from Corollary 5.24 that for any semi-norm $f$ on $\mathbb{R}^{n}$, one has sub-exponential concentration on the sphere $\mathbb{S}^{n-1}$ :

$$
P(f(\theta) \geq \mathbb{E} f+t) \leq C e^{-\frac{c t}{\mathbb{E} f}}
$$

Here the expectation is with respect to Haar measure on the sphere. One could of course get the reverse bound as well:

$$
P(f(\theta) \leq \mathbb{E} f-t) \leq C e^{-\frac{c t}{\mathbb{E} f}} .
$$

However, the above inequality does not pack in a lot of information, since the bound on the right is just of some constant order, potentially larger than 1.

This begs a natural question: is there a general small-ball type estimate for norms on the sphere? That is, could one get a bound of the type

$$
P(f(\theta) \leq \epsilon \mathbb{E} f) \leq \delta(\epsilon)
$$

when $f$ is a (semi-)norm and $\theta$ is uniformly distributed on the sphere? This question was answer by Klartag and Vershynin [110] (see also later works by Paouris, Tikhomirov, Valettas [154]). In this subsection, we shall outline this result.

We start with the following standard notation introduced by Milman:
Definition 5.50. Consider a norm $\|\cdot\|$ on $\mathbb{R}^{n}$. Let $M>0$ be its average over the sphere, so that

$$
M=\int_{\mathbb{S}^{n}-1}\|\theta\| d \sigma(\theta)=: \mathbb{E}_{\mathbb{S}^{n-1}}\|\theta\|
$$

Definition 5.51. One sets $d(\|\cdot\|)$ to be a "special dimension" defined by

$$
d(\|\cdot\|)=\min \left\{M,-\log \sigma\left(\|\theta\| \leq \frac{1}{2} M\right)\right\}
$$

For example, for the Euclidean norm $\|\cdot\|=|\cdot|$ one may see that $d=n$. As a general rule, one has $d(\|\cdot\|) \rightarrow \infty$ as $n \rightarrow \infty$ for various reasonable norms (see [110]).

We are ready to state the main Theorem of this subsection:
Theorem 5.52 (Klartag-Vershynin [110]). For any norm \| $\cdot \|$ one has

$$
\sigma\left(\theta \in \mathbb{S}^{n-1}:\|\theta\| \leq \epsilon M\right) \leq(c \epsilon)^{c \cdot d(\|\cdot\|)}
$$

for all $\epsilon \in(0,1)$. In particular, one can bound from below the negative moment of a norm:

$$
\left(\int_{\mathbb{S}^{n}-1}\|\theta\|^{-\ell} d \sigma(\theta)\right)^{1 / \ell} \geq c M \quad(\forall \ell \in[0, c \cdot d(\|\cdot\|)])
$$

Remark 5.53. Note that by Hölder's inequality one has

$$
\left(\int_{\mathbb{S}^{n-1}}\|\theta\|^{-\ell} d \sigma(\theta)\right)^{1 / \ell} \leq M
$$

and therefore the inequality in the Theorem above is nearly optimal.
The following Lemma is needed for the proof of this Theorem:
Lemma 5.54. Suppose $K$ is a centrally-symmetric convex body. We can then compare the spherical measure of $K$ with the Gaussian measure of $K$ via the inequality

$$
\frac{1}{2} \sigma\left(\mathbb{S}^{n-1} \cap \frac{1}{2} K\right) \leq \gamma(\sqrt{n} K) \leq \sigma\left(\mathbb{S}^{n-1} \cap 2 K\right)+e^{-c n}
$$

where $c$ is some absolute constant.

Proof. There are two facts used in the proof. First, one has $\gamma\left(2 \sqrt{n} B_{2}^{n}\right) \geq \frac{1}{2}$. Indeed, if $X$ is a random vector then $P(|X|>2 \sqrt{n})=P\left(|X|^{2}>4 n\right) \leq \frac{\mathbb{E}_{\gamma|X|^{2}}}{4 n}$. The LHS is $1-\gamma\left(2 \sqrt{n} B_{2}^{n}\right)$ and the RHS is precisely $\frac{1}{4}$. The other fact needed is that $\gamma\left(\frac{1}{2} \sqrt{n} B_{2}^{n}\right) \leq e^{-c n}$, which follows from writing the LHS in polar coordinates and using the Laplace method.

Next, we can bound from below by

$$
\begin{aligned}
\gamma(\sqrt{n} K) & \geq \gamma\left(2 \sqrt{n} B_{2}^{N} \cap \sqrt{n} K\right) \\
& \geq \gamma\left(2 \sqrt{n} B_{2}^{n}\right) \cdot \tilde{\sigma}\left(2 \sqrt{n} \mathbb{S}^{n-1} \cap \sqrt{n} K\right)
\end{aligned}
$$

Here $\tilde{\sigma}$ is the Haar measure on $2 \sqrt{n} \mathbb{S}^{n-1}$. Recall that $\gamma\left(2 \sqrt{n} B_{2}^{n}\right) \geq \frac{1}{2}$ and hence after renormalizing again,

$$
\gamma(\sqrt{n} K) \geq \frac{1}{2} \sigma\left(\mathbb{S}^{n-1} \cap \frac{1}{2} K\right) .
$$

For the upper bound, we remark that

$$
\gamma(\sqrt{n} K) \leq \sigma\left(\frac{1}{2} \sqrt{n} B_{2}^{n}\right)+\sigma\left(\frac{1}{2} \sqrt{n} \mathbb{S}^{n-1} \cap \sqrt{n} K\right)
$$

so that then

$$
\gamma(\sqrt{n} K) \leq e^{-c n}+\sigma\left(\mathbb{S}^{n-1} \cap 2 K\right)
$$

Remark 5.55. In fact, for the Lemma to hold it suffices to assume that $K$ is star-shaped.
Proof of the Theorem. Take $\|\cdot\|$ with unit ball $K$. One has $\gamma\left(e^{t} K\right)$ being log-concave by Cordero-Erasquin, Fredeliki, Mauren, et al. Then

$$
\gamma\left(a^{\lambda} b^{1-\lambda} K\right) \geq \gamma(a K)^{\lambda} \gamma(b K)^{1-\lambda} \quad \forall a, b>0
$$

Consider $\operatorname{Med}(\|\cdot\|)$ to be the median of the norm, so that Med $\leq 2 \cdot M$ (this is apparently a well-known fact). Set $L:=\operatorname{Med} \cdot \sqrt{n} \cdot K$ to be a symmetric convex body, then we have

$$
\begin{aligned}
\gamma(2 L) & \geq \frac{1}{2} \sigma\left(\mathbb{S}^{n-1} \cap \mathrm{Med} \cdot K\right) \\
& \geq \frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}
\end{aligned}
$$

(by definition of median); we can also estimate $\gamma(L / 8)$ from above so that

$$
\begin{aligned}
\gamma(L / 8) & \leq \sigma\left(\mathbb{S}^{n-1} \cap \frac{1}{4} \cdot \mathrm{Med} \cdot K\right)+e^{-c n} \\
& =\sigma\left(x \in \mathbb{S}^{n-1}:\|x\| \leq \frac{1}{4} \mathrm{Med}\right)+e^{-c n} \\
& \leq \sigma\left(x \in \mathbb{S}^{n-1}:\|x\| \leq \frac{1}{2} M\right)+e^{-c n}
\end{aligned}
$$

by our previous point. By definition of $d$, we then have this begin no greater than $e^{-d(\|\cdot\|)}+$ $e^{-c n}$, which in turn is bounded above by $2 e^{-c d(\|\cdot\|)}$ since $d \leq n$. Applying the B-theorem for $a=\epsilon b=2, \lambda=\frac{2}{\log (1 / \epsilon)}$ we then get

$$
\begin{aligned}
\gamma(\epsilon L)^{3 / \log (1 / \epsilon)} \cdot \gamma(2 L)^{1-\frac{3}{\log (1 / \epsilon)}} & \leq \gamma\left(\epsilon^{3 / \log (1 / \epsilon)} \cdot 2^{1-\frac{3}{\log (1 / \epsilon)}} L\right) \\
& \leq \gamma(L / 8)
\end{aligned}
$$

for sufficiently small $\epsilon$. Then

$$
\begin{aligned}
\gamma(\epsilon L)^{3 / \log (1 / \epsilon)} & \leq \frac{\gamma(L / 8)}{\gamma(2 L)^{1-e / \log (1 / \epsilon)}} \\
& \leq\left(c^{\prime} \epsilon\right)^{c \cdot d(\|\cdot\|)}
\end{aligned}
$$

for another constant $c^{\prime}$. Finally, if we want to estimate the spherical measure of the set, we observe that

$$
\begin{aligned}
\sigma(\|x\| \leq \epsilon M) & \leq \gamma(\epsilon L) \\
& \leq\left(c^{\prime} \epsilon\right)^{c d(\|\cdot\|)} .
\end{aligned}
$$

Thus

$$
\int_{\mathbb{S}^{n-1}}\left\|\frac{\theta}{M}\right\|^{-c d / 10} d \sigma \leq C
$$

which completes the proof of the Theorem.

### 5.14 Home work

Question 5.56 (3 points). Recall that a spherical cap is a non-empty set of the form

$$
\mathbb{S}^{n-1} \cap\{\langle x, v\rangle \geq t\},
$$

for some $v \in \mathbb{R}^{n}$ and $t \in \mathbb{R}$.
Prove the isoperimetric inequality on the sphere $\mathbb{S}^{n-1}:$ given $A \subset \mathbb{S}^{n-1}$ with $\sigma(A)=\alpha \in$ $(0,1)$ (where $\sigma$ is the Haar measure on the sphere), prove that the perimeter of $A$ (which we defined in class) is larger that that of a spherical cap of measure $\alpha$.

Hint: use an analogue of Steiner symmetrizations, for example, or some other approach.
Question 5.57 (1 point). Using the Question 5.56, deduce the sharp concentration inequality on the sphere (which we stated in class).

Hint: use an approach similar to how we deduce the Gaussian sharp concentration from the Gaussian isoperimetry (we will do it in a few weeks).

Question 5.58 (Rahul's question, 2 points). Could you prove a concentration result on the sphere of the type

$$
\sigma\left(A_{t}\right) \geq C_{1} e^{-c_{2} n^{2} t^{4}}
$$

for some range of $t$ and some constants? Use the same ideas as what we discussed in class.
Question 5.59 (2 points). a) Prove the Efron-Stein inequality: for any measurable function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and any random vector $X=\left(X_{1}, \ldots, X_{n}\right)$, one has

$$
\operatorname{Var} f(X) \leq \mathbb{E} \sum_{i=1}^{n} \operatorname{Var}_{i} f(X),
$$

where Var stands for the variance with respect to the distribution of $X, \mathbb{E}$ stands for the expectation with respect to the distribution of $X$, and $V^{2} r_{i}$ is the variance with respect to $X_{i}$ (so $\operatorname{Var}_{i} f(X)$ is a random variable.)
b) Prove the tensorisation property of the Poincare inequality: let $\mu_{1}, \ldots, \mu_{m}$ be a collection of measures on $\mathbb{R}^{k_{1}}, \ldots, \mathbb{R}^{k_{m}}$ respectively, so that $k_{1}+\ldots+k_{m}=n$. Let the measure $\mu=$ $\mu_{1} \times \ldots \times \mu_{m}$ on $\mathbb{R}^{n}$. Then the Poincaré constant of $\mu$ equals the maximum of the Poincaré constants of $\mu_{1}, \ldots, \mu_{m}$.

Question 5.60 (1 point). a) Recall that for a random vector $X$ distributed according to the measure $\mu$ and a function $f$, we denote $\operatorname{Ent} f(X)=\int f(x) \log f(x) d \mu(x)-\int f d \mu$. $\log \left(\int f d \mu\right)$. Prove that for any measurable function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and any random vector $X=\left(X_{1}, \ldots, X_{n}\right)$, one has

$$
E n t f(X) \leq \mathbb{E} \sum_{i=1}^{n} E n t_{i} f(X)
$$

where Ent stands for the entropy with respect to the distribution of $X, \mathbb{E}$ stands for the expectation with respect to the distribution of $X$, and Ent ${ }_{i}$ is the entropy with respect to $X_{i}$ (so $E n t_{i} f(X)$ is a random variable.)

Hint: use the variational characterization of entropy that we proved in class.
b) Recall that we say that a measure $\mu$ on $\mathbb{R}^{n}$ satisfies the Log-Sobolev inequality with constant $\beta$ if for any locally Lipschitz function $f$ one has $\operatorname{Entf}^{2}(X) \leq 2 \beta \mathbb{E}|\nabla f(X)|^{2}$, and $\beta>0$ is the smallest number that works here.

Prove the tensorisation property of the Log-Sobolev inequality: let $\mu_{1}, \ldots, \mu_{m}$ be a collection of measures on $\mathbb{R}^{k_{1}}, \ldots, \mathbb{R}^{k_{m}}$ respectively, so that $k_{1}+\ldots+k_{m}=n$. Let the measure $\mu=$ $\mu_{1} \times \ldots \times \mu_{m}$ on $\mathbb{R}^{n}$. Then the Log-Sobolev constant of $\mu$ equals the maximum of the LogSobolev constants of $\mu_{1}, \ldots, \mu_{m}$.

Question 5.61 (5 points). (Intentionally vague question, allowing for some freedom). Find any interesting extension or generalization of the Herbst argument in the situation of, say, Generalized Log-Sobolev inequality, or in some other more general situation.

Question 5.62 (1 point). Confirm that if $\psi$ is a convex function on $\left(\mathbb{R}^{n}\right)^{+}$then the function $\psi\left(x_{1}^{2}, \ldots, x_{n}^{2}\right)$ is convex as well. (this concludes the proof of the Thin Shell inequality in the unconditional case, as we discussed in class).

Question 5.63 (1 point). Confirm that for any Borel probability measure $\mu$ on a metric space $X$, one has $\alpha_{\mu}(t) \rightarrow_{t \rightarrow \infty} 0$.

Question 5.64 (1 point). Confirm that if the function $f$ on $\mathbb{R}^{n}$ is $p$-Lipschitz (that is, $|f(x)-f(y)| \leq p|x-y|$ for all $\left.x, y \in \mathbb{R}^{n}\right)$ then one has $|\nabla f| \leq p$.

Question 5.65 (3 points). a) For a log-concave probability measure $\mu$ with density $f$ on $\mathbb{R}^{n}$, define $b_{p}: \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ via

$$
b_{p}(\theta)=\int_{0}^{\infty} f(t \theta) t^{p} d t
$$

Prove that $B_{p}(x)=|x| b_{p}\left(\frac{x}{|x|}\right)$ is convex for $p \geq 1$, and therefore is a Minkowski functional on $\mathbb{R}^{n}$ of some convex body $K_{\mu}$. This body is called Ball's body as it was introduced by Keith Ball.
b) Suppose $\mu$ is isotropic. For which $p$ is $K_{\mu}$ isotropic (after being normalized to have volume one)?
c) Prove that verifying the thin shell conjecture for all log-concave measures is equivalent to verifying it only for uniform measures on convex bodies.
d) Prove that verifying the KLS conjecture for all log-concave measures is equivalent to verifying it only for uniform measures on convex bodies.

Question 5.66 (3 points). Confirm that the KLS conjecture is equivalent to showing that for any isotropic convex body $K$ in $\mathbb{R}^{n}$, cutting $K$ into two parts of equal volume is achieved, up to a multiple of an absolute constant, by an affine hyperplane cut. (For 1 point out of 3, show that the KLS conjecture implies this fact.)
Hint: Explain that (in some sense) for a (nice) closed connected region $M$ one has

$$
|\partial M|=\int_{\partial M}\left|\nabla 1_{M}\right|
$$

Question 5.67 (1 point). Let $f: S^{n-1} \rightarrow \mathbb{R}$ be a seminorm. Then,

$$
\left(\int_{S^{n-1}}|f(\theta)|^{q} d \sigma(\theta)\right)^{1 / q} \leq \frac{c q}{p} \sqrt{\frac{n+p}{n+q}}\left(\int_{S^{n-1}}|f(\theta)|^{p} d \sigma(\theta)\right)^{1 / p}
$$

Hint: use the reverse Hölder inequality that we proved and integration by parts.
Question 5.68 (1 point). Let $\mu$ be any even log-concave measure on $\mathbb{R}^{n}$. Show that for any symmetric measurable set $A$ in $\mathbb{R}^{n}$ and any $t>0$ one has

$$
\mu\left(x \in \mathbb{R}^{n}: \exists y \in A:\langle x, y\rangle>-t\right) \geq 1-\frac{1}{\mu(A)} e^{-t}
$$

Hint: Use Klartag's extension of the functional Blaschke-Santalo inequality.

Question 5.69 (2 points). Let $\mu$ be a measure on a metric space $X$ with Laplace functional $E_{\mu}(\lambda)$. Suppose the diameter of $X$ is bounded from above by $D<\infty$. Then for any $\lambda \geq 0$ one has

$$
E_{\mu}(\lambda) \leq e^{\frac{D^{2} \lambda^{2}}{2}}
$$

Question 5.70 (1 point). Show that whenever for a measure $\mu$ one has for all $\lambda \geq 0$ that $E_{\mu} \leq e^{\frac{\lambda^{2}}{2 c}}$ for some constant $c>0$ then for every $t>0$ one has $\alpha_{\mu}(t) \leq e^{-\frac{c t^{2}}{8}}$.

Hint: use the result we proved in class.
Remark. This shows that using Payne-Weinberger inequality as means of obtaining concentration bounds is sub-optimal.

Question 5.71 (1 point). Prove the (close to optimal) sub-Gaussian concentration bound for the discrete cube $\{-1,1\}^{n}$ equipped with the uniform measure and the Hamming distance (as defined in class): show

$$
\alpha_{\mu}(t) \leq e^{-\frac{t^{2} n}{8}} .
$$

Hint: use questions 5.69 and 5.70.
Question 5.72 (2 points). Show that Paouris's inequality (that we stated in class) follows from the following result of Guedon and Milman: for an isotropic log-concave random vector $X$ on $\mathbb{R}^{n}$ and any $p \in \mathbb{R}$ such that $1 \leq|p-2| \leq c_{1} n^{\frac{1}{6}}$ one has

$$
1-C \frac{|p-2|}{n^{\frac{1}{3}}} \leq n^{-\frac{1}{2}}\left(\mathbb{E}|X|^{p}\right)^{\frac{1}{p}} \leq 1+C \frac{|p-2|}{n^{\frac{1}{3}}}
$$

and for any $p \in \mathbb{R}$ such that $c_{1} n^{\frac{1}{6}} \leq|p-2| \leq c_{2} n^{\frac{1}{2}}$ one has

$$
1-C \frac{|p-2|^{\frac{1}{2}}}{n^{\frac{1}{4}}} \leq n^{-\frac{1}{2}}\left(\mathbb{E}|X|^{p}\right)^{\frac{1}{p}} \leq 1+C \frac{|p-2|^{\frac{1}{2}}}{n^{\frac{1}{4}}}
$$

Remark. Note that this estimates include negative values of $p$, unlike the reverse Hölder inequality that we proved in class.

Question 5.73 (1 point). Find an example of a non-Lipschitz function $f$ on the sphere $\mathbb{S}^{n-1}$ which violates the concentration around the median inequality.

Question 5.74 (1 point). Confirm the following fact: suppose for all Lipschitz functions on the space $(X, d, \mu)$ one has

$$
\mu(f \geq \mathbb{E} f+t) \leq \alpha(t)
$$

for some function $\alpha$ on $\mathbb{R}^{+}$. Then $\alpha_{\mu}(t) \leq \alpha\left(\frac{t}{2}\right)$ (where as usual $\alpha_{\mu}$ denotes the concentration function).

Question 5.75 (2 points). - Show that if the diameter of the metric space $(X, d)$ is bounded by $R>0$ then for any probability measure $\mu$ on $X$ one has $E_{\mu}(\lambda) \leq e^{c D^{2} \lambda^{2}}$. Here $E_{\mu}$ stands for the Laplace functional, as defined in class.

- Deduce (using also a result we proved in class) the nearly sharp sub-Gaussian concentration on the Hamming cube.

Question 5.76 (2 points). Prove that $d\left(\|\cdot\|_{\infty}\right) \rightarrow \infty$ when the dimension tends to infinity. Here $d(\|\cdot\|)$ is the Klartag-Vershynin dimension of a norm, as defined in class.

Question 5.77 (1 point). Estimate from below the Poincaré constant of the domain consisting of two unit balls in $\mathbb{R}^{n}$ connected with a neck of width $\epsilon$.

Question 5.78 (1 point). Prove that in dimension 1, for any log-concave measure $\mu$ on $\mathbb{R}$, the isoperimetric sets are rays.

## 6 Gaussian Measure and its special properties

### 6.1 A general discussion

Recall that the Gaussian Measure on $\mathbb{R}^{n}$ is given by

$$
d \gamma(x)=\frac{1}{\sqrt{2 \pi}^{n}} e^{\frac{-|x|^{2}}{2}} d x .
$$



We have already seen that it has many wonderful properties, including:

- It is a Log-concave isotropic probability measure; to check the isotropicity, note

$$
\int\langle x, \theta\rangle^{2} d \gamma=\int x_{i}^{2} d \gamma=1
$$

- It is the only measure both product and rotation invariant
- Linear images of Gaussian random vectors are determined by their Covariance matrix
- Gaussian measure plays the main role in the Central Limit Theorem (and is preserved by convolutions)
- It is extremal for Log-Sobolev inequality
- It is extremal for Reverse Log-Sobolev inequality
- It corresponds to the equality case in the functional Blaschke-Santaló inequality
- It is extremal for the Entropy Power Inequality
- There is a nice "Gaussian Fourier system" called Hermite polynomials
- It satisfies the Poincaré inequality with constant 1
- It satisfies the B-theorem and an improved Poincaré inequality for symmetric functions with constant $\frac{1}{2} \ldots$

It is worthwhile mentioning also:
Theorem 6.1 (Gaussian Correlation Inequality, Royen [159]). If $A, B$ are symmetric convex sets in $\mathbb{R}^{n}$, then

$$
\gamma(A \cap B) \geq \gamma(A) \cdot \gamma(B)
$$

In adition to Royen [159], see also the exposition by Latala, Matlak [122] in regards to the above breakthrough result.

