HOME WORK FOR THE TOPICS COURSE IN CONCENTRATION OF MEASURE PHENOMENA AND CONVEXITY, FALL 2023, GEORGIA TECH

GALYNA V. LIVSHYTS

Please upload solutions via Canvas in pdf anytime, any number of times. While one only needs 5 points total to pass the course with an A, interested students are encouraged to solve more problems. The deadline for the home work is December 4.

If you find typos or have questions, please let me know!

The problems will keep being added throughout the semester, I anticipate their number reach about a 100 or more, with potential 200 (or so) points available. The problems are always added in the end of each section, so the number of a problem stays the same; however, problems could be added to various sections at any time.

You are encouraged to take a note of the home work questions, since a chunk of worthwhile material will be left as a home work, in order to keep the course going with some pace.

1. Laplace method and basic volume computations in high dimensions

Question 1.1 (1 point)**.** Let $F:\mathbb{R}\to\mathbb{R}$ be a real-valued function, and m be a positive number. We make the following assumptions.

- F attains the absolute maximum at the point s_0, and for every $s\neq s_0$ we have $F(s)<F(s_0)$.
- Further, assume that there exist numbers $a,b>0$ such that $F(s)<F(s_0)-b$ whenever $|s-s_0|>a$.
- Suppose that the integral $\int e^{F(s)}ds<\infty$.
- Suppose that F is twice differentiable in some neighborhood of s_0.
- Suppose that $F''(s_0)<0$.

Prove that when $m\to\infty$, the integral

$$\int e^{mF(s)}ds=(1+o(1))e^{mF(s_0)}\frac{\sqrt{2\pi}}{\sqrt{-mF''(s_0)}}.$$

Hint 1: Observe that WLOG $s_0=F(s_0)=0$, and that F is equal to $-\infty$ outside of the support.

Hint 2: Pick any $\epsilon>0$ and note that one may find a $\delta>0$ so that for all $s\in(-\delta,\delta)$ we have

$$|F(s)-\frac{F''(0)s^2}{2}|\leq\epsilon.$$

Hint 3: Find an estimate for

$$\int_{-\delta}^{\delta}e^{mF(s)}ds.$$

Hint 4: Note that the assumptions imply that for every $\delta>0$ there is $\eta(\delta)>0$ such that $F(s)<F(s_0)-\eta(\delta)$;

Hint 5: Find an estimate for $\int_\delta^\infty e^{mF(s)}ds$ and $\int_{-\infty}^{-\delta}e^{mF(s)}ds$; to do that, use the previous hint, and also note that $e^{mF(s)}=e^{(m-1)F(s)}e^{F(s)}$. Use the assumption about the converging integral as well.

Hint 6: Carefully make sure that the assumptions allow you to let $m\to\infty$ and $\epsilon\to 0$.

Question 1.2 (1 point)**.** All the questions below require an answer up to a multiplicative factor of $1+o(1)$, when $n\to\infty$.

a) Find $\frac{|B_2^n|_n}{|B_2^{n-1}|_{n-1}}$.

Hint: Use the formula from Question 1 and the Fubbini theorem. Note that this method is alternative to the one we used in class to express $|B_2^n|_n$.

b) Find the volume of
$$\{x \in \mathbb{R}^n : |x| \leq 2, x_1 \in [a, b]\},$$
where b1) $a = 0$, $b = 0.1$; b2) $a = -\frac{1}{\sqrt{n}\log n}$, $b = \frac{1}{n}$.

Hint: Use the expression for $|B_2^k|_k$ which we derived in class.

c) Using any method you like, find the volume of
$$conv(\{x \in \mathbb{R}^n :\ |x| < 3, x_2 = 0\} \cup \{x \in \mathbb{R}^n :\ |x - e_2| < 1, x_2 = 1\}).$$

d) Let γ be the standard Gaussian measure on $\mathbb{R}^n$ with density $\frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}}$. For each $t \in (0, \infty)$, find $\gamma(\{x : |x| > t\})$, depending on t (find the best approximation you can for each range).

e) Let μ be the **probability** measure with density $C(n)e^{-|x|^3}$. Find $C(n)$.

f) Let μ be as above. Let $R \in (0, \infty)$ be such that $\mu(RB_2^n) = \frac{1}{2}$. Find R.

Question 1.3 (1 point)**.** Let A be a convex set in $\mathbb{R}^n$ satisfying $x_1 = 0$ for all $x \in A$. Find the volume of $conv(A, Re_1)$, in terms of $|A|_{n-1}$, R and n

Question 1.4 (1 point)**.** a) Using Laplace's method, prove that at least 99% of the volume of the $n-$dimensional Euclidean ball is contained in a strip of width $\frac{100}{\sqrt{n}}$ around any equator, for a sufficiently large n.
b) Prove the same fact on the sphere. (hint: use Fubbini's theorem directly on the sphere, but be careful about how the curvature of the sphere impacts your integral.)

Question 1.5 (2 points)**.** Find a function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every symmetric convex body K in $\mathbb{R}^n$ with $|K|_n = 1$, there exists a vector $u \in \mathbb{S}^{n-1}$ (possibly depending on the body), such that $|K \cap u^\perp|_{n-1} \geq F(n)$. Acceptable answers could be $F(t) = 20t^{-t}$, $F(t) = 5^{-t}$, $F(t) = 3t^{-2}$, $F(t) = \frac{1}{t}$, $F(t) = \frac{10}{\sqrt{t}}$, $F(t) = 100t^{-\frac{1}{4}}$, $F(t) = \frac{1}{\log t}$, $F(t) = 0.00001$, $F(t) = \sqrt{2}$, etc.