The following Proposition is a way to quantify that the Gaussian measure is " $a$ role model" for all isotropic probability measures (and especially for log-concave ones). Recall $C_{p}(\mu)$, the Poincare constant associated with a measure $\mu$, is the smallest number such that for all $f$ :

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq C_{p}(\mu) \int|\nabla f|^{2} d \mu
$$

Proposition 6.2. Suppose $\mu$ is an isotropic probability measure. Then

$$
C_{p}(\mu) \geq 1=C_{p}(\gamma)
$$

Proof. Recall the fact that $\mu$ is isotropic implies that $\int x d \mu=0$, i.e. $\forall i \int x_{i} d \mu=0$. Also $\forall \theta \in S^{n-1}, \int\langle x, \theta\rangle^{2} d \mu=0$, and in particular

$$
\int x_{i}^{2} d \mu=1
$$

So,

$$
1=\int x_{i}^{2} d \mu-\left(\int x_{i} d \mu\right)^{2} \leq C_{p}(\mu) \int 1 d \mu=C_{p}(\mu)
$$

In this section, we shall see several very strong isoperimetric-properties and phenomena which are unique to the Gaussian measure.

### 6.2 The isoperimetric profile

Recall the Isoperimetric problem for general (probability) measures $\mu$. The objective of this problem is to find

$$
\inf _{\mu(A)=a} \mu^{+}(\partial A)
$$

for a given $a \in[0,1]$, where the weighted perimeter is defined as

$$
\mu^{+}(\partial A)=\liminf _{\varepsilon \rightarrow 0} \frac{\mu\left(A+\varepsilon B_{2}^{n} \backslash A\right)}{\varepsilon}
$$

Definition 6.3. The isoperimetric profile of $\mu$ is defined as

$$
I_{\mu}(a)=\inf _{\mu(A)=a} \mu^{+}(\partial A)
$$

Below are some properties of $I_{\mu}(a)$.

- For non-atomic measures, $I_{\mu}(a) \geq 0, I_{\mu}(a) \rightarrow 0$ as $a \rightarrow 0$, and $I_{\mu}(a) \rightarrow 0$ as $a \rightarrow 1$
- $I_{\mu}\left(a-\frac{1}{2}\right)$ is even. One can see this by taking complements, i.e. $\mu(A)=1-\mu\left(A^{c}\right)$ but they have the same perimeters $\mu^{+}(\partial A)=\mu^{+}\left(\partial A^{c}\right)$.
- $I_{\mu}$ is convex for log-concave measures (proved by E. Milman [144])


Remark 6.4. Consider for example the Lebesgue measure $|A|=a$, then $|\partial A| \geq c_{n} \cdot a^{1 / n}$, and we have concavity. Recall that the proof of this followed from the Brunn-Minkowski inequality.

So to obtain an isoperimetry bound for a general log-concave measure, one may try to use the Prekopa-Leindler inequality.

$$
\begin{aligned}
\mu^{+}(\partial K) & =\liminf _{\varepsilon \rightarrow 0} \frac{\mu\left(K+\varepsilon B_{2}^{n}\right)-\mu(K)}{\varepsilon} \\
& =\liminf _{\varepsilon \rightarrow 0} \frac{\mu\left((1-t) \frac{K}{1-t}+t \frac{\varepsilon B_{2}^{n}}{t}\right)-\mu(K)}{\varepsilon} \\
& \geq \sup _{t} \liminf _{\varepsilon \rightarrow 0} \frac{\mu\left(\frac{K}{1-t}\right)^{1-t} \mu\left(\frac{\varepsilon B_{2}^{n}}{t}\right)^{t}-\mu(K)}{\varepsilon} .
\end{aligned}
$$

However, one cannot hope to get a sharp bound because the Prekopa-Leindler inequality is never tight!

### 6.3 The Ehrard Inequality

The Ehrard inequality is a fancier and tighter Gaussian version of the Prekopa-Leindler inequality.

Definition 6.5. The Gaussian cumulative distribution function (c.d.f.) is denoted as

$$
\Phi(s)=\int_{-\infty}^{s} \frac{1}{\sqrt{2 \pi}} e^{-t^{2} / 2} d t=\gamma_{1}(-\infty, s)
$$



Note that

$$
\Phi^{\prime}(s)=\frac{1}{\sqrt{2 \pi}} e^{-s^{2} / 2} .
$$

We will be considering the inverse $\Phi^{-1}:[0,1] \rightarrow \mathbb{R}$.


We are ready to state arguably the most important Gaussian Isoperiemtric-type inequality, proved by Ehrhard for convex sets [56], [57] then by Latala when one of the sets is convex [120], and finally by Borell [33], [32] for arbitrary Borel-measurable sets:

Theorem 6.6 (Ehrard-Borell). For all Borel measurable sets $A, B \subset \mathbb{R}^{n}, \forall \lambda \in(0,1)$,

$$
\Phi^{-1}(\gamma(\lambda A+(1-\lambda) B)) \geq \lambda \Phi^{-1}(\gamma(A))+(1-\lambda) \Phi^{-1}(\gamma(B))
$$

Other proofs are given by Neeman and Paouris [150], van Handel [80] and Ivanisvili [87], among others.

Let us compare this to Prekopa-Leindler. Gaussian measure is log-concave, which means that

$$
\log \gamma(\lambda A+(1-\lambda) B) \geq \lambda \log (\gamma(A))+(1-\lambda) \log (\gamma(B))
$$

Consider the function

$$
m(t)=\gamma((1-t) A+t B)
$$

Ehrard's inequality says that $\Phi^{-1} \circ m$ is concave, while Prekopa-Leindler says that $\log m$ is concave.

Claim 6.7. Ehrhard's inequality is stronger than the Prekopa-Leindler inequality for the Gaussian measure: for any strictly increasing function $m \Phi^{-1} \circ m$ is concave implies that $\log m$ is concave

Proof. We consider the local form of the functions. Notice that $\Phi^{-1} \circ m$ is equivalent to the fact that

$$
\left(\Phi^{-1} \circ m\right)^{\prime \prime} \leq 0,
$$

and $\log \circ m$ is concave is equivalent to the fact that

$$
(\log \circ m)^{\prime \prime} \leq 0
$$

In general, we have

$$
(f \circ m)^{\prime \prime}=f^{\prime \prime} \cdot\left(m^{\prime}\right)^{2}+f^{\prime} m^{\prime \prime} \leq 0,
$$

which is true if and only if

$$
\frac{m^{\prime \prime}}{\left(m^{\prime}\right)^{2}} \leq-\frac{f^{\prime \prime}}{f^{\prime}}
$$

for $f^{\prime}>0$. So relating this back to Ehrard, we have

$$
\frac{m^{\prime \prime}}{\left(m^{\prime}\right)^{2}} \leq-\frac{\left(\Phi^{-1}\right)^{\prime \prime}}{\left(\Phi^{-1}\right)^{\prime}}
$$

and for Prekopa-Leindler, we have

$$
\frac{m^{\prime \prime}}{\left(m^{\prime}\right)^{2}} \leq-\frac{(\log )^{\prime \prime}}{(\log )^{\prime}}
$$

To prove that Ehrard is stronger than Prekopa-Leindler, we prove the following key claim

$$
\begin{equation*}
\forall t:-\frac{\left(\Phi^{-1}(t)\right)^{\prime \prime}}{\left(\Phi^{-1}(t)\right)^{\prime}} \leq-\frac{(\log (t))^{\prime \prime}}{(\log (t))^{\prime}} \tag{21}
\end{equation*}
$$

We compute each component of this inequality

$$
\begin{gathered}
\frac{(\log (t))^{\prime \prime}}{(\log (t))^{\prime}}=-\frac{1 / t^{2}}{t / t}=-\frac{1}{t} \\
\left(\Phi^{-1}(t)\right)^{\prime}=\frac{1}{\Phi^{\prime}\left(\Phi^{-1}(t)\right)}=\sqrt{2 \pi} e^{\Phi^{-1}(t)^{2} / 2} \\
\left(\Phi^{-1}(t)\right)^{\prime \prime}=2 \pi \Phi^{-1}(t) \cdot e^{\Phi^{-1}(t)^{2} / 2} \\
\Longrightarrow \frac{\left(\Phi^{-1}(t)\right)^{\prime \prime}}{\left(\Phi^{-1}(t)\right)^{\prime}}=\sqrt{2 \pi} \Phi^{-1}(t) e^{\Phi^{-1}(t)^{2} / 2} .
\end{gathered}
$$

The fact that 21 implies the overall claim is equivalent to saying that

$$
\begin{equation*}
-\sqrt{2 \pi} \Phi^{-1}(t) e^{\Phi^{-1}(t)^{2} / 2} \geq \frac{1}{t} . \tag{22}
\end{equation*}
$$

Why is 22 true? Indeed, if $a=\Phi^{-1}(t)$, then 22 becomes

$$
-\frac{1}{\Phi(a)} \leq \sqrt{2 \pi} a e^{a^{2} / 2}
$$

If $a \geq 0$, the inequality is trivially true. If $a \leq 0$, we have to show that

$$
\int_{-\infty}^{a} e^{-t^{2} / 2} d t \leq-\frac{1}{a} e^{-a^{2} / 2}
$$

Change of variables $b=-a$ gives us

$$
\int_{b}^{\infty} e^{-t^{2} / 2} d t \leq \frac{1}{b} e^{-b^{2} / 2}
$$

when $b \geq 0$. To see why the above inequality is true, notice that

$$
\begin{aligned}
\int_{b}^{\infty} e^{-t^{2} / 2} d t & =\int_{b}^{\infty} t \cdot \frac{1}{t} e^{-t^{2} / 2} d t \\
& \leq \frac{1}{b} \int_{b}^{\infty} t e^{-t^{2} / 2} d t \\
& =\frac{1}{b} e^{-b^{2} / 2}
\end{aligned}
$$

So the claim is proved.

Remark 6.8. What is the geometric meaning of $\Phi^{-1}(a)$, for $a \in[0,1]$. Consider the half space

$$
H=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \alpha\right\} .
$$

Then $\gamma(H)=a$ means that $\Phi^{-1}(a)=\alpha$. Indeed,

$$
\Phi(\alpha)=\gamma_{1}(-\infty, \alpha)=\gamma_{n}(H)=a
$$



Compare this to $f(t)=t^{1 / n}$ - the (multiple of) the radius of the ball of Lebesgue volume $t$ (see the below remark).

Remark 6.9. van Handel, Shenfeld [81] fully characterized the equality cases in Ehrhard's inequality.

In particular, the equality in the Ehrhard inequality is attained when $A, B$ are parallel half-spaces. Indeed,

$$
\begin{aligned}
& A=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \Phi^{-1}(a)\right\}, \gamma(A)=a \\
& B=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \Phi^{-1}(b)\right\}, \gamma(B)=b
\end{aligned}
$$

Then

$$
\frac{A+B}{2}=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \frac{\Phi^{-1}(a)+\Phi^{-1}(b)}{2}\right\}
$$

So

$$
\Phi^{-1}\left(\gamma\left(\frac{A+B}{2}\right)\right)=\frac{\Phi^{-1}(\gamma(A))+\Phi^{-1}(\gamma(B))}{2}=\frac{\Phi^{-1}(a)+\Phi^{-1}(b)}{2}
$$

the point where $\frac{A+B}{2}$ intersects the $x$-axis.

### 6.4 Gaussian isoperimetric inequality

Ehrhard's inequality implies the following classical and important result:
Theorem 6.10 (The Gaussian Isoperimetric inequality, Sudokov-Tsirelson [165], Borell [32]). If $A$ is a Borel measurable in $\mathbb{R}^{n}$ with $\gamma(A)=a \in[0,1]$, then

$$
\gamma^{+}(\partial A) \geq \gamma^{+}\left(\partial H_{a}\right)=\frac{1}{\sqrt{2 \pi}} e^{-\Phi^{-1}(a)^{2} / 2}
$$

where $H_{a}$ is the halfspace of measure a

$$
H_{a}=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \Phi^{-1}(a)\right\} .
$$



In other words,

$$
I_{\gamma}(a)=\frac{1}{\sqrt{2 \pi}} e^{-\Phi^{-1}(a)^{2} / 2}
$$

Note that this implies that

$$
I_{\gamma}(a)=\frac{1}{\Phi^{-1}(a)^{\prime}},
$$

and that

$$
I_{\gamma}(a) \cdot I_{\gamma}^{\prime \prime}(a)=-1
$$

Proof. (of the Gaussian isoperiemtry via Ehrhard) Let $K$ be some Borel set in $\mathbb{R}^{n}$. Then

$$
\begin{aligned}
\gamma^{+}(\partial K) & =\liminf _{\varepsilon \rightarrow 0} \frac{\gamma\left(K+\varepsilon B_{2}^{n}\right)-\gamma(K)}{\varepsilon} \\
& =\sup _{\lambda>0} \liminf _{\varepsilon \rightarrow 0} \frac{\gamma\left((1-\lambda) \frac{K}{1-\lambda}+\lambda \frac{\varepsilon B_{2}^{n}}{\lambda}\right)-\gamma(K)}{\varepsilon} \\
& \geq \liminf _{\varepsilon \rightarrow 0, \lambda \rightarrow 0} \frac{\Phi\left((1-\lambda) \Phi^{-1}\left(\frac{K}{1-\lambda}\right)+\lambda \Phi^{-1}\left(\frac{\varepsilon B_{2}^{n}}{\lambda}\right)\right)-\gamma(K)}{\varepsilon}
\end{aligned}
$$

where the last inequality follows from Ehrard's inequality. Now let $t=\frac{\varepsilon}{\lambda}$. $t$ can be anything since $\varepsilon$ and $\lambda$ can tend to 0 at different rates. It turns out the optimal case is taking $t \rightarrow \infty$. Since $\gamma(K)=a$, the last line of the above becomes (by Taylor's Theorem)

$$
\begin{aligned}
& =\lim _{t \rightarrow \infty} \Phi^{\prime}\left(\Phi^{-1}(a)\right) \cdot \frac{\Phi^{-1}\left(\partial\left(t B_{2}^{n}\right)\right)}{t} \\
& =\lim _{t \rightarrow \infty} \Phi^{\prime}\left(\Phi^{-1}(a)\right) \\
& =\frac{1}{\sqrt{2 \pi}} e^{-\Phi^{-1}(a)^{2} / 2} \\
& =\gamma^{+}\left(\partial H_{a}\right) .
\end{aligned}
$$

The fact that $\lim _{t \rightarrow \infty} \frac{\Phi^{-1}\left(\partial\left(t B_{2}^{n}\right)\right)}{t}=1$ is left as homework.
One can then ask the question about an anisotropic version of this. In other words, instead of taking $B_{2}^{n}$, take some set $L$ (this is left as homework).

### 6.5 Gaussian concentration inequality and Borell's noise stability

Theorem 6.11 (Gaussian concentration inequality and Borell's noise stability).

$$
\gamma\left(A_{t}\right) \geq 1-\frac{1}{2} e^{-t^{2} / 2}
$$

if $\gamma(A) \geq 1 / 2$. Moreover, if $H$ is a half-space with $\gamma(A)=\gamma(H)=a \in[0,1]$, we have

$$
\gamma\left(A_{t}\right) \geq \gamma\left(H_{t}\right)=\Phi\left(\Phi^{-1}(a)+t\right)
$$

Proof. Let

$$
h(t)=\Phi^{-1}\left(\gamma\left(A_{t}\right)\right) .
$$

Note that

$$
h^{\prime}(t)=\sqrt{2 \pi} e^{\frac{\Phi^{-1}\left(\gamma\left(A_{t}\right)\right)^{2}}{2}} \cdot \frac{d}{d t} \gamma\left(A_{t}\right) \geq \frac{\gamma^{+}\left(\partial A_{t}\right)}{I_{\gamma}\left(\gamma\left(A_{t}\right)\right)} \geq 1
$$

Above, the second to last inequality follows from the Gaussian isoperimetric inequality, and the last inequality follows from the definition of the isoperimetric profile. We will now apply Newton's formula

$$
\begin{gathered}
h(t)=h(0)+\int_{0}^{t} h^{\prime}(s) d s \geq h(0)+\int_{0}^{t} d s=h(0)+t \\
\Longrightarrow \Phi^{\prime}\left(\gamma\left(A_{t}\right)\right) \geq \Phi^{-1}(\gamma(A))+t
\end{gathered}
$$

which implies

$$
\gamma\left(A_{t}\right) \geq \Phi\left(\Phi^{-1}(\gamma(A))+t\right)=\gamma\left(H_{t}\right)
$$

where $\gamma(A)=\gamma(H)$.
Next, suppose $\gamma(A) \geq 1 / 2$. Then $\Phi^{-1}(\gamma(A)) \geq 0$, and

$$
\gamma\left(A_{t}\right) \geq \Phi\left(\Phi^{-1}(\gamma(A))+t\right) \geq \Phi(t)=\frac{1}{\sqrt{2 \pi}} \int_{-t}^{\infty} e^{-s^{2} / 2} d s
$$

or

$$
1-\gamma\left(A_{t}\right) \leq \frac{1}{\sqrt{2 \pi}} \int_{t}^{\infty} e^{-s^{2} / 2} d s
$$

It now suffices to show

$$
\frac{1}{\sqrt{2 \pi}} \int_{t}^{\infty} e^{-s^{2} / 2} d s \leq \frac{1}{2} e^{-t^{2} / 2}
$$

Consider for $t>0$,

$$
F(t)=\frac{1}{\sqrt{2 \pi}} \int_{t}^{\infty} e^{-s^{2} / 2} d s-\frac{1}{2} e^{-t^{2} / 2}
$$

and thus

$$
F^{\prime}(t)=-\frac{1}{\sqrt{2 \pi}} e^{-t^{2} / 2}-\frac{t}{2} e^{-t^{2} / 2}
$$

and

$$
F^{\prime \prime}(t)=\frac{t}{\sqrt{2 \pi}} e^{-t^{2} / 2}-\frac{1}{2} e^{-t^{2} / 2}+\frac{t^{2}}{2} e^{-t^{2} / 2}
$$

Observe that

$$
F^{\prime}\left(\sqrt{\frac{2}{\pi}}\right)=0
$$

and

$$
F^{\prime \prime}(t) \geq 0 \text { if and only if } t \geq \sqrt{\frac{2}{\pi}}
$$

Note also that

$$
F \rightarrow 0 \text { as } t \rightarrow \infty,
$$

and that $F(0)=0$. This means that $F$ is concave and non-decreasing on $\left[0, \sqrt{\frac{2}{\pi}}\right]$ (and thus is non-negative on that interval), and convex and non-increasing on $\left[\sqrt{\frac{2}{\pi}}, \infty\right]$, and as it also tends to zero at infinity, we conclude that it must remain non-negative. This concludes the proof.

### 6.6 Isoperimetry on the cube

We will now be discussing another application of the Gaussian isoperimetry.
Recall the notion of the isoperimetric profile for a measure $\mu$ :

$$
I_{\mu}(a)=\inf _{\mu(A)=a} \mu^{+}(\partial A)
$$

For specific nice convex bodies $K$, one can ask what is the isopereimetric profile for $d \mu$ uniform on $K$. In other words, we ask the question: if for a measurable set $A \subset \mathbb{R}^{n}$, we have $|K \cap A|=a$, then what is the lower bound for $|K \cap \partial A|$ ? This is known for $K=B_{2}^{n}$ : the optimal $A$ are balls, appropriately positioned (see picture below).

How about $K=[0,1]^{n}$ ? The following seemingly simple question is open.
Conjecture 6.12 (Isoperimetry on the cube). Suppose $A \subset \mathbb{R}^{n}$ is a measurable set such that $\left|[0,1]^{n} \cap A\right|=a \in[0,1]$. Then

$$
\left|\partial A \cap(0,1)^{n}\right| \geq \min \left(\left|\partial C_{k}(a) \cap(0,1)^{n}\right|,\left|\partial C_{k}^{c}(a) \cap(0,1)^{n}\right|\right)
$$

where $C_{k}(a)=r \cdot B_{2}^{k} \times \mathbb{R}^{n-k}+x_{0}$ is a $k$-dimensional round cylinder such that

$$
\left|C_{k}(a) \cap B_{\infty}^{n}\right|=a .
$$



In fact, while the exact isoperimetric problem for the cube is open (except in dimension 2 ), the following lower bound can be achieved:

Theorem 6.13 (Barthe-Maurey [18], Hadwiger). Let $\mu$ be the uniform measure on $[0,1]^{n}$. Then

$$
I_{\mu}(a) \geq \sqrt{2 \pi} I_{\gamma}(a)
$$

In other words, for all measurable sets $A$ in $\mathbb{R}^{n}$, if

$$
\left|[0,1]^{n} \cap A\right|=a \in[0,1]
$$

then

$$
\left|(0,1)^{n} \cap \partial A\right| \geq e^{-\frac{\Phi^{-1}(a)^{2}}{2}}
$$



See also Glaudo [72].
Remark 6.14. Let $\mu, \nu$ be any pair of probability measures such that

$$
T_{\#} \mu=\nu
$$

where $T$ is L-Lipschitz. Then

$$
\alpha_{\nu}(t) \leq \frac{1}{L} \alpha_{\mu}(t)
$$

and

$$
I_{\nu}(a) \geq \frac{1}{L} I_{\mu}(a)
$$

Indeed, to see this, consider $A, \mu(A)=\mu(T A)$. If $T$ is 1-Lipschitz, then $T\left(A_{t}\right) \subset(T A)_{t}$, which implies the inequalities above.

Corollary 6.15. Let $d \mu=e^{-v} d x, \nabla^{2} v \geq k \cdot I d$. Then

$$
\begin{gathered}
\alpha_{\mu}(t) \leq \frac{1}{k} \alpha_{\gamma}(t)=\frac{1}{2 k} e^{-t^{2} / 2} \\
I_{\mu}(a) \geq k \cdot I_{\gamma}(a)=\frac{k}{\sqrt{2 \pi}} e^{-\Phi^{-1}(a)^{2} / 2} .
\end{gathered}
$$

Proof. This corollary follows from Corollary 6.15 and from Cafarelli's theorem 4.50.
Next, we shift our discussion to the uniform measure on the unit cube.
Lemma 6.16. There is a map $\pi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ that is $\frac{1}{\sqrt{2 \pi}}$-Lipschitz such that

$$
\pi_{\#} \gamma=u n i f\left([0,1]^{n}\right)
$$

Proof. Recall (Example 4.41) that $\Phi_{\#} \gamma_{1}=$ unif $[0,1]$, where as usual $\Phi$ stands for the Gaussian cdf:

$$
\int_{0}^{1} f(x) d x=\int_{-\infty}^{\infty} f(\Phi(s)) \cdot \operatorname{jac}(\Phi) d s=\int_{-\infty}^{\infty} f(\Phi(s)) d \gamma(s)
$$

We construct $\pi$ as follows

$$
\left(x_{1}, \ldots, x_{n}\right) \xrightarrow{\pi}\left(\Phi\left(x_{1}\right), \ldots, \Phi\left(x_{n}\right)\right) .
$$

So $\pi_{\#} \gamma=[0,1]^{n}$. Next, to confirm the Lipschitzness, note

$$
\begin{aligned}
(\pi(x)-\pi(y))^{2} & =\sum\left(\Phi\left(x_{i}\right)^{2}-\Phi\left(y_{i}\right)^{2}\right) \\
& \leq \sup \left(\Phi^{\prime}\right)^{2}|x-y|^{2}
\end{aligned}
$$

and recall

$$
\Phi^{\prime}(s)=\frac{1}{\sqrt{2 \pi}} e^{-s^{2} / 2} \leq \frac{1}{\sqrt{2 \pi}}
$$



Picture (c) Klartag

Proof. (of the Barthe-Maurey theorem) The result follows from combining the Lemma 6.16 and Corollary 6.15.

We will now present some more results related to the isoperimetry on the cube conjecture. Corollary 6.17. $A \subset \mathbb{R}^{n},\left|A \cap[0,1]^{n}\right|=\frac{1}{2} \Longrightarrow\left|\partial A \cap(0,1)^{n}\right| \geq 1$, and it is attained for

$$
A=\left\{x_{1} \leq \frac{1}{2}\right\}
$$

In fact, as a corollary we get the following non-trivial result:
Corollary 6.18 (Vaaler [168]). For any $\theta \in S^{n-1}$,

$$
\left|\theta^{\perp} \cap\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}\right|_{n-1} \geq 1
$$

with equality when $\theta=(1,0, \ldots, 0)$.

Remark 6.19. As for the converse estimate, Ball [7] showed that

$$
\left|\theta^{\perp} \cap\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}\right| \leq \sqrt{2}
$$

where the optimal $\theta$ is

$$
\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right) .
$$

### 6.7 Gaussian symmetrization and the proof of the Ehrhard inequality

The following concept was introduced by Ehrhard [56], [57], see also Borell [33] and Bogachev [27].

Definition 6.20 (Gaussian symmetrization). Fix an integer $k, 1 \leq k \leq n$. Fix also $L$, a subspace of $\mathbb{R}^{n}$ with $\operatorname{dim} L=n-k$. Fix any $e \perp L$. Then for a Borel measurable set $A \subset \mathbb{R}^{n}$, consider the Gaussian symmetrization of $A$ denoted by

$$
S(L, e)(A)
$$

such that for all $x \in L$

$$
S(L, e)(A) \cap\left(x+L^{\perp}\right)=\{y:\langle y, e\rangle \geq r\} \cap\left(x+L^{\perp}\right)
$$

where $r=r(x)$ is chosen so that

$$
\gamma_{k}\left(S(L, e)(A) \cap\left(x+L^{\perp}\right)\right)=\gamma_{k}\left(A \cap\left(x+L^{\perp}\right)\right)
$$

Here are some examples:

- If $L=e_{1}^{\perp}$ and $e=e_{1}$, then $S\left(e_{1}^{\perp}, e_{1}\right)(A)=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \Phi_{1}^{-1}(\gamma(A))\right\}$;
- If $k=n-1$, then this corresponds to matching $(n-1)$-dimensional slices of set $A$ to rays $J$ of a 2 -dimensional set.


Some properties of the Gaussian symmetrization are left as homework:
Lemma 6.21. Let $A$ and $B$ be Borel-measurable sets in $\mathbb{R}^{n}$ and $v \in \mathbb{R}^{n}$. Then

- $\gamma(S(L, e)(A))=\gamma(A)$;
- $A \subset B \Longrightarrow S(L, e)(A) \subset S(L, e)(B)$;
- $S(L, e)(A+v)=S(L, e)(A)+v$.

Also,
Lemma 6.22. Let $L_{1}, L_{2}$ be linear subspaces such that $\left(L_{1} \cap L_{2}\right)^{\perp} \cap L_{1}$ and $\left(L_{1} \cap L_{2}\right)^{\perp} \cap L_{2}$ are orthogonal. Then

$$
S\left(L_{1}, e\right) \circ S\left(L_{2}, e\right)=S\left(L_{2}, e\right) \circ S\left(L_{1}, e\right)=S\left(L_{1} \cap L_{2}, e\right)
$$

This implies:
Corollary 6.23. Let $n \geq 3$ and $k \geq 2$. Then for all $k$-symmetrizations $S=S(L, e)$, there is a sequence of 2-symmetrizations $S_{1}, \ldots, S_{k-1}$ such that $S=S_{1} \circ \ldots \circ S_{k-1}$.

Furthermore,
Lemma 6.24. In dimension 2, there exists a sequence $\theta_{1}, \ldots, \theta_{k}, \ldots$ such that

$$
S\left(\theta_{k}^{\perp}, \theta_{k}\right) \circ S\left(\theta_{k-1}^{\perp}, \theta_{k-1}\right) \circ \ldots \circ S\left(\theta_{1}^{\perp}, \theta_{1}\right)(A)
$$

converges in Hausdorff distance to a half-space of the same Gaussian measure as $A$.
Combining Corollary 6.23 and Lemma 6.24 we get
Corollary 6.25. The statement of Lemma 6.24 is true in any dimension.
Remark 6.26. If $A$ is a half-space, then it is invariant under any symmetrizations.
Next, we formulate:
Theorem 6.27. If $A$ is a closed set in $\mathbb{R}^{n}$, then for all $L$ and for all $e \in L^{\perp}$,

$$
S(L, e)(A)+r B_{2}^{n} \subset S(L, e)\left(A+r B_{2}^{n}\right)
$$

Proof. Home work!
Remark 6.28. The previous Theorem implies the Gaussian isoperimetric inequality, when combined with Corolalry 6.25 (without going via Ehrhard's inequality).

Finally, the following result is crucial in our proof of Ehrhard's inequality, and its proof is based on all the results above, and is left as a home work:

Theorem 6.29. If $A$ is an open convex set in $\mathbb{R}^{n}$, then for all $L$ and for all $e \in L^{\perp}$, then $S(L, e)(A)$ is convex.

Proof. Home work!
We are now ready to prove the Ehrhard inequality for convex sets $A$ and $B$. Recall that it states that for any $\lambda \in[0,1]$,

$$
\begin{equation*}
\Phi^{-1}(\gamma(\lambda A+(1-\lambda) B)) \geq \lambda \Phi^{-1}(\gamma(A))+(1-\lambda) \Phi^{-1}(\gamma(B)) \tag{23}
\end{equation*}
$$

The idea is to consider $n$-dimensional convex sets $A$ and $B$ as parallel sections of an $(n+1)$ dimensional convex set, symmetrize it into a 2-dimensional convex set, and the convexity of this set (which follows from Theorem 6.29) is exactly the statement of Ehrhard's inequality (23).

Proof. (of Ehrhard's inequality for convex sets.)
Consider $A, B$ as subsets of $\mathbb{R}^{n+1}$ : let

$$
\begin{aligned}
& \widetilde{A}=A \times\{0\}, \\
& \widetilde{B}=B \times\{0\}
\end{aligned}
$$

and $C=\operatorname{conv}(\widetilde{A}, \widetilde{B})$. Then

$$
C \cap\left\{e_{n+1}^{\perp}+\frac{1}{2} e_{n+1}\right\}=\frac{A+B}{2} \cap\left\{e_{n+1}^{\perp}+\frac{1}{2} e_{n+1}\right\}
$$



Take $n$-symmetrizations in $\mathbb{R}^{n+1}$ of $C$ such that intersections with $n$-dimensional hyperplanes are preserved. Let

$$
C_{\lambda}=e_{n+1}^{\perp} \cap\left(C-\lambda e_{n+1}\right)=e_{n+1}^{\perp} \cap(\lambda A+(1-\lambda) B)
$$

and

$$
f(\lambda)=\Phi^{-1}\left(\gamma\left(C_{\lambda}\right)\right)
$$

Then by definition of symmetrization

$$
\left(\lambda e_{n+1}+e_{n+1}^{\perp}\right) \cap S(C)=\left(e_{n+1}+e_{n+1}^{\perp}\right) \cap\left\{x \in \mathbb{R}^{n}:\langle x, e\rangle \geq r\right\}
$$

where $r=-f(\lambda)$. By Theorem 6.29, the set $S(C)$ is convex, or equivalently $f(\lambda)$ is concave. Therefore,

$$
\Phi^{-1}\left(\gamma\left(C_{\lambda}\right)\right) \text { is convex }
$$

yielding

$$
\Longleftrightarrow \Phi^{-1}(\gamma(\lambda A+(1-\lambda) B)) \geq \lambda \Phi^{-1}(\gamma(A))+(1-\lambda) \Phi^{-1}(\gamma(B))
$$

### 6.8 The Latała's Functional Ehrhard inequality

In this subsection, we present the funcitonal version of the Ehrhard inequality which was observed by Latała [125]

Theorem 6.30 (Functional Ehrhard's inequality, Latała [125]). Let $\lambda \in[0,1]$, and suppose $F, G, H: \mathbb{R}^{n} \rightarrow[0,1]$ are such that for all $x, y \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\Phi^{-1}(H(\lambda x+(1-\lambda) y)) \geq \lambda \Phi^{-1}(F(x))+(1-\lambda) \Phi^{-1}(G(y)) \tag{24}
\end{equation*}
$$

Then

$$
\Phi^{-1}\left(\int_{\mathbb{R}^{n}} H d \gamma\right) \geq \lambda \Phi^{-1}\left(\int_{\mathbb{R}^{n}} F d \gamma\right)+(1-\lambda) \Phi^{-1}\left(\int_{\mathbb{R}^{n}} G d \gamma\right)
$$

Therefore, for convex $f, g$,

$$
\Phi^{-1}\left(\int \Phi\left(-(\lambda f+(1-\lambda) g)^{*} d \gamma\right) \geq \lambda \Phi^{-1}\left(\int \Phi\left(-f^{*}\right) d \gamma\right)+(1-\lambda) \Phi^{-1}\left(\int \Phi\left(-g^{*}\right) d \gamma\right)\right.
$$

In other words, $\Phi^{-1}\left(\int \Phi\left(-(f+t g)^{*} d \gamma\right)\right.$ is concave. Here, as before, $f^{*}$ stands for Legendre transform.

Remark 6.31. As before, one may note that $\left(\lambda f^{*}+(1-\lambda) g^{*}\right)^{*}=f \square_{\lambda} g$ satisfies (24) and this is why one can reformulate it in terms of Legendre transform.

Proof. Consider $A, B \subset \mathbb{R}^{n} \times \mathbb{R}=\mathbb{R}^{n+1}$ given by

$$
\begin{aligned}
A & =\left\{(x, y): y \leq \Phi^{-1}(F(x))\right\} \\
B & =\left\{(x, y): y \leq \Phi^{-1}(G(x))\right\}
\end{aligned}
$$

$A$ and $B$ are subgraphs, and $x \in \mathbb{R}^{n}, y \in \mathbb{R}$. Then the condition of the theorem implies

$$
\begin{equation*}
\lambda A+(1-\lambda) B \subset\left\{(x, y): y \leq \Phi^{-1}(H(x))\right\} \subset \mathbb{R}^{n+1} \tag{25}
\end{equation*}
$$

Ehrhard's inequality in $\mathbb{R}^{n+1}$ implies

$$
\begin{equation*}
\Phi^{-1}(\gamma(\lambda A+(1-\lambda) B)) \geq \lambda \Phi^{-1}(\gamma(A))+(1-\lambda) \Phi^{-1}(\gamma(B)) \tag{26}
\end{equation*}
$$

Then (25) and (26) imply

$$
\begin{gathered}
\Phi^{-1}\left(\gamma\left((x, y): y \leq \Phi^{-1}(H)\right)\right) \geq \\
\lambda \Phi^{-1}\left(\gamma\left((x, y): y \leq \Phi^{-1}(F)\right)\right)+(1-\lambda) \Phi^{-1}\left(\gamma\left((x, y): y \leq \Phi^{-1}(G)\right)\right) .
\end{gathered}
$$

This implies the desired inequality, in view of the fact that by Fubini

$$
\gamma\left((x, y): y \leq \Phi^{-1}(F)\right)=\int F d \gamma
$$

Remark 6.32. Functional Ehrhard also tensorizes (this is left as homework). But the base case of the induction (the 1-dimensional case) is difficult.

### 6.9 Generalized Bobkov's inequality via linearizing functional Ehrhard's inequality

In this subsection we will do the same procedure with Ehrhard's inequality that allowed us to deduce the Generalized Log-Sobolev inequality from the Prekopa-Leindler inequality, following Barthe, Cordero-Erausquin, Ivanisvili, Livshyts [15]. We remark that an alternative procedure which involved linearization of the geometric Ehrhard inequality directly (rather than its functional version) was done by Kolesnikov and Milman [115], and a number of interesting geometric corollaries was obtained. It remains unclear if there are direct links between the work in [115] and what we are about to present.

Consider

$$
\begin{aligned}
\alpha(t) & =\Phi^{-1}\left(\int \Phi\left(-((1-t) f+t g)^{*} d \gamma\right)-(1-t) \Phi^{-1}\left(\int \Phi\left(-f^{*}\right) d \gamma\right)\right. \\
& -t \Phi^{-1}\left(\int \Phi\left(-g^{*}\right) d \gamma\right)
\end{aligned}
$$

Then Functional Ehrhard's inequality Theorem 6.30 implies

$$
\alpha(t) \geq 0 \text { for all } t \in[0,1]
$$

and

$$
\begin{gathered}
\alpha(0)=0 \\
\Longrightarrow \alpha^{\prime}(0) \geq 0 .
\end{gathered}
$$

Recall

$$
\Phi^{\prime}(s)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{s^{2}}{2}}
$$

and

$$
\frac{d}{d a} \Phi^{-1}(a)=\frac{1}{\Phi^{\prime}\left(\Phi^{-1}(a)\right)}=\sqrt{2 \pi} e^{\frac{\Phi^{-1}(a)^{2}}{2}}=\frac{1}{I(a)}
$$

where

$$
I(a)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{\Phi^{-1}(a)^{2}}{2}}
$$

is the Gaussian isoperimetric profile. Recall that $\frac{d}{d t}((1-t) f+t g)^{*}=(f-g)\left(\nabla f^{*}\right)$ (see Lemma 3.42). We then write

$$
\begin{aligned}
\alpha^{\prime}(0) & =\frac{1}{I\left(\int \Phi\left(-f^{*}\right)\right)} \cdot \int \frac{1}{\sqrt{2 \pi}} e^{-\frac{f^{* 2}}{2}} \cdot-\left(f\left(\nabla f^{*}\right)-g\left(\nabla f^{*}\right)\right) d \gamma \\
& +\Phi^{-1}\left(\int \Phi\left(-f^{*}\right) d \gamma\right)-\Phi^{-1}\left(\int \Phi\left(-g^{*}\right) d \gamma\right) \\
& \geq 0
\end{aligned}
$$

So we have

$$
\boldsymbol{A}(f)+\int g\left(\nabla f^{*}\right) \cdot \frac{1}{\sqrt{2 \pi}} e^{-\frac{f^{* 2}}{2}} d \gamma \geq I \quad\left(\int \Phi\left(-f^{*}\right) d \gamma\right) \cdot \Phi^{-1}\left(\int \Phi\left(-g^{*}\right) d \gamma\right)
$$

where $\boldsymbol{\varphi}(f)$ is a function that depends on only $f$ and not $g$. Set $G=g^{*}, f^{*}=-\Phi^{-1}(h)$ for function $h$. Then

$$
\begin{gathered}
\nabla f^{*}=-\frac{\nabla h}{I(h)}, \\
I(h)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{f^{* 2}}{2}}, \\
\boldsymbol{\oplus}(h)+\int G^{*}\left(-\frac{\nabla h}{I(h)}\right) \cdot I(h) d \gamma \geq I \quad\left(\int h d \gamma\right) \cdot \Phi^{-1}\left(\int \Phi(-G) d \gamma\right) .
\end{gathered}
$$

Recall

$$
\begin{aligned}
\left.\left(\lambda G\left(\frac{x}{\lambda}\right)\right)^{*}\right|_{z} & =\sup _{y}\left(\langle y, z\rangle-\lambda G\left(\frac{y}{\lambda}\right)\right) \\
& =\lambda \sup _{t}(\langle t, z\rangle-G(t)) \\
& =\lambda G^{*}(z)
\end{aligned}
$$

where we did a change of variables $t=\frac{y}{\lambda}$. Then for every $\lambda$,

$$
\boldsymbol{\oplus}(h)+\int G^{*}\left(-\frac{\nabla h}{I(h)}\right) \cdot I(h) d \gamma \geq I \quad\left(\int h d \gamma\right) \cdot \Phi^{-1}\left(\int \Phi\left(-\lambda G\left(\frac{x}{\lambda}\right)\right) d \gamma\right) .
$$

We divide both sides by $\lambda$ and let $\lambda \rightarrow \infty$. Note that $\frac{\boldsymbol{(})(h)}{\lambda} \rightarrow 0$, and we get:

Theorem 6.33 ("Generalized Bobkov's inequality", Barthe, Cordero-Erasquin, Ivanisvili, Livshyts [15]). For all convex $G$ and for all $h$ (such that the integrals make sense)

$$
\int G^{*}\left(-\frac{\nabla h}{I(h)}\right) \cdot I(h) d \gamma \geq I\left(\int h d \gamma\right) \cdot \lim _{\lambda \rightarrow \infty} \frac{\Phi^{-1}\left(\int \Phi\left(-\lambda G\left(\frac{x}{\lambda}\right)\right) d \gamma\right)}{\lambda}
$$

Remark 6.34. If $G$ is ray-increasing, we have

$$
G\left(\frac{x}{\lambda}\right) \geq G(0)
$$

Thus

$$
\begin{aligned}
\lim _{\lambda \rightarrow \infty} \frac{\Phi^{-1}\left(\int \Phi\left(-\lambda G\left(\frac{x}{\lambda}\right)\right) d \gamma\right)}{\lambda} & \leq \lim _{\lambda \rightarrow \infty} \frac{\Phi^{-1}\left(\int \Phi(-\lambda G(0)) d \gamma\right)}{\lambda} \\
& =\lim _{\lambda \rightarrow \infty} \frac{-\lambda G(0)}{\lambda} \\
& =-G(0)
\end{aligned}
$$

In fact, often $\geq$ holds as well. Note that ray increasing means that $\forall t>0, \forall \theta \in S^{n-1}, G(t \theta)$ is increasing in $t$.

We will consider the following example

$$
\begin{gathered}
G(x)= \begin{cases}-\sqrt{1-|x|^{2}} & \text { if }|x| \leq 1 \\
\infty & \text { if }|x|>1\end{cases} \\
\Phi\left(-\lambda G\left(\frac{x}{\lambda}\right)\right) d \gamma=\int_{\lambda B_{2}^{n}} \Phi\left(\sqrt{\lambda^{2}-x^{2}}\right) d \gamma \approx \Phi(\lambda),
\end{gathered}
$$

and

$$
\lim _{\lambda \rightarrow \infty} \frac{\Phi^{-1}(\Phi(\lambda))}{\lambda}=1
$$

We leave the details of this limit as a home work.
Recall (Example 3.35, part 5) that $G^{*}(x)=\sqrt{1+|x|^{2}}$. Note that

$$
I(h) G^{*}\left(\frac{\nabla h}{I(h)}\right) \geq I(h) \sqrt{1+\frac{|\nabla h|^{2}}{I(h)^{2}}}=\sqrt{I(h)^{2}+|\nabla h|^{2}} .
$$

Plugging this $G$ into Theorem 6.33 we deduce the following celebrated inequality of Bobkov (which was originally proved via different means).

Theorem 6.35 (Bobkov [21]).

$$
\int_{\mathbb{R}^{n}} \sqrt{I(h)^{2}+|\nabla h|^{2}} \geq I\left(\int_{\mathbb{R}^{n}} h d \gamma\right)
$$

The inequality tenzorizes, so one can use induction in dimension and so-called 2-point symmetrizations for the proof, as was done in [21]. Several alternative proofs were given by Barthe, Ivanisvili [16], Carlen, Kerce [42], Neeman, Paouris [150], among others. The proof that was presented in these notes is by Barthe, Cordero-Erausquin, Ivanisvili, Livshyts [15].

Remark 6.36. Bobkov's inequality is implies (and in fact follows from) the Gaussian isoperimetric inequality. Indeed, let $h=\mathbf{1}_{K}$ and so $|\nabla h|=h_{x} \mathbf{1}_{\{x \in \partial K\}}$. The LHS of the inequality is $\gamma^{+}(\partial K)$ and the RHS is $I(\gamma(K))$. So we have

$$
\gamma^{+}(\partial K) \geq I(\gamma(K))=\gamma^{+}(\partial H)
$$

where $H$ is a half-space and $\gamma(H)=\gamma(K)$. See e.g. Neeman [149] for the opposite implication.

So we get the following "diagram":


Gaussian Isoperimetry

Prekopa-Leindler
$\downarrow$
Generalized log-Sobolev

Log-Sobolev

Classical isoperimetry

Consider now another example:

$$
G(x)= \begin{cases}-1 & \text { if }\|x\|_{K} \leq 1 \\ \infty & \text { if }\|x\|_{K}>1\end{cases}
$$

Then $G^{*}(x)=1+h_{K}(x)=1+\|x\|_{K^{o}}$, and we get:
Corollary 6.37.

$$
I\left(\int h d \gamma\right) \cdot \lim _{\lambda \rightarrow \infty} \frac{\Phi^{-1}(\Phi(\lambda) \gamma(\lambda K))}{\lambda} \leq \int|\nabla h| d \gamma+\int I(h) d \gamma .
$$

In particular, for $K=B_{2}^{n}$

$$
I\left(\int h d \gamma\right)-\int I(h) d \gamma \leq \int|\nabla h| d \gamma
$$

Remark 6.38. The last inequality is weaker than Bobkov's inequality since

$$
\int_{\mathbb{R}^{n}} \sqrt{I(h)^{2}+|\nabla h|^{2}} \leq I\left(\int h d \gamma\right)-\int I(h) d \gamma
$$

Remark 6.39. More generally for

$$
G(x)= \begin{cases}-\sqrt[p]{1-|x|^{p}} & \text { if }\|x\|_{K} \leq 1 \\ \infty & \text { if }\|x\|_{K}>1\end{cases}
$$

one can obtain p-Bobkov inequalities.

### 6.10 An Ehrhard-Brascamp-Lieb type inequality

We will now differentiate Ehrhard's inequality twice to obtain a version of Ehrhard-BrascampLieb inequality. Note that Theorem 6.30 implies that

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \Phi^{-1}\left(\int \Phi\left(-(f+t g)^{*}\right) d \gamma\right) \leq 0 \tag{27}
\end{equation*}
$$

The left hand side of the above equals to

$$
\frac{d}{d t}\left[\frac{1}{I\left(\int \Phi\left(-f_{t}^{*}\right)\right)} \cdot \int e^{-\frac{f_{t}^{* 2}}{2}} \cdot \frac{1}{\sqrt{2 \pi}}(-1) \frac{d}{d t} f_{t}^{*} d \gamma\right]
$$

Recall $\frac{d}{d t} f_{t}^{*}=-g\left(\nabla f_{t}^{*}\right)$. So this becomes (after evaluating at $t=0$ ).

$$
\begin{aligned}
& =-\frac{I^{\prime}\left(\int \Phi\left(-f_{t}^{*}\right)\right)}{I^{2}\left(\int \Phi\left(-f_{t}^{*}\right)\right)} \cdot\left(\int e^{-\frac{f^{* 2}}{2}} \cdot \frac{1}{\sqrt{2 \pi}} g\left(\nabla f^{*}\right) d \gamma\right)^{2} \\
& +\frac{1}{I\left(\int \Phi\left(-f_{t}^{*}\right)\right)} \cdot\left(\int-f^{*} e^{-\frac{f^{* *}}{2}} \cdot \frac{-1}{\sqrt{2 \pi}} g\left(\nabla f^{*}\right)^{2} d \gamma\right) \\
& +\frac{1}{I\left(\int \Phi\left(-f_{t}^{*}\right)\right)} \cdot\left(\int e^{-\frac{f^{* 2}}{2}} \cdot \frac{-1}{\sqrt{2 \pi}} \frac{d^{2}}{d t^{2}} f_{t}^{*} d \gamma\right)
\end{aligned}
$$

We recall, by Lemma 3.42:

$$
\frac{d^{2}}{d t^{2}} f_{t}^{*}=-\left\langle\nabla^{2} f_{\nabla f^{*}} \nabla g(\nabla f), \nabla g(\nabla f)\right\rangle=-\left\langle\left(\nabla^{2} W\right)^{-1} \nabla \phi, \nabla \phi\right\rangle
$$

where $\phi=g\left(\nabla f^{*}\right)$ and $W=f^{*}$. With this change of variables, and in view of the computation above, we see that (27) amounts to:
Theorem 6.40 (Barthe, Cordero-Erasquin, Ivanisvili, Livshyts [15]). Consider convex $W \geq$ 0 and consider the probability measure $d \mu=e^{-\frac{W^{2}}{2}+C} d \gamma$, let

$$
a=\int \Phi(-W) d \gamma \in[0,1]
$$

and

$$
A=\int e^{-\frac{W^{2}}{2}} d \gamma \cdot \Phi^{-1}(a) e^{\Phi^{-1}(a)^{2} / 2}
$$

Then for any locally Lipschitz function $h$,

$$
\int h^{2} W d \mu-A\left(\int h d \mu\right)^{2} \leq \int\left\langle\left(\nabla^{2} W\right)^{-1} \nabla h, \nabla h\right\rangle d \mu
$$

Remark 6.41. In fact, one could deduce Theorem 6.40 by linearizing Theorem 6.33, similar to how we deduced the Brascamp-Lieb inequality from the Generalized Log-Sobolev inequality; this is left as a home work. In fact, as we pointed out before, one could also deduce BrascampLieb by taking the second derivative of Prekopa-Leindler inequality; this was also left as a home work.

### 6.11 Minimizing centered in-radius of a convex set of fixed Gaussian measure

We now discuss a few other isoperimetric inequalities related to the Gaussian measure.
Definition 6.42. Let $K$ be a convex set in $\mathbb{R}^{n}$ that contains the origin. Then the centered in-radius of $K$ is defined as

$$
r(k)=\sup \left\{r>0: r B_{2}^{n} \subset K\right\} .
$$

Proposition 6.43. Let $K$ be a convex set with $\gamma(K)=a>\frac{1}{2}$. Then

$$
r(K) \geq r\left(H_{a}\right)
$$

where $H_{a}=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \Phi^{-1}(a)\right\}, \gamma\left(H_{a}\right)=a$.
Proof. Consider $K$ with $r(K)=r$. Then there exists a hyperplane that supports both $K$ and $r B_{2}^{n}$. Consider $H=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle \leq r\right\}$ where $\theta$ is the outer unit normal. Then $K \subset H$ and $\gamma(K) \subset \gamma(H)$.

Proposition 6.44. Let $K$ be a symmetric convex set in $\mathbb{R}^{n}$ with $\gamma(K)=a \in[0,1]$. Then

$$
r(K) \geq r\left(S_{K}\right)
$$

where $S_{K}$ is a symmetric strip of Gaussian measure a.

### 6.12 Gaussian barycenter inequality and some extensions

Proposition 6.45. Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function.
Let $K$ be a convex set in $\mathbb{R}^{n}$. Then $\forall \theta \in S^{n-1}$,

$$
\int_{K} F(\langle x, \theta\rangle) d \gamma \geq \int_{H_{K}} F(\langle x, \theta\rangle) d \gamma
$$

where $H_{K}(\theta)=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle \leq \Phi^{-1}(\gamma(K))\right\}, \gamma\left(H_{K}(\theta)\right)=\gamma(K)$.

Lemma 6.46. Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Let $G$ be given by $G^{\prime}(t)=F(t) e^{-\frac{t^{2}}{2}}$. Then $G \circ \Phi^{-1}$ is convex on $(0,1]$.

Proof.

$$
\left.\left(G \circ \Phi^{-1}\right)^{\prime \prime}\right|_{t}=2 \pi\left(G^{\prime \prime}(s)+G^{\prime}(s) s\right) e^{s^{2}}
$$

where $s=\Phi^{-1}(t)$. Note that

$$
\begin{aligned}
G^{\prime \prime}(s)+G^{\prime}(s) \cdot s & =e^{-s^{2} / 2} \cdot\left(F(s) s-F(s) s+F^{\prime}(s)\right) \\
& =F^{\prime}(s) e^{-s^{2} / 2} \geq 0
\end{aligned}
$$

We now prove the previous proposition.
Proof. WLOG, let $\theta=e_{1}$. For $x \in e_{1}^{\perp}$, denote $I_{x}=K \cap\left\{x+t e_{1}\right\}=\left[\Phi^{-1}\left(\alpha_{x}\right), \Phi^{-1}\left(\alpha_{x}+a_{x}\right)\right]$.

$$
\begin{align*}
\int_{K} F\left(x_{1}\right) d \gamma & =\int_{\mathbb{R}^{n-1}} \frac{1}{\sqrt{2 \pi}} \int_{I_{x}} F(t) e^{-t^{2} / 2} d t d \gamma_{n-1}  \tag{28}\\
& =\int_{\mathbb{R}^{n-1}} \frac{1}{\sqrt{2 \pi}}\left(G \circ \Phi^{-1}\left(\alpha_{x}+a_{x}\right)-G \circ \Phi^{-1}\left(\alpha_{x}\right)\right) d \gamma_{n-1}
\end{align*}
$$

where $G$ is such that $G^{\prime}=F e^{-t^{2} / 2}$. Note that $G \circ \Phi^{-1}$ is convex by the previous Lemma, and therefore

$$
G \circ \Phi^{-1}\left(\alpha_{x}+a_{x}\right)-G \circ \Phi^{-1}\left(\alpha_{x}\right) \geq G \circ \Phi^{-1}\left(a_{x}\right)-G \circ \Phi^{-1}(0) .
$$

So (28) is

$$
\begin{aligned}
& \geq \int_{\mathbb{R}^{n-1}} \frac{1}{\sqrt{2 \pi}}\left(G \circ \Phi^{-1}\left(a_{x}\right)-G \circ \Phi^{-1}(0)\right) d \gamma_{n-1} \\
& \geq \frac{1}{\sqrt{2 \pi}} G \circ \Phi^{-1}\left(\int_{\mathbb{R}^{n-1}} a_{x} d \gamma_{n-1}\right)-G \circ \Phi^{-1}(0) \\
& =\frac{1}{\sqrt{2 \pi}} G \circ \Phi^{-1}(\gamma(K))-G \circ \Phi^{-1}(0)=\int_{H_{K}\left(e_{1}\right)} F\left(x_{1}\right) d \gamma
\end{aligned}
$$

where the last inequality follows from Jensen's inequality (by convexity of $G \circ \Phi^{-1}$ ). Also, equality is attained if $K$ is the appropriate half-space.

As a corollary, we deduce:
Theorem 6.47 (Bobkov [22]). The $L^{2}$-norm of the Gaussian barycenter of a convex set $K$ is maximized by a half-space when $\gamma(K)$ is prescribed.

$$
\left|\int_{K} x d \gamma\right| \leq\left|\int_{H_{K}} x d \gamma\right|
$$

where $H_{K}$ is any half-space with $\gamma\left(H_{K}\right)=\gamma(K)$.

Proof. WLOG say $\int_{K} x d \gamma=-t e_{1}, t \geq 0$. Then

$$
\begin{aligned}
\left|\int_{K} x d \gamma\right| & =t \\
& =-\int_{K} x_{1} d \gamma \\
& \leq-\int_{H_{K}} x_{1} d \gamma \\
& =\left|\int_{H_{K}} x d \gamma\right|
\end{aligned}
$$

where the last inequality follows from Proposition 6.45 and $H_{K}=\left\{x \in \mathbb{R}^{n}: x_{1} \leq \Phi^{-1}(\gamma(K))\right\}$.

### 6.13 Gaussian measure of dilates of convex sets in a direction, and an improved Gaussian Poincare inequality for linear functions on convex sets

Proposition 6.48. Let $K$ be a convex set and $\theta \in S^{n-1}$. Let, as before,

$$
H_{K}(\theta)=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle \leq \Phi^{-1}(\gamma(K))\right\} .
$$

Then

$$
\int_{K}\langle x, \theta\rangle^{2} d \gamma \leq \int_{H_{K}(\theta)}\langle x, \theta\rangle^{2} d \gamma
$$

Before we prove this proposition, let us outline an interesting implication. Throughout this subsection denote

$$
T_{t}=\left(\begin{array}{cccc}
t & 0 & \cdots & 0  \tag{29}\\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{array}\right)
$$

the dilation operator in the direction $e_{1}$. We get:
Corollary 6.49. Let $K$ be a convex set in $\mathbb{R}^{n}$ and $H_{K}\left(e_{1}\right)$ be defined as before. Then $\forall t \geq 0$,

$$
\gamma\left(T_{t} K\right) \geq \gamma\left(T_{t} H_{K}\left(e_{1}\right)\right)
$$

Proof. It is enough to show

$$
\left.\frac{d}{d t} \gamma\left(T_{t} K\right)\right|_{t=1} \geq\left.\frac{d}{d t} \gamma\left(T_{t} H_{K}\left(e_{1}\right)\right)\right|_{t=1}
$$

We have

$$
\begin{aligned}
\left.\frac{d}{d t} \gamma\left(T_{t} K\right)\right|_{t=1} & =\left(\frac{1}{\sqrt{2 \pi}} \int_{T_{t} K} e^{-x^{2} / 2} d x\right)_{1}^{\prime} \\
={ }_{y=T_{t}^{-1} x} & \left(\frac{1}{\sqrt{2 \pi}^{n}} \int_{K} e^{-t y_{1}^{2} / 2-\left(\sum_{i=2}^{n} y_{i}^{2}\right) / 2} t d y\right)_{1}^{\prime} \\
& =\int_{K}\left(1-x_{1}^{2}\right) d \gamma \\
& =\gamma(K)-\int_{K} x_{1}^{2} d \gamma \\
& \geq \gamma\left(H_{K}\left(e_{1}\right)\right)-\int_{H_{K}\left(e_{1}\right)} x_{1}^{2} d \gamma \\
& =\frac{d}{d t} \gamma\left(T_{t} H_{K}\right)_{t=1}
\end{aligned}
$$

Remark 6.50. Let $K$ be a asymmetric convex set in $\mathbb{R}^{n}$. Then for all $t \geq 1, \gamma\left(T_{t} K\right)=\alpha(t)$ increases in $t$, where $T_{t}$ is defined in (29).

Indeed,

$$
\alpha^{\prime}(t)_{t=1}=\gamma(K)-\int_{K} x_{1}^{2} d \gamma \geq 0
$$

We know by Poincare inequality,

$$
f_{K} x_{1}^{2} d \gamma-\left(f_{K} x_{1} d \gamma\right)^{2} \leq 1
$$

Since $f_{K} x_{1} d \gamma=0$,

$$
\int_{K} x_{1}^{2} d \gamma \leq \gamma(K)
$$

This property is rather special for the Gaussian measure.
Definition 6.51. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function, $f \geq 0$. Denote $f^{*}$ a non-increasing function such that

$$
\gamma_{1}(f>t)=\gamma_{1}\left(f^{*}>t\right)
$$

Lemma 6.52. Let $K \subset R$ be connected. $f: K \rightarrow \mathbb{R}, f \geq 0$, measurable, $f \in L^{1}(K, \gamma)$. Suppose $\{f>t\}$ are connected for each $t$. Then

$$
\int_{K} f x^{2} d \gamma_{1}(x) \leq \int_{K} f^{*} x^{2} d \gamma_{1}
$$

Proof. $a=\int_{0}^{\infty} \mathbf{1}_{\{t<a\}} d t$.

$$
\begin{align*}
\int_{K} f x^{2} d \gamma_{1} & =\int_{K} \int_{0}^{\infty} \int_{0}^{\infty} \mathbf{1}_{\{f(x)>s\}} \cdot \mathbf{1}_{\left\{x^{2}>t\right\}} d t d s d \gamma_{1}(x)  \tag{30}\\
& =\int_{0}^{\infty} \int_{0}^{\infty} \gamma_{1}\left(\{f(x)>s\} \cap\left\{x^{2}>t\right\}\right) d t d s
\end{align*}
$$

Now notice for all $A \subset \mathbb{R}$,

$$
\gamma_{1}(A \cap[-\alpha, \alpha]) \geq \gamma_{1}\left(A^{*} \cap[-\alpha, \alpha]\right)
$$

where $A^{*}=\left(-\infty, \Phi^{-1}(\gamma(A))\right)$. So (30) is

$$
\begin{aligned}
& \leq \int_{0}^{\infty} \int_{0}^{\infty} \gamma_{1}\left(\left\{f^{*}>t\right\} \cap\left\{x^{2}>s\right\}\right) d t d s \\
& =\int_{K} f^{*} x^{2} d \gamma
\end{aligned}
$$

We will now prove the previous Proposition.
Proof. WLOG let $\theta=e_{1}$. Let $f(t)=\gamma_{1}\left(K \cap\left\{x_{1}=t\right\}\right)$.

$$
\begin{aligned}
\int_{K} x_{1}^{2} d \gamma & =\int_{\mathbb{R}} x^{2} f(x) d \gamma_{1}(x) \\
& \leq \int_{\mathbb{R}} x^{2} f^{*}(x) d \gamma_{1}(x) \\
& =\int_{H_{K}} x_{1}^{2} d \gamma
\end{aligned}
$$

Finally, we deduce the following result which will appear in [138]:
Theorem 6.53 ([138]). Let $K$ be any convex set, $\gamma(K)=a \in[0,1]$.

$$
\begin{equation*}
\eta(a)=\sqrt{2 \pi} a \Phi^{-1}(a) e^{\Phi^{-1}(a)^{2} / 2} \in[-1,0] \tag{31}
\end{equation*}
$$

Then

$$
f_{K}\langle x, \theta\rangle^{2} d \gamma+\eta(a)\left(f_{K}\langle x, \theta\rangle d \gamma\right)^{2} \leq 1 .
$$

Proof.

$$
f_{K}\langle x, \theta\rangle^{2} d \gamma \leq f_{H_{K}(\theta)}\langle x, \theta\rangle^{2} d \gamma
$$

WLOG $f_{K}\langle x, \theta\rangle^{2} d \gamma \leq 0 \Longrightarrow f_{K}\langle x, \theta\rangle d \gamma \geq \int_{H_{K}}\langle x, \theta\rangle d \gamma$. So the LHS of the above is

$$
\leq f_{K}\langle x, \theta\rangle^{2} d \gamma+\eta(a)\left(f_{H_{K}(\theta)}\langle x, \theta\rangle d \gamma\right)^{2}=1
$$

where the last equality is left as homework (one can deduce this from Ehrhard's inequality).

Remark 6.54. This is a stronger version of Poincare inequality for linear functions.

### 6.14 Gaussian measure of dilates of convex sets

For $t \geq 1, K$ a set, and $\gamma(K)$ fixed, for what:
( $\boldsymbol{\sim}$ is

$$
\begin{equation*}
\gamma(t K) \geq \gamma(t \boldsymbol{\oplus}) ? \tag{32}
\end{equation*}
$$

80 is

$$
\begin{equation*}
\gamma(t K) \geq \gamma(t \boldsymbol{\infty}) ? \tag{33}
\end{equation*}
$$

Lemma 6.55. (32) is equivalent to $\gamma(K)=\gamma(\boldsymbol{\oplus})$,

$$
\int_{K}|x|^{2} d \gamma \leq \int_{\boldsymbol{a}}|x|^{2} d \gamma
$$

(33) is equivalent to $\gamma(K)=\gamma(\boldsymbol{\rho})$,

$$
\int_{K}|x|^{2} d \gamma \geq \int_{\boldsymbol{a}}|x|^{2} d \gamma
$$

Proof. (32) is equivalent to

$$
\begin{aligned}
& \frac{d}{d t} \gamma(t K)_{t=1} \geq \frac{d}{d t} \gamma(t \boldsymbol{\varphi})_{t=1} \\
& \frac{d}{d t} \gamma(t K)=\left(c_{n} \int_{t K} e^{-x^{2} / 2} d x\right)_{t=1}^{\prime} \\
&=\left(c_{n} \int_{K} e^{-t^{2} y^{2} / 2} t^{n} d y\right)_{t=1}^{\prime} \\
&=n \gamma(K)-\int_{K} x^{2} d \gamma
\end{aligned}
$$

Remark 6.56. $x^{2}=\langle x, x\rangle=|x|^{2}$

Proposition 6.57. Let $K$ be a measurable set in $\mathbb{R}^{n}$ such that

$$
\gamma(K)=\gamma\left(R B_{2}^{n}\right)
$$

Then

$$
\int_{K} x^{2} d \gamma \geq \int_{R B_{2}^{n}} x d \gamma
$$

Proof.

$$
\begin{gathered}
\sqrt{2 \pi}^{n}\left(\int_{K} x^{2} d \gamma-\int_{R B_{2}^{n}} x^{2} d \gamma\right)=\int_{S^{n-1}} \int_{0}^{\rho_{K}(\theta)} t^{n+1} e^{-t^{2} / 2} d t d \theta-\int_{S^{n-1}} \int_{0}^{R} t^{n+1} e^{-t^{2} / 2} d t d \theta \\
=\int_{\left\{\theta: \rho_{K}(\theta) \geq R\right\}} \int_{R}^{\rho_{K}(\theta)} t^{n+1} e^{-t^{2} / 2} d t d \theta-\int_{\left\{\theta: \rho_{K}(\theta) \leq R\right\}} \int_{\rho_{K}(\theta)}^{R} t^{n+1} e^{-t^{2} / 2} d t d \theta \geq \\
R^{2}\left(\int_{S^{n-1}} \int_{0}^{\rho_{K}(\theta)} t^{n-1} e^{-t^{2} / 2} d t d \theta-\int_{S^{n-1}} \int_{0}^{R} t^{n-1} e^{-t^{2} / 2} d t d \theta\right)=R^{2}\left(\gamma(K)-\gamma\left(R B_{2}^{n}\right)\right)=0
\end{gathered}
$$

Corollary 6.58. $\gamma(K)=a, K \subset \mathbb{R}^{n}$ some measurable set. Then $\forall t \geq 1$,

$$
\gamma(t K) \leq \gamma\left(t R B_{2}^{n}\right)
$$

with $\gamma\left(R B_{2}^{n}\right)=a$. And

$$
\gamma(t K) \geq \gamma\left(t \cdot\left(\mathbb{R}^{n} \backslash r B_{2}^{n}\right)\right)
$$

with $\gamma\left(\mathbb{R}^{n} \backslash r B_{2}^{n}\right)=a$.
Next, we ask ourselves the following question: which convex set $K$ minimizes $\gamma(t K)$ for $t \geq 1$, while $\gamma(K)$ is fixed? Without the convexity assumption, we just saw that the answer would be - the complement of a centered ball. But with the convexity assumption?

Theorem 6.59. Let $K$ be convex and $t \geq 1$. Let $H$ be a half space with $\gamma(H)=\gamma(K)$ and suppose $\gamma(K) \geq \frac{1}{2}$. Then

$$
\gamma(t K) \geq \gamma(t H)
$$

Proof. As before, it suffices to establish the inequality for derivatives, and get the conclusion by Newton's theorem. This time, we shall use another expression for the derivative:

$$
\begin{aligned}
\frac{d}{d t} \gamma(t K) & =\frac{d}{d t} \gamma(t K \backslash K)=\frac{d}{d t} \int_{t K \backslash K} d \gamma=\int_{\partial K} h_{K}\left(n_{x}\right) d \gamma \\
& \geq r(K) \cdot \int_{\partial K} d \gamma=r(K) \cdot \gamma^{+}(\partial K) \geq r(H) \cdot \gamma^{+}(\partial H)=\frac{d}{d t} \gamma(t H),
\end{aligned}
$$

where we used the fact that $h_{K}\left(n_{x}\right) \geq r(K)$, and the last inequality comes from the Gaussian isoperimetric inequality and the fact that $r(K) \geq r(H)$ (which we proved in Proposition 6.43.)

Remark 6.60. Note that earlier we showed that on one hand,

$$
\frac{d}{d t} \gamma(t K)_{t=1}=n \gamma(K)-\int_{K} x^{2} d \gamma
$$

and on the other hand,

$$
\frac{d}{d t} \gamma(t K)_{t=1}=\int_{\partial K} h_{K}\left(n_{x}\right) d \gamma=\int_{\partial K}\left\langle x, n_{x}\right\rangle d \gamma
$$

The two expressions could also be shown to be equal using integration by parts with boundary. However, in this course we strive to avoid integrating by parts with boundary.

Remark 6.61. Theorem 6.59 is also true for $\gamma(K) \leq \frac{1}{2}$, as can be shown via slightly modifying the argument (see home work). This result could also be deduced directly from the Ehrhard inequality.

### 6.15 Gaussian measure of dilates of symmetric convex sets: the S-inequality of R. Latała, K. Oleszkiewicz [123]

Next, we ask the same question as in the last subsection, but restrict attention to symmetric convex sets: for $t \geq 1$, which symmetric convex set minimizes $\gamma(t K)$ while $\gamma(K)$ is fixed? The answer to this question is: the symmetric strip, and this is a famous result of R. Latała, K. Oleszkiewicz [123].

Theorem 6.62 (Latała-Oleszkiewicz, $S$-inequality). Let $K$ be a symmetric convex set, $\gamma(K)=a \in[0,1]$. Then for any $t \geq 1$,

$$
\gamma(t K) \geq \gamma\left(t S_{K}\right)
$$

where $S_{K}$ is the symmetric strip with same Gaussian measure as $K$, that is

$$
S_{K}=\left\{x \in \mathbb{R}^{n}:|\langle x, \theta\rangle| \leq \alpha\right\} \quad \text { so that } \quad \gamma\left(S_{k}\right)=a
$$

Corollary 6.63. If $\gamma(K)=\gamma\left(S_{k}\right)$ for a symmetric strip $S_{K}$, then

$$
\int_{K} x^{2} d \gamma \leq \int_{S_{K}} x^{2} d \gamma
$$

for symmetric convex $K$.
The key result that they proved in [123] is the following
Theorem 6.64 (L-O). If $r(K)=\sup \left\{r>0: r B_{2}^{n} \subset K\right\}, K$ a symmetric convex set, then

$$
r(K) \gamma^{+}(\partial K) \geq r\left(S_{K}\right) \gamma^{+}\left(\partial S_{K}\right)
$$

where $\gamma(K)=\gamma\left(S_{K}\right)$ for a symmetric strip $S_{K}$.

Remark 6.65. Unlike in the non-symmetric case, for symmetric sets this statement is highly non-trivial.

Indeed, if $K$ is symmetric and convex, $r(K) \geq r\left(S_{K}\right)$. But we have

$$
\gamma^{+}(\partial K) \nsupseteq \gamma^{+}\left(\partial S_{K}\right),
$$

in general.
Let us mention a related
Conjecture 6.66 (Morgan; Heilman [83]). Let $K$ be a symmetric set in $\mathbb{R}^{n}$ with $\gamma(K)=a$. Then,

$$
\gamma^{+}(\partial K) \geq \min _{k=1, \ldots, n}\left\{\gamma^{+}\left(\partial C_{k}(a)\right), \gamma^{+}\left(\partial C_{k}^{c}(a)\right)\right\}
$$

where $C_{k}(a)$ is the $k$-round cylinder, i.e.

$$
C_{k}(a)=R_{k} B_{2}^{k} \times \mathbb{R}^{n-k},
$$

such that $\gamma\left(C_{k}(a)\right)=a$.
For sets of $\gamma(K)=1-\varepsilon$ for $\varepsilon$ small, indeed $\gamma^{+}(\partial K) \geq \gamma^{+}\left(\partial S_{K}\right)$ is known, see Barchiesi, Julin [19].

Proof of S-inequality from the Theorem 6.64. As before,

$$
\frac{d}{d t} \gamma(t K) \geq r(K) \gamma^{+}(\partial K) \geq r\left(S_{K}\right) \gamma^{+}\left(\partial S_{K}\right)=\frac{d}{d t} \gamma\left(t S_{K}\right) .
$$

Theorem 6.64 in fact follows from the 2-dimensional fact:
Lemma 6.67. Let $K$ be a convex set in $\mathbb{R}^{2}$ which is symmetric about the $y$-axis. Then,

$$
r(K) \gamma^{+}(\partial K) \geq r\left(S_{K}\right) \gamma^{+}\left(\partial S_{K}\right)
$$

where $S_{K}$ is a symmetric strip.
Proof of the Theorem 6.64 from the Lemma. We use Ehrhard symmetrization. For $K \subset \mathbb{R}^{n}$ symmetrize into 2 -dimensional set $\tilde{K}$ which is convex, symmetric about an axis, $\gamma^{+}(\partial K) \geq$ $\gamma^{+}(\partial \tilde{K})$ and $r(K) \geq r(\tilde{K})$. Notice that

$$
\gamma^{+}(\partial K) r(K) \geq \gamma^{+}(\partial \tilde{K}) r(\tilde{K}) \geq \gamma^{+}\left(\partial \tilde{S_{K}}\right) r\left(\tilde{S_{K}}\right)=\gamma^{+}\left(\partial S_{K}\right) r\left(S_{K}\right)
$$

As you see, once one thinks of the proof scheme (which is, of course, the hard part), the proof of the S-inequality readily follows from Lemma 6.15. However, this Lemma is the most difficult and highly technical step in the proof, and the argument is computer-assisted. We do not reproduce it here, but refer the interested reader to [123].

### 6.16 Home work

Question 6.68 (3 points). Find an alternative proof of Bobkov's inequality by approximating the Gaussian measure by the uniform measure on the Hamming cube.

Question 6.69 (1 point). Verify that

$$
\frac{\phi^{-1}\left(\gamma\left(t B_{2}^{n}\right)\right)}{t} \rightarrow_{t \rightarrow \infty} 1
$$

(recall that we used this fact to deduce the Gaussian Isoperimetric Inequality from Ehrhard's inequality).

Question 6.70 (1 point). Deduce the Gaussian Isoperimetric Inequality directly from Bobkov's inequality.

Question 6.71 (2 points). Prove Kahane's inequality: let $g_{1}, \ldots, g_{k}, \ldots$ be a sequence of i.i.d. $N(0,1)$ random variables. For any $q \geq p>0$, any $n \geq 1$ and any $z_{1}, \ldots, z_{n} \in \mathbb{R}^{n}$ we have

$$
\left(\mathbb{E}\left\|\sum_{i=1}^{n} g_{i} z_{i}\right\|^{q}\right)^{\frac{1}{q}} \leq \frac{\alpha_{q}}{\alpha_{p}}\left(\mathbb{E}\left\|\sum_{i=1}^{n} g_{i} z_{i}\right\|^{p}\right)^{\frac{1}{p}}
$$

where

$$
\alpha_{p}=\left(\mathbb{E}\left|g_{i}\right|^{p}\right)^{\frac{1}{p}} .
$$

Question 6.72 (2 points). Prove the following properties of Ehrhard symmetrizations. Let $S=S(L, e)$ be a Gaussian symmetrization and $A$ and $B$ be arbitrary closed sets. Then

- $\gamma(S(A))=\gamma(A)$ provided that $A$ is Borel measurable
- If $A \subset B$ then $S(A) \subset S(B)$
- For a vector $v, S(A+v)=S(A)+v$
- If $A_{1} \subset A_{2} \subset \ldots$ are open sets and $A=\cup_{i=1}^{\infty} A_{i}$ then $S(A)=\cup_{i=1}^{n} S\left(A_{i}\right)$

Question 6.73 (1 point). Let $L_{1}$ and $L_{2}$ be two sub-spaces in $\mathbb{R}^{n}$ such that $\left(L_{1} \cap L_{2}\right)^{\perp} \cap L_{1}$ and $\left(L_{1} \cap L_{2}\right)^{\perp} \cap L_{2}$ are orthogonal. Then

$$
S\left(L_{1}, e\right) \circ S\left(L_{2}, e\right)=S\left(L_{2}, e\right) \circ S\left(L_{1}, e\right)=S\left(L_{1} \cap L_{2}, e\right)
$$

Question 6.74 (1 point). Let $n \geq 3$ and $k \geq 2$. Show that for every $k$-symmetrization $S$ there exist 2-symmetrizations $S_{1}, \ldots, S_{k-1}$ such that $S=S_{1} \circ \ldots \circ S_{k=1}$. Hint: use Question 6. 73.

Question 6.75 (1 point). In dimension 2, show that there is a sequence $\theta_{1}, \ldots, \theta_{k}, \ldots \in \mathbb{S}^{n-1}$ such that letting $S_{i}=S\left(\theta_{i}^{\perp}, \theta_{i}\right) \circ \ldots \circ S\left(\theta_{1}^{\perp}, \theta_{1}\right)$, one has for every set $A$, that $S_{i}(A)$ converges to a half-space of the same Gaussian measure as $A$.

Question 6.76 (2 points). Prove, for any $\epsilon>0$, any Gaussian symmetrization $S$ and any set $A$ :

$$
S(A)+\epsilon B_{2}^{n} \subset S\left(A+\epsilon B_{2}^{n}\right)
$$

Conclude that the Ehrhard symmetrization decreases the Gaussian Perimeter. Using Questions 6.75 and 6.74, conclude the Gaussian Isoperimetric Inequality (directly without passing via the Ehrhard inequality).

Question 6.77 (2 points). Prove that the Gaussian symmetrization of any convex set is also convex. (recall that this was a crucial step in proving Ehrhard's inequality.)

Question 6.78 (2 points). Find lower estimates on the isoperimetric profile of some product measures of your choice (beyond the uniform measure on the cube and the Gaussian).

Question 6.79 (4 points). Solve the isoperimetric problem on the square in dimension 2: prove that if $\left|A \cap[0,1]^{2}\right|=a \in[0,1]$ then $\left|\partial A \cap[0,1]^{2}\right|$ is bounded from below by the case of A being either an appropriately shifted ball, or a half-space.

Question 6.80 (3 points). Let $L$ be a convex body. Find a lower estimate for the anisotropic Gaussian perimeter of a set $A$ with $\gamma(A)=a$, that is

$$
\liminf _{\epsilon \rightarrow 0} \frac{\gamma(A+\epsilon L)-\gamma(A)}{\epsilon}
$$

For which $L$ is it sharp?
Question 6.81 (2 points). Prove the simple case of the Gaussian Correlation Inequality (called the Sidak Lemma): let $K$ and $L$ be a pair of symmetric strips. Then $\gamma(K \cap L) \geq$ $\gamma(K) \gamma(L)$.

Hint: use the Prekopa-Leindler inequality.
Question 6.82 (1 point). Prove the Gaussian Log-Sobolev inequality by linearizing Bobkov's inequality.

Question 6.83 (1 point). Show that the functional Ehrhard inequality tensorizes, i.e. that from knowing it in dimensions $k$ and $m$ one can deduce it in the dimension $k+m$.

Question 6.84 (5 points). Try and find the proof of Functional Ehrhard Inequality in dimension one, without using the geometric Ehrhard.

Question 6.85 (1 point). Verify that for $a \in[0,1]$,

$$
\begin{equation*}
\eta(a)=\sqrt{2 \pi} a \Phi^{-1}(a) e^{\Phi^{-1}(a)^{2} / 2} \geq-1 \tag{34}
\end{equation*}
$$

Question 6.86 (3 points). In class we showed that if $K$ is any convex set, $\gamma(K)=a \in[0,1]$, then letting $\eta(a)$ as in (34) we have

$$
\frac{1}{\gamma(K)} \int_{K}\langle x, \theta\rangle^{2} d \gamma+\frac{\eta(a)}{\gamma(K)^{2}}\left(\int_{K}\langle x, \theta\rangle d \gamma\right)^{2} \leq 1
$$

Find an alternative proof of this fact using Ehrhard's inequality, or perhaps the consequences of Ehrhard's inequality - the generalized Bobkov inequality or the Ehrhard-Brascamp-Lieb inequality which we deduced in class.

Question 6.87 (1 point). Modify the proof of the Theorem 6.59 to conclude that also in the case $\gamma(K) \leq \frac{1}{2}$, one has $\gamma(t K) \geq \gamma(t H)$ for all $t \geq 1$, were $K$ is a convex set and $H$ is a half-space with $\gamma(K)=\gamma(H)$.

## 7 Hörmander's L2 method

### 7.1 The Bochner formula and its extension

Let $d \mu=e^{-v(x)} d x$ on $\mathbb{R}^{n}$ for a convex function $v$ and $L u=\Delta u-\langle\nabla u, \nabla v\rangle$ for $u \in C^{2}\left(\mathbb{R}^{n}\right)$. Recall that when the integrals make sense, then we have the following integration by parts formula:

$$
\int v L u d \mu=-\int\langle\nabla u, \nabla v\rangle d \mu
$$

Furthermore, the following classical "double integration by parts" formula is very useful:
Theorem 7.1 (Bochner-Lichnerovich). Assuming that the integrals exist and $u \in C^{2}\left(\mathbb{R}^{n}\right)$, we have

$$
\int(L u)^{2} d \mu=\int\left\|\nabla^{2} u\right\|^{2}+\left\langle\nabla^{2} v \nabla u, \nabla u\right\rangle d \mu
$$

where $\nabla^{2} u=\left(\frac{\partial^{2} u}{\partial x_{i} \partial x_{j}}\right)_{i j}$ is the Hessian matrix and $\|A\|^{2}=\sum_{i, j} a_{i j}^{2}$ is the Hilbert-Schmidt norm.

Standard proof. We have

$$
\begin{aligned}
\int(L u)^{2} d \mu=\int L u L u d \mu=-\int\langle\nabla u, \nabla L u\rangle d \mu & =\int\left\langle\nabla^{2} v \nabla u, \nabla u\right\rangle-\sum_{i=1}^{n} \partial_{i} u L \partial_{i} u d \mu \\
& =\int\left\langle\nabla^{2} v \nabla u, \nabla u\right\rangle+\sum_{i=1}^{n} \int\left\langle\nabla \partial_{i} u, \nabla \partial_{i} u\right\rangle d \mu \\
& =\int\left\|\nabla^{2} u\right\|^{2}+\left\langle\nabla^{2} v \nabla u, \nabla u\right\rangle d \mu
\end{aligned}
$$

where we have used $\nabla(L u)=L \nabla u-\nabla^{2} v \nabla u$ with $L \nabla u=\left(L \partial_{1} u, \ldots, L \partial_{n} u\right)^{T}$.
Furthermore, the following is true:
Theorem 7.2. Assuming that the integrals exist, $u \in C^{2}\left(\mathbb{R}^{n}\right)$ and $f \in C^{1}\left(\mathbb{R}^{n}\right)$, we have

$$
\int f\left((L u)^{2}-\left\|\nabla^{2} u\right\|^{2}-\left\langle\nabla^{2} v \nabla u, \nabla u\right\rangle\right) d \mu=2 \int\left\langle\nabla^{2} u \nabla u, \nabla f\right\rangle d \mu
$$

Remark 7.3. When $f \equiv 1$, we get Bochner's formula. Furthermore, this result can be obtained from Bochner's formula by changing the measure to $f d \mu$ whose potential is $V-\log f$.

Non-standard proof. We shall outline the same argument that we used to prove integration by parts. We have

$$
\int f d \mu=\int f(x) e^{-v(x)} d x=\int f(y+t \nabla u(y)) e^{-v(y+t \nabla u(y))} \operatorname{det}\left(I d+t \nabla^{2} u\right) d y=: \alpha(t)
$$

where we substituted $x=y+t \nabla u(y)$ Note that $\alpha$ does not depend on $t$. Therefore,

$$
\frac{d}{d t} \alpha(0)=0 .
$$

This implies, as we saw before,

$$
\int f L u d \mu=-\int\langle\nabla f, \nabla u\rangle d \mu
$$

We also have

$$
\frac{d^{2}}{d t^{2}} \alpha(0)=0
$$

which implies the theorem (HW). Hint: $\frac{d}{d t} \operatorname{det} A(t)=\operatorname{det}(A(t)) \operatorname{tr}\left(A^{-1}(t) \frac{d}{d t} A(t)\right)$ which gives

$$
\operatorname{det}\left(I d+t \nabla^{2} u\right)=1+t \Delta u+\frac{t^{2}}{2}\left((\Delta u)^{2}-\left\|\nabla^{2} u\right\|^{2}\right)
$$

### 7.2 Sobolev Spaces

For $f \in C^{1}\left(\mathbb{R}^{n}\right)$, we set

$$
\|f\|:=\sqrt{\int|\nabla f|^{2} d \mu}
$$

and define the Sobolev space as

$$
W^{1,2}(\mu):=\left\{\int f d \mu<\infty\right\} \cap\left\{f: \partial_{i} f \in L^{2}(\mu)\right\}
$$

where the closure is taken with respect to $\|\cdot\|$. Note that if $\int f d \mu<\infty$ and $\partial_{i} f \in L^{2}(\mu)$, then by Poincaré inequality

$$
\int f^{2} d \mu \leq\left(\int f d \mu\right)^{2}+C(\mu) \int|\nabla f|^{2} d \mu<\infty
$$

More generally, we can define space $W^{k, m}(\mu)$ of functions whose $k$-th (generalized) derivatives are integrable to the power $m$ with respect to $\mu$. The integration by parts
formula and Bochner formula are valid in $W^{1,2}$ and $W^{2,2}$, respectively. In fact, the operator $L$ on the Soblev space is defined via the integration by parts formula being valid for all test functions, since Sobolev functions might not be differentiable.

The upshot is that we will work in Sobolev spaces and will freely use all our tools related to integration by parts, and will freely work with differential operators. See Evans [62] for more details.

### 7.3 Density of the image of $L$ in $W^{1,2}(\mu)$

In this subsection, we prove the following very important fact: for any function $f \in W^{1,2}(\mu)$ with $\int f d \mu=0$, we can find a function $u$ such that $L u \approx f$, where $\approx$ is understood in some acceptable sense. Namely, we outline the following classical fact:
Theorem 7.4. $L\left(W^{2,2}(\mu)\right)$ is dense (w.r.t. the $W^{1,2}$-norm) in $W^{1,2}(\mu) \cap\left\{g: \int g d \mu=0\right\}$. In other words, for any $g \in W^{1,2}(\mu)$ such that $\int g d \mu=0$ and $\varepsilon>0$, there exists $\tilde{g}$ such that $\int \tilde{g} d \mu=0, \tilde{g} \in W^{1,2}(\mu)$ and $u \in W^{2,2}(\mu)$ with $L u=\tilde{g}$ and $\|g-\tilde{g}\|_{W^{1,2}(\mu)}<\varepsilon$.
Remark 7.5. Note that $\int g d \mu=0$ is important. Indeed,

$$
\int L u d \mu=-\int\langle\nabla 1, \nabla u\rangle d \mu=0
$$

and thus $L u=\tilde{g}$ implies $\int \tilde{g}=0$.
Lemma 7.6 (Lax-Milgram). Let $H$ be a Hilbert space with norm $\|\cdot\|, Q$ a symmetric bilinear form on $H$, $\ell$ be a linear functional on $H$. In addition, let

- $Q(f, g) \leq C_{1}\|f\|\|g\|$ (continuity),
- $Q(f, f) \geq C_{2}\|f\|^{2}$ (coercivity) and
- $\left\|\ell(f) \leq C_{3}\right\| f \|$ (continuity).

Then, there exists a unique $h \in H$ such that

$$
Q(f, h)=\ell(f)
$$

Sketch of proof. Set $H=W^{1,2}(\mu), Q(f, g)=\int\langle\nabla f, \nabla g\rangle d \mu$. Then we have $Q(f, g) \leq\|f\|\|g\|$ and $Q(f, f)=\|f\|^{2}$. If we set $\ell(f)=-\int f g_{0} d \mu$ for some fixed function $g_{0} \in W^{1,2}(\mu)$ with $\int g_{0} d \mu=0$. This implies

$$
|\ell(f)|=\left|\int\left(f-\int f d \mu\right) g_{0} d \mu\right| \leq\left\|f-\int f\right\|\left\|g_{0}\right\| \leq C(g) C_{\text {Poin }}(\mu)\|f\|
$$

By Lax-Milgram there exists $h \in W^{1,2}(\mu)$ such that

$$
-\int u g_{0} d \mu=\int\langle\nabla h, \nabla f\rangle, q q u a d \forall f \in W^{1,2}
$$

which is equivalent to

$$
g_{0}=L h, \quad \int u L h d \mu=\int\langle\nabla h, \nabla f\rangle
$$

by integration by parts. The theorem follows by using elliptic regularity methods.
We remark that a somewhat different proof (which is a bit more complicated, but it does not swap elliptic regularity under the rag) can be found in Cordero-Erausquin, Fradelizi, Maurey [50].

### 7.4 Review of dual norms

Definition 7.7. Let $(B,\|\cdot\|)$ be a Banach space. Then the dual norm is defined by

$$
\|x\|_{*}:=\sup _{y \in B:\|y\| \leq 1}\langle x, y\rangle .
$$

Some examples:

1. In $\mathbb{R}^{n}$, suppose $\|\cdot\|=\|\cdot\|_{K}$ for a symmetric convex body $K$. Then $\|\cdot\|_{*}=\|\cdot\|_{K^{\circ}}$.
2. $\|f\|_{p}=\left(\int|f|^{p}\right)^{1 / p}$ for $p \geq 1$, then

$$
\left(\|f\|_{p}\right)_{*}=\sup _{g: \int|g|^{p} d \mu \leq 1} \int f g d \mu \leq\|f\|_{q}\|g\|_{p}=\|f\|_{q} .
$$

And equality is achieved for $f=c g$. So $L^{p}(\mu)^{*}=L^{q}(\mu)$.
3. Let $\mathcal{F}$ be a function space over $\mathbb{R}^{n}$. Set $X=\mathcal{F} \times \ldots \times \mathcal{F}$ ( $n$-times) a space of vector functions. Let $A$ be a function matrix of size $n \times n$. Suppose $A$ is positive definite. Then, for $F \in X, F=\left(f_{1}, \ldots, f_{n}\right)$,

$$
\|F\|:=\sqrt{\int\langle A F, F\rangle d \mu}
$$

is a norm (HW). The dual norm is (HW)

$$
\|F\|_{*}=\sqrt{\int\left\langle A^{-1} F, F\right\rangle d \mu}
$$

### 7.5 The $H^{-1}$-norm

Recall

$$
\|f\|_{W^{1,2}(\mu)}=\sqrt{\int|\nabla f|^{2} d \mu}=:\|f\|_{H^{1}(\mu)}
$$

What is the dual norm?

$$
\|f\|_{H^{-1}(\mu)}:=\left(\|f\|_{H^{1}(\mu)}\right)_{*}=\sup \left\{\int u f d \mu: \int|\nabla u|^{2} d \mu \leq 1\right\} .
$$

Remark 7.8. Note that $\|f\|_{H^{-1}(\mu)}$ only makes sense when $\int f d \mu=0$, otherwise it is infinite. We shall only consider $f$ with $\int f d \mu=0$ when we look at their $H^{-1}$ norms.

Remark 7.9. Suppose that $\int f d \mu=0$ and $f=L v$ for some $v \in W^{2,2}(\mu)$. Then

$$
\int f g d \mu=\int L v g d \mu=-\int\langle\nabla v, \nabla g\rangle d \mu \leq \sqrt{\int|\nabla v|^{2} d \mu} \sqrt{\int|\nabla g|^{2} d \mu}
$$

If $\|g\|_{H^{1}} \leq 1$, then

$$
\int f g d \mu \leq \sqrt{\int|\nabla v|^{2} d \mu}
$$

Furthermore,

$$
\sup _{g:\|g\|_{H^{1}} \leq 1} \int f g d \mu=\sqrt{\int|\nabla v|^{2} d \mu}
$$

since the supremum is attained when $g$ is proportional to $v$.
In other words, when the equation $L v=f$ has a solution, then

$$
\|f\|_{H^{-1}(\mu)}=\|v\|_{H^{1}(\mu)} \text { where } L v=f
$$

As we have recently discussed, every $f \in W^{1,2}(\mu)$ with $\int f d \mu=0$ can be approximated arbitrarily closely by $\tilde{f}$ such that the equation $L v=\tilde{f}$ has a solution.

Remark 7.10. Note that if $\int f d \mu \neq 0$, then $\|f\|_{H^{-1}}=\infty$. Indeed, in this case, we can take $u=\operatorname{sgn}\left(\int f d \mu\right) R$ where $R>0$. Then, $\int f u d \mu=\left|\int f d \mu\right| R \xrightarrow{R \rightarrow \infty} \infty$ and $\int|\nabla u|^{2} d \mu=$ $0 \leq 1$.

Therefore we shall consider the $H^{-1}-$ norm only for functions which are mean zero.
Let us now discuss the geometric meaning of the $H^{-1}$ norm. The following result can be found in Villani [170], see also Klartag [101] (the proof below is taken from the appendix of [101]).

Theorem 7.11 (Relation of the $H^{-1}$ norm to mass transport). Let $\mu$ be a finite compact Borel measure on $\mathbb{R}^{n}$, and $h: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a bounded measurable function satisfying $\int h d \mu=$ 0 . For sufficiently small $\varepsilon>0$, define $\mu_{\varepsilon}$ to be the measure $d \mu_{\varepsilon}=(1+\varepsilon h) d \mu$. Then

$$
\|h\|_{H^{-1}(\mu)}=\liminf _{\varepsilon \rightarrow 0} \frac{W_{2}\left(\mu, \mu_{\varepsilon}\right)}{\varepsilon}
$$

Recall

$$
W_{2}(\mu, \nu)^{2}=\inf _{\substack{\gamma \in \Gamma(\mu, \nu) \\ \mu, \nu}} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|x-y|^{2} d \gamma(x, y)
$$

Proof. We outline the proof of the $\leq$ inequality. The other direction is left to the reader. We want to show for all $\varphi \in C^{\infty}(\mu)$,

$$
\int h \varphi d \mu \leq \sqrt{\int|\nabla \varphi|^{2} d \mu} \liminf _{\varepsilon \rightarrow 0} \frac{W_{2}\left(\mu, \mu_{\varepsilon}\right)}{\varepsilon} .
$$

Without loss of generality, assume $\varphi$ is compactly supported. By Taylor's theorem, we have

$$
\varphi(y)-\varphi(x) \leq|\nabla \varphi(x)||x-y|+R|x-y|^{2} \quad \forall x, y \in \mathbb{R}^{n}
$$

where $R$ depends on $\varphi$.
Consider $0<\varepsilon<\frac{1}{\sup |h|}$. Then, $\mu_{\varepsilon}$ is a measure. Let $\pi$ be any coupling of $\mu$ and $\mu_{\varepsilon}$. We have

$$
\int_{\mathbb{R}^{n}} h \varphi d \mu=\frac{1}{\varepsilon} \int_{\mathbb{R}^{n}} \varphi\left(\mu_{\varepsilon}-\mu\right) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}}|\varphi(y)-\varphi(x)| d \pi(x, y) .
$$

By the Cauchy-Schwartz inequality and the estimate from Taylor's theorem,

$$
\begin{aligned}
& \frac{1}{\varepsilon} \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}}|\varphi(y)-\varphi(x)| d \pi(x, y) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}}|\nabla \varphi||x-y| d \pi(x, y)+\int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} R|x-y|^{2} d \pi(x, y) \\
& \leq \frac{1}{\varepsilon} \sqrt{\int_{\mathbb{R}^{n} \times \mathbb{R}^{n}}|\nabla \varphi|^{2} d \pi(x, y)} \sqrt{\int_{\mathbb{R}^{n} \times \mathbb{R}^{n}}|x-y|^{2} d \pi(x, y)}+\frac{R}{\varepsilon} \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}}|x-y|^{2} d \pi(x, y) .
\end{aligned}
$$

Taking supremum over all $\pi$ which are couplings of $\mu$ and $\mu_{\epsilon}$, we get

$$
\int_{\mathbb{R}^{n}} h \varphi d \mu \leq \frac{1}{\varepsilon} \sqrt{\int|\nabla \varphi|^{2} d \mu} W_{2}\left(\mu, \mu_{\varepsilon}\right)+R \frac{W_{2}\left(\mu, \mu_{\varepsilon}\right)^{2}}{\varepsilon} .
$$

One can check that $\frac{W_{2}\left(\mu, \mu_{\varepsilon}\right)^{2}}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$, which completes the proof.

### 7.6 The Hörmander Duality Lemma

We say that norm $\|\cdot\|_{1}$ dominates norm $\|\cdot\|_{2}$ when any sequence of functions which converges to zero in $\|\cdot\|_{1}$, also converges to zero in $\|\cdot\|_{2}$.

The Lemma below will be the cornerstone of everything that follows.
Lemma 7.12. As usual, let $\mu$ be a log-concave probability measure on $\mathbb{R}^{n}$ with density $e^{-V}$ and the associated Laplace operator L. Suppose for all $u \in W^{2,2}(\mu)$, we have $\int(L u)^{2} d \mu \geq$ $\|\nabla u\|^{2}$ for some norm $\|\cdot\|$ on $\underbrace{W^{1,2}(\mu) \times \ldots \times W^{1,2}(\mu)}_{n \text { times }}$ such that $\|\cdot\|_{*}$ is dominated by $\|\cdot\|_{H^{1}}$. Then, for all $f \in W^{1,2}(\mu)$,

$$
\begin{equation*}
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq\|\nabla f\|_{*}^{2} \tag{35}
\end{equation*}
$$

where $\|\cdot\|_{*}$ is the dual norm. The reverse implication holds as well.

Proof. Without loss of generality, assume $\int f d \mu=0$ (since Equation (35) is invariant under addition of constants to $f$ ) and that there exists $u \in W^{2,2}(\mu)$ such that $L u=f$. Indeed, recall by Theorem 7.4 that $L\left(W^{2,2}(\mu)\right)$ is dense in $W^{1,2}(\mu)$, i.e., for all $\varepsilon>0$, there exists $\widetilde{f} \in W^{1,2}(\mu)$ and $u \in W^{2,2}(\mu)$ such that $L u=\widetilde{f}$ and $\int|\nabla f-\nabla \widetilde{f}|^{2} d \mu<\varepsilon$. Since $\|\cdot\|_{*}$ is dominated by $\|\cdot\|_{H^{1}}$, this implies that $\|\nabla f-\nabla \tilde{f}\|_{*} \leq \delta(\epsilon)$ where $\delta(\epsilon) \rightarrow_{\epsilon \rightarrow 0} 0$. By passing to a sequence of functions and sending $\epsilon \rightarrow 0$ we get the conclusion for $f$, provided that we can get the conclusion for each of the functions $\tilde{f}$ in the sequence, for which the solution to the PDE $L u=\tilde{f}$ exists. Therefore, to keep the notation clean, let us assume that the solution to $L u=f$ exists.

Then, integrating by parts and using the assumption $\int(L u)^{2} d \mu \geq\|\nabla u\|^{2}$, we get:

$$
\begin{aligned}
\int f^{2} d \mu & =2 \int L u \cdot f d \mu-\int(L u)^{2} d \mu \\
& =-2 \int\langle\nabla u, \nabla f\rangle d \mu-\int(L u)^{2} d \mu \\
& \leq \int-2\langle\nabla u, \nabla f\rangle d \mu-\|\nabla u\|^{2} \leq\|\nabla f\|_{*}^{2},
\end{aligned}
$$

where in the last passage we used the fact that for any norm $\|\cdot\|$ one has

$$
\|\nabla u\| \cdot\|\nabla f\|_{*} \geq\langle\nabla u, \nabla f\rangle
$$

and therefore, by the AM-GM inequality,

$$
\begin{equation*}
\|\nabla u\|^{2}+\|\nabla f\|_{*}^{2}-2\langle\nabla u, \nabla f\rangle \geq 0 \tag{36}
\end{equation*}
$$

Remark 7.13. Note that if we replace the pair of dual norms $\|\cdot\|$ and $\|\cdot\|_{\star}$ with any pair of functionals that satisfy (36) then the conclusion of Lemma 7.12 still remains valid.

Recall Bochner's formula Theorem 7.1 which tells us that

$$
\int(L u)^{2} d \mu=\int\left\|\nabla^{2} u\right\|^{2}+\left\langle\nabla^{2} V \nabla u, \nabla u\right\rangle d \mu
$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm. This immideately gives us two situations in which Lemma 7.12 is applicable.

On one hand, by convexity of $V$ we know that $\nabla^{2} V$ is non-negative definite, we have

$$
\begin{equation*}
\int(L u)^{2} d \mu \geq \int\left\|\nabla^{2} u\right\|^{2} d \mu=\|\nabla u\|_{H^{1}(\mu)}^{2} \tag{37}
\end{equation*}
$$

Another estimate is

$$
\begin{equation*}
\int(L u)^{2} d \mu \geq \int\left\langle\nabla^{2} V \nabla u, \nabla u\right\rangle d \mu \tag{38}
\end{equation*}
$$

Let us discuss the implications of both of these estimates in detail in the coming subsections.

### 7.7 Klartag's $H^{-1}$ inequality

Combining (37) with Hörmander's lemma, we get
Corollary 7.14 (Klartag's $H^{-1}$ inequality [101], see also Barthe-Klartag [17]). For any $f \in W^{1,2}(\mu)$,

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq\|\nabla f\|_{H^{-1}}^{2}
$$

The following corollary of Corollary 7.14 and Theorem 7.11 is now immideate.
Corollary 7.15. For all $f \in W^{1,2}(\mu)$,

$$
\operatorname{var}_{\mu}(f) \leq \liminf _{\varepsilon \rightarrow 0} \sum_{i=1}^{n}\left(\frac{W_{2}\left(\mu, \mu\left(1+\varepsilon \partial_{i} f\right)\right)}{\varepsilon}\right)^{2}
$$

Suppose for a measure $\mu$ and $\varepsilon>0$ one manages to construct a transport map $T: \mu \rightarrow$ $\mu(1+\varepsilon \psi)$ such that $\int|x-T x|^{2} d \mu \leq C(\mu) \int \psi^{2} d \mu$. Then, in view of Corollary 7.15, we have for all $f \in W^{1,2}(\mu)$,

$$
\operatorname{var}_{\mu}(f) \leq C(\mu) \mathbb{E}_{\mu}|\nabla f|^{2}
$$

Using these ideas, Klartag [101] showed:
Theorem 7.16 (Klartag). If $\mu$ is an unconditional log-concave probability measure, then for any $f \in W^{1,2}(\mu)$,

$$
\operatorname{var}_{\mu}(f) \leq C \sqrt{\log n} \int|\nabla f|^{2} d \mu
$$

where $C$ is an absolute constant.
We have previously discussed this result in the context of other conjectures and results in section 5.7.

### 7.8 The $L 2$ proof of the Brascamp-Lieb inequality, which gives another proof of the Prekopa-Leindler and the Brunn-Minkowski inequality

On the other hand, combining (38) with Hörmander's lemma, and recalling the third example of dual norms from subsection 7.4, we get a new proof of the very familiar Theorem 3.75:

Corollary 7.17 (Brasscamp-Lieb inequality (1976.)). For any $f \in W^{1,2}(\mu)$,

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq \int\left\langle\left(\nabla^{2} V\right)^{-1} \nabla f, \nabla f\right\rangle d \mu
$$

Recall that he above statement is equivalent to $\frac{d^{2}}{d t^{2}} \log \int e^{-(F+t G)^{*}} d x \leq 0$, i.e. the Prekopa-Leindler inequality (see the proof of Theorem 3.75 earlier in the notes). Therefore, Hörmander's $L 2$ approach gives us another proof of the Prekopa-Leindler inequality Theorem 3.20, and therefore the Brunn-Minkowski inequality Theorem 3.1 (which followed as a corollary of Theorem 3.20)!

In fact, a remarkable direct proof of the Brunn-Minkowski inequality (omitting the Prekopa-Leindler) using this circle of ideas was given by Kolesnikov and Milman [114] (in fact, their results hold in much greater generality in a Riemannian setting). Hörmander's $L 2$ method on measure spaces with boundary requires significant modifications, and yields some additional freedom and a new circle of ideas. We do not go into detail here but refer the interested reader to Kolesnikov, Milman [112], [113], [114], [115], [116], Kolesnikov, Livshyts [117], [118], as well as the aforementioned paper by Klartag [101].

Let us point out also another result which follows from the generalized Bochner formula Theorem 7.2, and the proof is left as a home work.

Theorem 7.18. ([138]) Suppose $K$ is a convex set, $g$ is a concave function, and $\mu$ is $a$ log-concave probability measure. Then,

$$
\int g f^{2} d \mu-\left(\int g f d \mu\right)^{2} \leq \int g\left\langle\left(\nabla^{2} v\right)^{-1} \nabla f, \nabla f\right\rangle d \mu
$$

where $d \mu=e^{-v} d x$.

### 7.9 The $L 2$ proof of the symmetric Gaussian Poincaré inequality restricted to a symmetric convex set

Earlier this semester, we proved the symmetric Gaussian Poincaré inequality restricted to a symmetric convex set by linearlizing the Blaschke-Santaló inequality, and referring to Cafarelli's contraction theorem (see section 4.11). However, the Cafarelli theorem was only stated without proof. Finally, we prove this inequality honestly. This subsection follows the work of Cordero-Erausquin, Fradelizi, Maurey [50].

First, we shall need the uniqueness of the solution to our PDE:
Lemma 7.19. For any function $f \in W^{1,2}(\mu)$, in case the solution to $L u=f$ exists in $W^{2,2}(\mu)$, it is unique up to adding a constant to $u$.

Proof. Suppose not. Let $u, v \in W^{2,2}(\mu)$ be such that $L u=L v$ and let $h=u-v$. We get $L h=0$. Therefore, integrating by parts we see

$$
0=\int h L h d \mu=-\int|\nabla h|^{2} d \mu
$$

and therefore $h=C$ almost everywhere for some constant $C$ (this little step we leave as a home work). The conclusion follows.

As a corollary, we get
Corollary 7.20. Suppose $u \in W^{2,2}(\mu)$ is a solution to $L u=f$ for some $f \in W^{1,2}(\mu)$ such that $f$ is an even function, and $\mu$ is an even measure. Then $u$ is also an even function.

Proof. Suppose $u$ is not even. Note that $v(x)=u(-x)$ satisfies $L v=f$ also, since $f$ and $V$ are even. Thus when $u$ is not even, the uniqueness of the solution to our PDE (which we saw in Lemma 7.19) is violated, and we get a contradiction.

Finally, we deduce from Hörmander's Lemma 7.12:
Corollary 7.21. Suppose $K$ is a symmetric convex set, $\gamma$ is the Gaussian measure, and $f \in W^{1,2}(\mu)$ is an even function. Then,

$$
f_{K} f^{2} d \gamma-\left(f_{K} f d \gamma\right)^{2} \leq \frac{1}{2} f_{K}|\nabla f|^{2} d \gamma
$$

where $f_{K} \wp d \gamma=\frac{1}{\gamma(K)} \int_{K} \wp d \gamma$.
Proof. Let $d \mu=e^{-w} d \gamma$ where $w$ is an even convex function, and $\int d \mu=1$. Then, $L u=$ $\Delta u-\langle\nabla w+x, \nabla u\rangle$ and using Bochner's formula Theorem 7.1,

$$
\begin{aligned}
\int(L u)^{2} d \mu & =\int\left\|\nabla^{2} u\right\|^{2}+\left\langle\left(\nabla^{2} w+\mathrm{Id}\right) \nabla u, \nabla u\right\rangle d \mu \\
& \geq \int\left\|\nabla^{2} u\right\|^{2}+\langle\operatorname{Id} \nabla u, \nabla u\rangle d \mu \\
& =\int\left\|\nabla^{2} u\right\|^{2}+|\nabla u|^{2} d \mu
\end{aligned}
$$

Since $f$ is an even function, and $w+\frac{x^{2}}{2}$ is an even function, we may infer that $u$ is an even function by Corollary 7.20. Then, $\int \nabla u d \mu=0$. We have the estimate

$$
\int\left\|\nabla^{2} u\right\|^{2} d \mu \geq \int|\nabla u|^{2} d \mu-\left(\int \nabla u d \mu\right)^{2}=\int|\nabla u|^{2} d \mu
$$

Therefore,

$$
\int(L u)^{2} d \mu \geq \int\left\|\nabla^{2} u\right\|^{2}+|\nabla u|^{2} d \mu \geq 2 \int|\nabla u|^{2} d \mu .
$$

Consider the norm $\|\cdot\|=\sqrt{2}\|\cdot\|_{L^{2}(\mu)}$, so that $\|\cdot\|_{*}=\frac{1}{\sqrt{2}}\|\cdot\|_{L^{2}(\mu)}$. By Hörmander's lemma,

$$
\operatorname{var}_{\mu}(f) \leq \frac{1}{2}\|\nabla f\|_{L^{2}(\mu)}^{2}=\frac{1}{2} \mathbb{E}_{\mu}|\nabla f|^{2} .
$$

Finally, set $e^{-w}=\mathbf{1}_{K}$ to reach the conclusion.

### 7.10 Connections to the B-conjecture.

Recall the B-conjecture, previously stated as Conjecture 4.54: for all even log-concave probability measures $\mu$ and for all symmetric convex sets $K$ in $\mathbb{R}^{n}$, the function $\log \mu\left(e^{t} K\right)$ is concave in $t>0$. Recall section 4.12 where we outlined that this conjecture is true when $\mu=\gamma$, as was shown by Cordero-Erausquin, Fradelizi and Maurey. Recall that this follows from Corollary 7.21 which we have just proved using the $L 2$-method.

It turns out that the $L 2$-method is relevant for the B-conjecture in general. We start by outlining the equivalent local version of the B-conjecture, which is a generalization of the assertion of Corollary 7.21 (but this more general inequality is currently not known in general).

Proposition 7.22. For $d \mu=e^{-V} d x$, we have $\frac{d^{2}}{d t^{2}} \log \mu\left(e^{t} K\right) \leq 0$ if and only if

$$
\begin{equation*}
f_{K}\langle\nabla V, x\rangle^{2} d \mu-\left(f_{K}\langle\nabla V, x\rangle d \mu\right)^{2} \leq f_{K}\left\langle\left(\nabla^{2} V\right) x, x\right\rangle+\langle\nabla V, x\rangle d \mu \tag{39}
\end{equation*}
$$

Proof. Home work! Very similar to the Gaussian case which we outlined while proving Theorem 4.56.

Below we shall see that (39) would follow from the following conjecture.
Conjecture 7.23. For any even log-concave probability measure $\mu$ given by $d \mu=e^{-V} d x$ ), for all symmetric convex sets $K$, and for all even $f \in W^{1,2}(\mu)$, we have

$$
\begin{equation*}
f_{K} f^{2} d \mu-\left(f_{K} f d \mu\right)^{2} \leq f_{K}\left\langle\left(\nabla^{2} V+T\right)^{-1} \nabla f, \nabla f\right\rangle d \mu \tag{40}
\end{equation*}
$$

for some positive definite functional matrix $T=T(x)$ such that $T x=\nabla V$.
Remark 7.24. Equation (40) is stronger than the Brasscamp-Lieb inequality since

$$
\left(\nabla^{2} V+T\right)^{-1} \leq\left(\nabla^{2} V\right)^{-1}
$$

Next, we outline
Proposition 7.25. We claim that Conjecture 7.23 implies (39) for any even function $f$ and even $\mu$, and therefore (in view of Proposition 39), it implies the $B$-conjecture for an even $\mu$.

Proof. Take $f=\langle\nabla V, x\rangle$. Then, $\nabla f=\nabla^{2} V x+\nabla V$. Assuming Conjecture 7.23, we have

$$
\begin{gathered}
\operatorname{var}_{\left.\mu\right|_{K}}(\langle\nabla V, x\rangle) \leq f_{K}\left\langle\left(\nabla^{2} V+T\right)^{-1}\left(\nabla^{2} V x+\nabla V\right), \nabla^{2} V x+\nabla V\right\rangle d \mu= \\
f_{K}\left\langle x, \nabla^{2} V x+\nabla V\right\rangle d \mu .
\end{gathered}
$$

For some $T$ such that $T x=\nabla V$, notice $\nabla^{2} V x+\nabla V=\left(\nabla^{2} V+T\right) x$, so

$$
\operatorname{var}_{\left.\mu\right|_{K}}(\langle\nabla V, x\rangle) \leq f_{K}\left\langle x, \nabla^{2} V x+\nabla V\right\rangle d \mu .
$$

In remains to note, as before, that to conclude that $\mu\left(e^{t} K\right)$ is log-concave for all symmetric convex $K$ it suffices to show that $\frac{d^{2}}{d t^{2}} \log \mu\left(e^{t} K\right) \leq 0$.

The next result once again shall utilize Hörmander's duality lemma.
Proposition 7.26 (Corollary of Hörmander's lemma). Suppose for a given even log-concave probability measure $d \mu=e^{-V} d x$ on $\mathbb{R}^{n}$, for all even $u \in W^{2,2}(\mu)$, for all symmetric convex sets $K \subset \mathbb{R}^{n}$, we have

$$
\int_{K}\left\|\nabla^{2} u\right\|^{2} d \mu \geq \int_{K}\langle T \nabla u, \nabla u\rangle d \mu
$$

for some positive definite $T$ such that $T x=\nabla V$. Then, Conjecture 7.23 is true, i.e., for all even $f \in W^{1,2}(\mu)$, we have

$$
f_{K} f^{2} d \mu-\left(f_{K} f d \mu\right)^{2} \leq f_{K}\left\langle\left(\nabla^{2} V+T\right)^{-1} \nabla f, \nabla f\right\rangle d \mu .
$$

Proof. Recall Hörmander's lemma 7.12: if $\int(L u)^{2} d \nu \geq\|\nabla u\|^{2}$, then $\operatorname{var}_{\nu}(f) \leq\|\nabla f\|_{*}^{2}$. Here $d \nu=e^{-w} d \mu$ for some convex $w, \int d \nu=1$. We have

$$
\begin{aligned}
\int(L u)^{2} d \nu & =\int\left\|\nabla^{2} u\right\|^{2} d \nu+\int\left\langle\left(\nabla^{2} V+\nabla^{2} w\right) \nabla u, \nabla u\right\rangle d \nu \\
& \geq \int\left\|\nabla^{2} u\right\|^{2} d \nu+\int\left\langle\left(\nabla^{2} V\right) \nabla u, \nabla u\right\rangle d \nu \\
& \geq \int\langle T \nabla u, \nabla u\rangle+\left\langle\left(\nabla^{2} V\right) \nabla u, \nabla u\right\rangle d \nu \\
& =\int\left\langle\left(T+\nabla^{2} V\right) \nabla u, \nabla u\right\rangle d \nu \\
& =\|\nabla u\|^{2}
\end{aligned}
$$

where $\|\nabla f\|_{*}^{2}=\int\left\langle\left(T+\nabla^{2} V\right)^{-1} \nabla f, \nabla f\right\rangle d \nu$ (by example 3 from subsection 7.4). It remains to apply this with $d \nu(x)=\frac{1}{\mu(K)} \mathbf{1}_{K} d \mu(x)$.

We conclude this subsection with the following ultimate combination of Propositions 7.25 and 7.26 :

Corollary 7.27. Suppose for a given even log-concave probability measure $d \mu=e^{-V} d x$ on $\mathbb{R}^{n}$, for all even $u \in W^{2,2}(\mu)$, for all symmetric convex sets $K \subset \mathbb{R}^{n}$, we have

$$
\int_{K}\left\|\nabla^{2} u\right\|^{2} d \mu \geq \int_{K}\langle T \nabla u, \nabla u\rangle d \mu
$$

for some positive definite $T$ such that $T x=\nabla V$. Then, the $B$-conjecture is true for this measure $\mu$, i.e. for any symmetric convex $K$ in $\mathbb{R}^{n}$, we have $\mu\left(e^{t} K\right)$ is log-concave in $t>0$.

### 7.11 B-conjecture for Rotation-Invariant Measures.

Finally, we outline the proof, due to Cordero-Erausquin and Rotem [49], of the B-conjecture for rotation-invariant log-concave measures, which is the state of the art in the general topic of the $L 2$ method.

Definition 7.28. A measure $\mu$ is rotation-invariant if for all measurable $\Omega \subset \mathbb{R}^{n}$, we have $\mu(\Omega)=\mu(R \Omega)$ where $R$ is a rotation operator.

Some examples include the Gaussian, $e^{-|x|}, e^{-|x|^{p} / p}$, and unif $\left(B_{2}^{n}\right)$.
The following result is a fundamental new estimate which has already found more than one application:

Theorem 7.29 (Cordero-Erausquin and Rotem (2021.)). If $d \mu=e^{-V} d x$ is a log-concave probability measure, $V(x):=v(|x|)$ where $v: \mathbb{R}^{+} \rightarrow \mathbb{R}$, then for all odd functions $g \in W^{1,2}(\mu)$ and all symmetric convex sets $K$, we have

$$
\int_{K}|\nabla g|^{2} d \mu \geq \int_{K} \frac{v^{\prime}(|x|)}{|x|} g^{2} d \mu
$$

This Theorem implies:
Corollary 7.30. The B-conjecture is true for all rotation-invariant log-concave measures. In other words, for all log-concave rotation-invariant probability measures $\mu$ on $\mathbb{R}^{n}$, and all symmetric convex sets $K$, the map $t \mapsto \mu\left(e^{t} K\right)$ is log-concave in $t$.

Proof of Corollary 7.30 from Theorem 7.29. By Corollary 7.27 it is enough to show, for some positive-definite $T$ with $T x=\nabla V$,

$$
\begin{equation*}
\int_{K}\left\|\nabla^{2} u\right\|^{2} d \mu \geq \int_{K}\langle T \nabla u, \nabla u\rangle d \mu \tag{41}
\end{equation*}
$$

for all even $u$. We take $T=\frac{v^{\prime}(|x|)}{|x|} \cdot \operatorname{Id}$, and then indeed $T x=v^{\prime}(|x|) \cdot \frac{x}{|x|}=\nabla V=\nabla v(|x|)$ as required.

We apply Theorem 7.29 to all the partial derivatives $\partial_{i} u$ of $u, 1 \leq i \leq n$. We get

$$
\sum_{i=1}^{n} \int_{K}\left(\nabla \partial_{i} f\right)^{2} d \mu \geq \sum_{i=1}^{n} \int_{K} \frac{v^{\prime}(|x|)}{|x|}\left(\partial_{i} u\right)^{2} d \mu
$$

from which (41) follows.
We shall now prove the key Theorem 7.29. The proof will be based on two lemmas.
Lemma 7.31 (One-dimensional inequality). If $w, v:[0, \infty) \rightarrow \mathbb{R}$ are continuous functions, that are $C^{1}$ on $(0, \infty)$, and $f$ is compactly supported, $f(0)=0$, and $f \in C^{1}([0, \infty))$, then for all $\alpha \geq 0$,

$$
\int_{0}^{\infty} \frac{v^{\prime}(t)}{t} f^{2} t^{\alpha} e^{-w-v} d t \leq \int_{0}^{\infty}\left(\left(f^{\prime}\right)^{2}+\alpha(f / t)^{2}-v^{\prime} f^{2} / t\right) t^{\alpha} e^{-w-v} d t
$$

Remark 7.32. If $\alpha=0$, then

$$
\int_{0}^{\infty} \frac{v^{\prime}(t)}{t} f^{2} d \nu \leq \int_{0}^{\infty}\left(\left(f^{\prime}\right)^{2}-v^{\prime} f^{2} / t\right) d \nu \leq \int_{0}^{\infty}\left(f^{\prime}\right)^{2} d \nu
$$

if $f$ is our odd function, thus this Lemma implies Theorem 7.29 in dimension 1 right away.
Proof of Lemma 7.31. We approximate $f$ with a $C^{2}$ function. Let $f(t)=t g(t)$. Since $f(0)=0, g$ is continuous, and $g \in C^{1}(0, \infty)$.

The left hand side is

$$
\begin{aligned}
\int_{0}^{\infty} \frac{v^{\prime}(t)}{t} f^{2} t^{\alpha} e^{-w-v} d t & =\int_{0}^{\infty} v^{\prime}(t) g^{2} t^{\alpha+1} e^{-w-v} d t \\
& =-\int_{0}^{\infty}\left(g^{2} t^{\alpha+1} e^{-w}\right)\left(e^{-v}\right)^{\prime} d t \\
& =\int_{0}^{\infty} e^{-v}\left(2 g g^{\prime} t^{\alpha+1} e^{-w}+(\alpha+1) g^{2} t^{\alpha} e^{-w}-w^{\prime} g^{2} t^{\alpha+1} e^{-w}\right) d t \\
& =\int_{0}^{\infty}\left(2 g g^{\prime} t+(\alpha+1) g^{2}-w^{\prime} g^{2} t\right) t^{\alpha} e^{-w-v} d t
\end{aligned}
$$

where the last equality follows from integration by parts.
The right hand side is

$$
\begin{aligned}
\int_{0}^{\infty}\left(\left(f^{\prime}\right)^{2}+\alpha(f / t)^{2}-v^{\prime} f^{2} / t\right) t^{\alpha} e^{w-v} d t & =\int_{0}^{\infty}\left(\left(g+t g^{\prime}\right)^{2}+\alpha g^{2}-w^{\prime} t g^{2}\right) t^{\alpha} e^{-w-v} d t \\
& =\int_{0}^{\infty}\left(g^{2}+2 t g g^{\prime}+t^{2}\left(g^{\prime}\right)^{2}+\alpha g^{2}-w^{\prime} t g^{2}\right) t^{\alpha} e^{w-v} d t
\end{aligned}
$$

The difference is

$$
\text { RHS }-\mathrm{LHS}=\int_{0}^{\infty} t^{2}\left(g^{\prime}\right)^{2} t^{\alpha} e^{-w-v} d t \geq 0
$$

which completes the proof.
Next, the following result is a consequence of the log-concavity of $\mu$, and does not formally rely on the evenness assumption. It was formally deduced, in a much greater generality, by Kolesnikov and Milman [114].

Lemma 7.33. Let $w: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a convex $C^{1}$ function on $\mathbb{R}^{n}$ and $d \mu=e^{-w(\theta)} d \theta$ on $S^{n-1}$. Then, for any $C^{1}$ function $g: S^{n-1} \rightarrow \mathbb{R}$ such that $\int_{S^{n-1}} g d \mu=0$, we have

$$
\int_{S^{n-1}}(n-1-\langle\nabla w(\theta), \theta\rangle) g^{2} d \mu \leq \int_{S^{n-1}}\left|\nabla_{S^{n-1}} g\right|^{2} d \mu
$$

Here, $\nabla_{S^{n-1}} g=\nabla g-\langle\nabla g(\theta), \theta\rangle \theta$ if $g \in C^{1}\left(\mathbb{R}^{n}\right)$.

Proof Sketch for Lemma 7.33. Let $K_{t}:=(1-t) B_{2}^{n}+t L$ for some convex body $L$. Then, we can write $h_{K_{t}}(\theta)=1+t g(\theta)$ for small enough $t$. Also, by the Prekopa-Leindler inequality,

$$
\mu\left(K_{t}\right)=\mu\left((1-t) B_{2}^{n}+t L\right) \geq \mu\left(B_{2}^{n}\right)^{1-t} \mu(L)^{t}
$$

Therefore, $\frac{d^{2}}{d t^{2}} \log \mu\left(K_{t}\right) \leq 0$ at $t=0$, which yields

$$
\begin{equation*}
\left.\mu\left(K_{0}\right) \mu^{\prime \prime}\left(K_{t}\right)\right|_{t=0} \leq\left(\left.\mu^{\prime}\left(K_{t}\right)\right|_{t=0}\right)^{2} \tag{42}
\end{equation*}
$$

We have $\mu\left(K_{0}\right)=\mu\left(B_{2}^{n}\right)$, and one might believe that

$$
\left.\mu^{\prime}\left(K_{t}\right)\right|_{t=0}=\left.\frac{d}{d t} \mu\left((1-t) B_{2}^{n}+t L\right)\right|_{t=0}=\int_{S^{n-1}} g d \mu
$$

where $h_{K_{t}}=1+t g$. Further, it was shown by Kolesnikov, Milman [114] that

$$
\left.\mu^{\prime \prime}\left(K_{t}\right)\right|_{t=0}=\int_{S^{n-1}}(n-1-\langle\nabla w(\theta), \theta\rangle) g^{2}-\left|\nabla_{S^{n-1}} g\right|^{2} d \mu
$$

Therefore, by Equation (42), $\left.\mu^{\prime \prime}\left(K_{t}\right)\right|_{t=0} \leq 0$. The proof is complete.
Remark 7.34. Alternatively, one might hope to deduce Lemma 7.33 using the log-concavity of $\int e^{-F^{*}} d \mu$, specifically for the class of rotation-invariant $F$. As we know, this is equivalent to

$$
\int\left\langle\left(\nabla^{2} F\right)^{-1} \nabla \varphi, \nabla \varphi\right\rangle d \nu \geq \operatorname{Var}_{\nu}(\varphi)
$$

where $d \nu=e^{-F-V} d x=e^{-F} d \mu$. Is it possible to select a particular rotation-invariant $F$ in this inequality, and either integrate in polar coordinates, or pass to a limit, and end up with Lemma 7.33? This is left as a home work.
Proof of the Theorem. With $d \nu=e^{-v(|x|)-w(x)} d x$, where $v$ is convex on $[0, \infty]$ and $w$ is even convex on $\mathbb{R}^{n}$, we have

$$
\int_{\mathbb{R}^{n}} \frac{v^{\prime}(|x|)}{|x|} h^{2} d \nu=c_{n} \int_{S^{n-1}} \int_{0}^{\infty} \frac{v^{\prime}(t)}{t} h^{2}(t \theta) t^{n-1} e^{-v(t)-w(t \theta)} d t d \theta
$$

We use Lemma 7.31 with $f_{\theta}(t)=h(t \theta), w_{\theta}(t)=w(t \theta)$ and $\alpha=n-1$. Then, $w_{\theta}^{\prime}(t)=$ $\langle\nabla w(t \theta), \theta\rangle$, and

$$
\boldsymbol{\phi} \leq c_{n} \int_{S^{n-1}} \int_{0}^{\infty}\left(\langle h(t \theta), \theta\rangle^{2}+(n-1)\left(\frac{h(t \theta)}{t}\right)^{2}-\frac{1}{t}\langle\nabla w(t \theta), \theta\rangle \frac{h(t \theta)^{2}}{t}\right) t^{n-1} e^{-v(t)-w(t \theta)} d t d \theta
$$

We use Lemma 7.33 and $\nabla_{S^{n-1}} h(x)=\nabla h(x)-\langle\nabla h, x /| x| \rangle x /|x|$, to get

$$
\begin{aligned}
& c_{n} \int_{S^{n-1}} \int_{0}^{\infty}\left((n-1)\left(\frac{h(t \theta)}{t}\right)^{2}-\frac{1}{t}\langle\nabla w(t \theta), \theta\rangle \frac{h(t \theta)^{2}}{t}\right) t^{n-1} e^{-v(t)-w(t \theta)} d t d \theta \\
& \leq c_{n} \int_{0}^{\infty} t^{n-1} e^{-v(t)} \int_{S^{n-1}}\left|\nabla_{S^{n-1}} h(t \theta)\right|^{2} e^{-w(t \theta)} d \theta d t
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int \frac{w^{\prime}(|x|)}{|x|} h^{2} d \nu & \leq c_{n} \int_{S^{n-1}} \int_{0}^{\infty}\left(\langle h(t \theta), \theta\rangle^{2}+\left|\nabla_{S^{n-1}} h(t \theta)\right|^{2}\right) t^{n-1} e^{-w(t \theta)-v(t)} d t d \theta \\
& =c_{n} \int_{S^{n-1}} \int_{0}^{\infty}|\nabla h(t \theta)|^{2} t^{n-1} e^{-w(t \theta)-v(t)} d t d \theta \\
& =\int_{\mathbb{R}^{n}}|\nabla h(x)|^{2} e^{-w(x)-v(|x|)} d x
\end{aligned}
$$

This concludes the proof of the B-conjecture in the case of rotation-invariant measures.
Remark 7.35. The following claim is left as a home work: $\mu\left(e^{t} K\right)$ is log-concave for any symmetric convex set $K$ and any rotation-invariant log-concave measure $\mu$, if and only if for all log-concave measures $\nu$, and for all $t>0, \nu\left(t B_{2}^{n}\right) \nu\left(\frac{1}{t} B_{2}^{n}\right) \leq \nu\left(B_{2}^{n}\right)^{2}$. In other words, the work of Cordero-Erausquin and Rotem [49] implies a particular case of the "Log-concave Blaschke-Santalo"" conjecture (Conjecture 4.22) that we discussed earlier.

Remark 7.36. In fact, Cordero-Erausquin and Rotem [49] showed that the B-conjecture is true for a larger class of rotation-invariant measures, which goes beyond log-concave measures.

### 7.12 Klartag's improvement of Lichnerowicz's inequality

Recall that the Poincaré constant $C_{\text {Poin }}(\mu)$ is the smallest $c>0$ such that for all locally Lipschitz $f$,

$$
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq c \int|\nabla f|^{2} d \mu
$$

Recall that the KLS conjecture 4.19 states that for any log-concave isotropic probability measure, $C_{\text {Poin }}(\mu) \leq C$, where $C>0$ is an absolute constant independent of the dimension. Equivalently, for any log-concave probability measure,

$$
\begin{equation*}
C_{\text {Poin }}(\mu) \leq C\|\operatorname{Cov}(\mu)\|_{o p}, \tag{43}
\end{equation*}
$$

where $C>0$ is an absolute constant, $\|\cdot\|_{o p}$ stands for an operator norm of a matrix, and $\operatorname{Cov}(\mu)$ is the covariance matrix of $\mu$. Recall that the $(i, j)$-entry of the covariance matrix is given by $\operatorname{Cov}(\mu)_{i j}=\mathbb{E}\left(\left(X_{i}-\mathbb{E} X_{i}\right)\left(X_{j}-\mathbb{E} X_{j}\right)\right)$, where $X$ is a random vector distributed according to $\mu$.

Recall also Lichnerowicz's inequality (which follows, for instance from the Brascamp-Lieb inequality Theorem 3.75, or alternatively from Cafarelli's Theorem 4.50): if $d \mu=e^{-V} d x$ be a log-concave probability measure on $\mathbb{R}^{n}$. with $\nabla^{2} V \geq t \cdot$ Id then one has $C_{\text {Poin }}(\mu) \leq 1 / t$, or in other words, for all locally Lipschitz $f$,

$$
\begin{equation*}
\int f^{2} d \mu-\left(\int f d \mu\right)^{2} \leq \frac{1}{t} \int|\nabla f|^{2} d \mu \tag{44}
\end{equation*}
$$

Recall also that the Gaussian measure minimizes the Poincaré constant among all isotropic measures (see Proposition 6.2). Therefore, if $d \mu$ is such that $\nabla^{2} V \geq$ Id then $\mu$ is not isotropic unless it is Gaussian. Therefore if one would like to use Lichnerowicz's inequality in order to obtain bounds on the Poincaré constant of isotropic log-concave measures (which was a foundation of some powerful recent advances on the subject, see Eldan [58], Lee, Vempala [124], Chen [44], Klartag, Lehec [106]), then something is missing in the estimate. Furthermore, one may note that if for $d \mu=e^{-V} d x$ we have $\nabla^{2} V \geq t \cdot$ Id then

$$
\begin{equation*}
\|\operatorname{Cov}(\mu)\|_{o p} \leq \frac{1}{t} \tag{45}
\end{equation*}
$$

Indeed, this can be seen by applying Lichnerowicz's inequality to linear functions; the details are left to (a one line) home work. The inequality (45) shows that the conjectured KLS bound is in fact stronger than Lichnerowicz's inequality (44), at least up to an absolute constant.

Recently, Klartag [103] improved Lichnerowicz's inequality and obtained the bound which is a geometric average of the Lichnerowicz's bound (44) and the conjectured KLS bound (43):

Theorem 7.37 (Klartag [103]). If $\mu$ is a probability log-concave measure $d \mu=e^{-V} d x$ and $\nabla^{2} V \geq t \cdot$ Id then $C_{\text {Poin }}(\mu) \leq \sqrt{\frac{\|\operatorname{Cov}(\mu)\|_{o p}}{t}}$, where $\|\cdot\|_{o p}$ is operator norm.

Without loss of generality we assume that $\mu$ is barycentered, i.e. $\int x d \mu=0$. Then $\operatorname{Cov}(\mu)=\left(\mathbb{E}_{\mu}\left(X_{i} \cdot X_{j}\right)\right)$. Note also

$$
\begin{equation*}
\|\operatorname{Cov}(\mu)\|_{o p}=\sup _{\theta \in \mathbb{S}^{n-1}}\langle\operatorname{Cov}(\mu) \theta, \theta\rangle=\sup _{\theta} \sum \mathbb{E}_{\mu} X_{i} X_{j} \theta_{i} \theta_{j}=\sup _{\theta \in \mathbb{S}^{n-1}} \int\langle x, \theta\rangle^{2} d \mu . \tag{46}
\end{equation*}
$$

Assume also without loss of the generality that the support of the measure $\mu$ is the whole of $\mathbb{R}^{n}$. Recall that the Poincaré constant $C_{\text {Poin }}(\mu)=\frac{1}{\lambda_{1}}$ where $\lambda_{1}$ is the 1 st eigenvalue of the operator

$$
L u=\Delta u-\langle\nabla V, \nabla u\rangle .
$$

In other words,

$$
\lambda_{1}=\inf _{f \neq 0, f \in W^{1,2}(\mu)} \frac{\int|\nabla f|^{2} d \mu}{\int f^{2} d \mu}=\inf _{f \neq 0, f \in W^{1,2}(\mu)} \frac{\int|\nabla f|^{2} d \mu}{\int f^{2} d \mu-\left(\int f d \mu\right)^{2}}
$$

If the infimum is attained then $\lambda_{1}>0$ is the smallest number such that $\exists f \neq 0$ such that $L_{f}=-\lambda_{1} f$. This function $f$ is called the first eigenfunction of $\mu$. Note that the first eigenfunction may not exist. The measure $d \mu=e^{-|x|} d x$ on $\mathbb{R}$ is an example (see home work). However, modulo technical details which were fully outlined in [103] we assume "by approximation" that $\mu$ has an eigenfunction $f \in W^{2,2}(\mu) \cap C^{\infty}$ (in fact, even a stronger statement, related to $\mu$-tempered eigenfunctions, was obtained in [103]).

The following Lemma is a key step in the proof:

Lemma 7.38. Suppose $f$ is a first eigenfunction of $\mu$ and $\int f d \mu=0$, with $\int f^{2} d \mu=1$. Then
(i)

$$
\left(\int \nabla f d \mu\right)^{2} \geq \frac{1}{\lambda} \int\left\langle\nabla^{2} V \nabla f, \nabla f\right\rangle d \mu
$$

and
(ii)

$$
\left(\int \nabla f d \mu\right)^{2} \leq \lambda^{2} \cdot\|\operatorname{Cov}(\mu)\|_{o p}
$$

Proof of the Lemma. We apply Bochner's formula (Theorem 7.1) to $f$, which states that

$$
\begin{equation*}
\int(L f)^{2}=\int\left\|\nabla^{2} f\right\|^{2}+\left\langle\nabla^{2} V \nabla f, \nabla f\right\rangle d \mu \tag{47}
\end{equation*}
$$

An application of the Poincaré inequality gives that

$$
\begin{equation*}
\int\left|\nabla \partial_{i} f\right|^{2} d \mu \geq \lambda\left(\int\left(\partial_{i} f\right)^{2}-\left(\int \partial_{i} f\right)^{2}\right) \tag{48}
\end{equation*}
$$

Further, summing up (48) over all $i=1, \ldots, n$, and applying the Cauchy-Schwarz inequality, we get

$$
\begin{equation*}
\int\left\|\nabla^{2} f\right\|^{2} d \mu \geq \lambda\left(\int|\nabla f|^{2}-\sum\left(\partial_{i} f\right)^{2}\right) \geq \lambda\left(\int|\nabla f|^{2}-\left(\int \nabla f\right)^{2}\right) \tag{49}
\end{equation*}
$$

In view of (49) and (47) we get

$$
\begin{equation*}
\int(L u)^{2} d \mu \geq \lambda \cdot\left(\int|\nabla f|^{2} d \mu-\left(\int \nabla f d \mu\right)^{2}\right)+\int\left\langle\nabla^{2} V \nabla f, \nabla f\right\rangle d \mu \tag{50}
\end{equation*}
$$

On the other hand, since $f$ is the normalized first eigenfunction, we have

$$
\int(L u)^{2} d \mu=\lambda^{2} \int f^{2} d \mu=\lambda^{2}
$$

and therefore, (50) implies

$$
\begin{equation*}
\lambda^{2} \geq \lambda \int|\nabla f|^{2}-\lambda\left(\int \nabla f\right)^{2}+\int\left\langle\nabla^{2} V \nabla f, \nabla f\right\rangle d \mu \tag{51}
\end{equation*}
$$

Furthermore, since $f$ is the normalized first eigenfunction, we have

$$
1=\int f^{2} d \mu-\left(\int f d \mu\right)^{2}=\frac{1}{\lambda} \int|\nabla f|^{2} d \mu
$$

or in other words, $\int|\nabla f|^{2} d \mu=\lambda$. Combining this fact with (51) we get the first part of the Lemma.

For part (ii), note that for all $\theta \in \mathbb{S}^{n-1}$ one has, using integration by parts along with the equation $L f=-\lambda f$ :

$$
\begin{aligned}
\int\langle\nabla f, \theta\rangle d \mu=\int\langle\nabla f, \nabla\langle x, \theta\rangle\rangle d \mu & =\int-L f \cdot\langle x, \theta\rangle d \mu \\
& =\int \lambda f\langle x, \theta\rangle d \mu \\
& \leq \lambda\|f\|_{L^{2}(\mu)} \cdot\left(\int\langle x, \theta\rangle^{2} d \mu\right)^{1 / 2} \\
& =\lambda\left(\int\langle x, \theta\rangle^{2} d \mu\right)^{1 / 2}
\end{aligned}
$$

Recall that $\|\operatorname{Cov}(\mu)\|_{o p}=\sup _{\theta \in \mathbb{S}^{n-1}} \int\langle x, \theta\rangle^{2} d \mu$, and thus we get

$$
\begin{aligned}
\left(\int \nabla f d \mu\right)^{2}=\sup _{\theta \in \mathbb{S}^{n-1}}\left(\int\langle f, \theta\rangle d \mu\right)^{2} & \leq \lambda^{2} \sup _{\theta \in \mathbb{S}^{n-1}} \int\langle x, \theta\rangle^{2} d \mu \\
& =\lambda^{2}\|\operatorname{Cov}(\mu)\|_{o p}
\end{aligned}
$$

Proof of Klartag's Theorem. Combining both parts of the Lemma 7.38 gives

$$
\lambda^{2}\|\operatorname{Cov}(\mu)\|_{o p} \geq \frac{1}{\lambda} \int\left\langle\nabla^{2} V \nabla f, \nabla f\right\rangle d \mu \geq \frac{t}{\lambda} \int|\nabla f|^{2} d \mu
$$

where we used that $\nabla^{2} \nu \geq t \cdot$ Id. Recall that $\int|\nabla f|^{2} d \mu=\lambda$. The Theorem now follows if we recall that $C_{\text {poin }}(\mu)=\frac{1}{\lambda}$.

Remark 7.39. Klartag used Theorem 7.37, in conjunction with the Stochastic Localization technique (pioneered by Eldan [58] and developed in this context by Lee, Vempala [124], Chen [44], Klartag, Lehec [106]), to deduce the best to date bound regarding the KLS conjecture 4.19: he proved that for any isotropic log-concave probability measure $\mu$, the Poincaré constant $C_{\text {Poin }}(\mu)$ is bounded above by $C \sqrt{\log n}$, where $C>0$ is an absolute constant that does not depend on the dimension.

Let us conclude with the following series of remarks. Note that linear functions $f(x)=$ $\langle x, \theta\rangle$ form the space of the first eigenfunctions for the Gaussian measure. Another example when the first eigenfunction is understood is the uniform probability measure on a
coordinate parallelepiped: the eigenfunctions are given by $f(x)=\sin \left(t_{i} x_{i}\right)$ for each $i$, with the appropriate choice of $t_{i}$. In both of these cases, the first eigenfunctions are odd and "one-dimensional" (that is, they only depend on one variable rather than $n$ variables).

Perhaps inspired by these examples as well the considerations similar to the ones from Lemma 7.38, Klartag implicitly made a conjecture:

Conjecture 7.40 (Klartag [103]). Suppose $\mu$ is a barycentered isotropic log-concave probability measure such that its first eigenfunction $f$ exists and is smooth (and tempered, see [103] for details). Then $f$ has a "preferred direction", i.e.

$$
\left(\int \nabla f d \mu\right)^{2} \geq c_{0} \int|\nabla f|^{2} d \mu
$$

Here $c_{0}$ is an absolute constant that does not depend on dimension or $\mu$.
We leave it as a homework to deduce that this conjecture would imply the KLS conjecture.

### 7.13 The Dimensional Brunn-Minkowski Conjecture

We close with a discussion of the following Conjecture, first made in a partial case by Gardner, Zvavitch [69], and then formulated in general by Colesanti, Livshyts, Marsiglietti [47]:

Conjecture 7.41 (Dimensional Brunn-Minkowski conjecture). Suppose $\mu$ is an even logconcave probability measure, and $K, L$ are symmetric convex sets. Take $\lambda \in[0,1]$. Then

$$
\begin{equation*}
\mu(\lambda K+(1-\lambda) L)^{1 / n} \geq \lambda \mu(K)^{1 / n}+(1-\lambda) \mu(L)^{1 / n} \tag{52}
\end{equation*}
$$

## Remarks and History

- We know by Prekopa-Leindler inequality (Theorem 3.20) that

$$
\begin{equation*}
\mu(\lambda K+(1-\lambda) L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda} \tag{53}
\end{equation*}
$$

Note that this is a weaker inequality than the statement of Conjecture 7.41, since for any $a, b>0$ and $p \geq q>0$ one has

$$
\left(\lambda a^{p}+(1-\lambda) b^{p}\right)^{1 / p} \geq\left(\lambda a^{q}+(1-\lambda) b^{q}\right)^{1 / q} \rightarrow_{q \rightarrow 0} a^{\lambda} b^{1-\lambda}
$$

In other words, the inequality

$$
\begin{equation*}
\mu(\lambda K+(1-\lambda) L)^{p} \geq \lambda \mu(K)^{p}+(1-\lambda) \mu(L)^{p} \tag{54}
\end{equation*}
$$

is stronger when $p>0$ is larger.

- As we discussed before, if $\mu$ is the Lebesgue measure then by homogeneity (52) and (53) are equivalent.
- Symmetry (or some structural assumption) is important. Indeed, If $K=B_{2}^{n}$ and $L=B_{2}^{n}+R e_{1}$ for large $R>0$ (so that $L$ is a ball shifted far away from the origin), then $\mu\left(\frac{K+L}{2}\right) \rightarrow_{R \rightarrow \infty} 0$ while $\mu(K)$ remains fixed. Therefore, (52) fails for this $K$ and $L$. Furthermore, this example shows that the inequality (54) cannot hold for all convex sets, without additional assumptions, for no $p>0$.
- If $p>\frac{1}{n}$ then we cannot hope to get (54) for all symmetric convex sets. This follows from the fact that the Lebesgue measure approximates any even log-concave measure $\mu$ near the origin, thus one cannot hope to have a larger $p$ in (54) than in the Lebesgue case (which is sharp).
- However, in some cases one may hope to have (54) for convex sets $K$ and $L$ with $\mu(K), \mu(K) \geq a$, for some $a \in[0,1]$, with the exponent $p=p(a) \rightarrow_{a \rightarrow 1} \infty$. Indeed, this is the case for the Gaussian measure $\gamma$, as follows from the Ehrhard inequality (see home work). This is also the case for strictly log-concave even measures, as was shown by Livshyts [137].
- Gardner, Zvavitch [69], who first formulated Conjecture 7.41 in the case of the Gaussian measure, showed that in the case $n=1$ the conjecture is true.
- Gardner, Zvavitch [69] also asked if it is enough to assume that the convex sets $K$ and $L$ contain the origin, for Conjecture 7.41 to be true in the case of the Gaussian measure. However, Nayar and Tkocz [147] proved that this is not the case: they constructed an example on the plane of $K$ and $L$ both containing the origin however the optimal $p$ in (54) for them is approximately $0.3<0.5$. See the picture below of their counterexample:

- Livshyts, Marsighetti, Nayar and Zvavitch [134], and later Hosle, Kolesnikov, Livshyts [86] showed that Conjecture 7.41 follows from the Log Brunn-Minkowski Conjecture 4.63. Therefore, Conjecture 7.41 is true in dimension 2. Also, Conjecture 7.41 is true if the log-concave measure $\mu$ and the convex sets $K, L$ are all unconditional.

Below, we shall discuss the following results concerning Conjecture 7.41. As a first application of the $L_{2}$ approach to this problem, it was shown:

Theorem 7.42 (Kolesnikov, Livshyts [117]). For all convex sets $K, L$ containing the origin and any $\lambda \in[0,1]$,

$$
\gamma(\lambda K+(1-\lambda) L)^{\frac{1}{2 n}} \geq \lambda \gamma(K)^{\frac{1}{2 n}}+(1-\lambda) \gamma(L)^{\frac{1}{2 n}}
$$

where $\gamma$ is the standard Gaussian measure.
Note that the sets are not required to be symmetric in this Theorem, although the exponent is $\frac{1}{2 n}$ rather than $\frac{1}{n}$ (which is weaker). This result was recently extended by Aishwarya, Rotem [1] to $p$-homogeneous potentials, and especially interestingly, they showed that the assumption of the convexity of the sets can be dropped! We will discuss their extension as well.

Next, we state the remarkable
Theorem 7.43 (Cordero-Erausquin, Rotem [49]). Suppose $\mu$ is rotation-invariant and logconcave and suppose that $K$ and $L$ are symmetric and convex sets. Let $\lambda \in[0,1]]$. Then

$$
\mu(\lambda K+(1-\lambda) L)^{\frac{1}{n}} \geq \lambda \mu(K)^{\frac{1}{n}}+(1-\lambda) \mu(L)^{\frac{1}{n}}
$$

In other words, Conjecture 7.41 holds for rotation-invariant log-concave measures.
Earlier, this result was proved in the Gaussian case $\mu=\gamma$ by Eskenazis and Moschidis [61], and a key idea from their argument was also used by Cordero-Erausquin and Rotem [49]. The key new ingredient in the work of [49] was Theorem 49, the same key fact which allowed Cordero-Erausquin and Rotem [49] to prove the B-conjecture in the rotation-invariant case. Furthermore, we shall see that the following more general fact is true:

Proposition 7.44. Let $\mu$ be an even log-concave probability measure. Suppose for all even functions $u \in W^{2,2}(\mu)$ one has

$$
\int\|\nabla u\|^{2} d \mu \geq \int\langle\nabla T \nabla u, \nabla u\rangle d \mu
$$

for some non-negative definite matrix $T$ (that depends on $x$ ) such that $T x=\nabla V$. Then Conjecture 7.41 is true for this measure $\mu$.

One should compare Proposition 7.44 to Proposition 7.26 , which states that the same exact condition would imply the B-conjecture for a given measure $\mu$.

Lastly, the following result shows that indeed the log-concavity (guaranteed by PrekopaLeindler's inequality) can be improved to power-concavity for all even log-concave probability measures, and for symmetric convex sets.

Theorem 7.45 (Livshyts [137]). For all even log-concave measures $\mu$ and symmetric convex sets $K, L$ one has

$$
\mu(\lambda K+(1-\lambda) L)^{c_{n}} \geq \lambda \mu(K)^{c_{n}}+(1-\lambda) \mu(L)^{c_{n}}
$$

where

$$
c_{n}=c n^{-4}(\log n)^{-1 / 2}
$$

TBC... all the proofs will be added soon.

### 7.14 Home work

Question 7.46 (1 point). Outline a second proof of Bochner's identity, via the change of variables $x=y+t \nabla u(y)$, and taking the second derivative, that we briefly discussed in class.

Question 7.47 (1 point). a) Consider the Banach space $X=W^{1,2}(\mu) \times \ldots \times W^{1,2}(\mu)(n$ times), let $F=\left(f_{1}, \ldots, f_{n}\right) \in X$ and consider

$$
\|F\|=\sqrt{\int\langle A F, F\rangle d \mu}
$$

where $A=A(x)$ is a positive definite matrix of functions in $W^{1,2}(\mu)$. Show that $\|F\|$ is a norm.
b) Show that its dual norm is

$$
\|F\|_{*}=\sqrt{\int\left\langle A^{-1} F, F\right\rangle d \mu}
$$

Question 7.48 (2 points). In class, we showed that for $\mu$, a finite Borel measure, and a bounded measurable function $h$ with $\int h d \mu=0, \varepsilon>0$, and $\mu_{\varepsilon}$ such that $d \mu_{\varepsilon}=(1+\varepsilon h) d \mu$, we have

$$
\|h\|_{H^{-1}(\mu)}=\liminf _{\varepsilon \rightarrow 0} \leq \frac{W_{2}\left(\mu, \mu_{\varepsilon}\right)}{\varepsilon}
$$

Show that the $\geq$ inequality also holds.

Question 7.49 (5 points). Suppose $\mu$ is an isotropic unconditional log-concave probability measure and $\psi$ - an unconditional $W^{1,2}(\mu)$ function. Construct a transport map $T$ from $\mu$ to $\left(1+\epsilon \partial_{i} \psi\right) \mu$ such that

$$
\int|T x-x|^{2} d \mu(x) \leq C \log n \int\left|\partial_{i} \psi\right|^{2} d \mu
$$

Conclude Klartag's $C \log n$ bound for the KLS conjecture in the case of unconditional logconcave measures, using also several tools that we discussed the class.

Question 7.50 (1 point). Explain why the conclusion of the Question implies the $B$ conjecture for unconditional log-concave measures.
Question 7.51 (1 point). Prove the reverse implication in the Lemma of Hörmander (reverse to the one we proved in class.)
Question 7.52 (2 points). Deduce (using the generalized Bochner formula that we discussed in class) the following extension of Brascamp-Lieb inequality: let $K$ be a convex set, $\mu$ - logconcave measure in $\mathbb{R}^{n}$ with potential $V$ such that $\mu(K)=1$, and let $g$ be a concave function on K. Then

$$
\int g f^{2} d \mu-\left(\int g f d \mu\right)^{2} \leq \int g\left\langle\left(\nabla^{2} V\right)^{-1} \nabla f, \nabla f\right\rangle d \mu
$$

Question 7.53 (1 point). Let $\mu$ be an even log-concave measure on $\mathbb{R}^{n}$. Show that the fact that $\mu\left(e^{t} K\right)$ is log-concave in $t>0$ for any symmetric convex $K$ is equivalent to the fact that for any symmetric convex $K$ one has

$$
\frac{1}{\mu(K)} \int_{K}\langle\nabla V, x\rangle^{2} d \mu-\left(\frac{1}{\mu(K)} \int_{K}\langle\nabla V, x\rangle d \mu\right)^{2} \leq \frac{1}{\mu(K)} \int_{K}\left\langle\nabla^{2} V x, x\right\rangle+\langle\nabla V, x\rangle d \mu .
$$

Question 7.54 (2 points). Show that the fact that $\mu\left(e^{t} K\right)$ is log-concave in $t$ for any symmetric convex $K$ and any rotation-invariant log-concave measure $\mu$ is equivalent to the fact that for any even log-concave measure $\nu$ one has

$$
\nu\left(t B_{2}^{n}\right) \nu\left(t^{-1} B_{2}^{n}\right) \leq \nu^{2}\left(B_{2}^{n}\right) .
$$

In other words, the result of Cordero-Erasquin and Rotem implies the conjectured log-concave Blaschke-Santalo inequality in a very partial case.

Question 7.55 (4 points). As discussed in class, prove Lemma 2 (from November 29) using the Brascamp-Lieb inequality.

Question 7.56 ( 1 point). Let $\mu$ be a log-concave probability measure on $\mathbb{R}^{n}$ with density $e^{-V}$ such that $\nabla^{2} V \geq t \cdot I d$. Suppose (for simplicity) that $\int x d \mu=0$ (the barycenter is at the origin). Recall that the covariance matrix is then $\operatorname{Cov}(\mu)=\left(\mathbb{E}_{\mu} X_{i} X_{j}\right)$. Prove that $\|\operatorname{Cov}(\mu)\| \leq \frac{1}{t}$.

Hint: Recall that the operator norm of the covariance matrix is $\sup _{\theta \in \mathbb{S}^{n-1}} \int\langle x, \theta\rangle^{2} d \mu$ and use similar ideas to the ones we used when showing that the Gaussian measure minimizes the Poincaré constant among the isotropic measures.

Question 7.57 (2 points). Show that there are log-concave probability measures for which the first eigenfunction does not exist.
Hint: consider $d \mu=\frac{1}{2} e^{-|x|} d x$ on $\mathbb{R}$.
Question 7.58 (1 point). Recall Klartag's "preferred direction" conjecture that we discussed in class: there exists an absolute constant $c>0$ such that if $\mu$ is an isotropic log-concave probability measure such that its first eigenfunction exists and is $C^{2}$, then

$$
\left(\int \nabla f d \mu\right)^{2} \geq c \int|\nabla f|^{2} d \mu
$$

Show that this conjecture implies the KLS conjecture (which states that the Poincaré constant of an isotropic log-concave probability measure is bounded from above by an absolute constant.)

Question 7.59 (4 points). Prove Klartag's "preferred direction" conjecture from Question 7.58 in dimension 1.

Question 7.60 ( 8 points). Try to find a lower bound for the $c>0$ in Klartag's "preferred direction" conjecture from Question 7.58 in all dimensions; it is OK if it depends on $n-$ what is the largest bound that you can get?

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[^0]:    ${ }^{1}$ Recall that we say that a random vector $X$ is isotropic if $\mathbb{E} X=0$ and for any $\theta \in S^{n-1}, \mathbb{E}\langle x, \theta\rangle^{2}=1$.