2. Basic Convexity, Brunn-Minkowski inequality, Steiner symmetrizations, Prekopa-Leindler inequality

Question 2.1 (1 point)**.** Prove that for any convex **body** K in $\mathbb{R}^n$ and for any point $x \in \mathbb{R}^n \setminus K$, there exists a vector $\theta \in \mathbb{S}^{n-1}$ and a number $\rho \in \mathbb{R}$ such that $\langle x, \theta\rangle > \rho$ and for all $y \in K$, $\langle y, \theta\rangle < \rho$. (this is a finite-dimensional version of the Khan-Banach Theorem)

Question 2.2 (1 point)**.** Prove that a convex hull of a finite number of points in $\mathbb{R}^n$ either has an empty interior, or can be expressed as an intersection of a finite number of half spaces.

Question 2.3 (1 point)**.** Show that the Minkowski functional of a symmetric convex body is a norm on $\mathbb{R}^n$.

Question 2.4 (1 point)**.** Show that for a convex set Q containing the origin we have
$$|Q| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_Q^n(\theta) d\theta = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\theta\|_Q^{-n} d\theta.$$

Question 2.5 (1 point)**.** a) Show that for any pair of convex bodies K, L we have
$$h_{K+L}(x) = h_K(x) + h_L(y).$$
b) Show that for $a > 0$, $h_K(ax) = ah_K(x) = h_{aK}(x)$.
c) Pick $v \in \mathbb{R}^n$. Show that $h_{[-v,v]}(x) = |\langle v, x\rangle|$. Here $[-v, v]$ is the interval connecting vectors $-v$ and v.

Question 2.6 (1 point)**.** Below S_u stands for Steiner symmetrization with respect to $u^\perp$; K stands for a convex body in $\mathbb{R}^n$ with non-empty interior. Show that
a) $S_u(aK) = aS_uK$ for all $a > 0$;
b) If $K \subset L$ then $S_u(K) \subset S_u(L)$; conclude that $S_u(K)$ is continuous with respect to Hausdorf metric;
c) $S_u(K) + S_u(L) \subset S_u(K + L)$.

Question 2.7 (1 point)**.** Recall that for a compact set $A \subset \mathbb{R}^n$, the diameter

$$diam(A) = max_{x,y\in A}|x-y|.$$

Prove that

$$diam(S_u(K)) \leq diam(K).$$

Conclude the *isodiametric* inequality: if the volume of a set is fixed, its diameter is minimized by a Euclidean ball.

Question 2.8 (1 point)**.** Prove that the Steiner symmetrization decreases the perimeter of a convex set. Note that this gives another proof of the isoperimetric inequality for convex sets.

Question 2.9 (1 point)**.** Recall that for a convex set $K \subset \mathbb{R}^n$, the in-radius of K is

$$r(K) = \sup\{t > 0 : \exists y \in \mathbb{R}^n : y + tB_2^n \subset K\},$$

and the circum-radius of K is

$$R(K) = \inf\{t > 0 : \exists y \in \mathbb{R}^n : K \subset y + tB_2^n\}.$$

a) Prove that $r(S_u(K)) \geq r(K)$.
b) Prove that $R(S_u(K)) \leq R(K)$.
Conclude that the Euclidean ball maximizes the in-radius and minimizes the circum-radius when the volume is fixed.

Question 2.10 (2 points)**.** Prove the Urysohn inequality. Define *mean width of a convex body K* as

$$w(K) = \frac{2}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} h_K(\theta)d\theta.$$

Show that if $|K| = |B_2^n|$ then $w(K) \geq 2$.
Hint: use the Brunn-Minkowski inequality and Steiner symmetrizations.

Question 2.11 (1 point)**.** Fix Borel measurable sets $K, L \subset \mathbb{R}^n$. Confirm what we discussed in class: the validity for every $\lambda \in [0,1]$ of the inequality

$$|\lambda K + (1-\lambda)L| \geq |K|^\lambda |L|^{1-\lambda}$$

implies the validity of

$$|\lambda K + (1-\lambda)L|^{\frac{1}{n}} \geq \lambda|K|^{\frac{1}{n}} + (1-\lambda)|L|^{\frac{1}{n}}.$$

Question 2.12 (1 point)**.** Show that for $a, b > 0$, one has $(\lambda a^p + (1-\lambda)b^p)^{\frac{1}{p}} \to_{p\to 0} a^\lambda b^{1-\lambda}$.

Question 2.13 (2 points)**.** a) Let $p \geq -\frac{1}{n}$, and suppose functions f, g and h on $\mathbb{R}^n$ satisfy

$$h(\lambda x + (1-\lambda)y) \geq ((1-\lambda)f^p(x) + \lambda g^p(y))^{\frac{1}{p}}.$$

Show that

$$\int h \geq \left((1-\lambda)\left(\int f\right)^{\frac{p}{np+1}} + \lambda\left(\int g\right)^{\frac{p}{np+1}}\right)^{\frac{np+1}{p}}.$$

Hint: try, for example, a similar proof to Lyusternik's proof of the Brunn-Minkowski inequality. b) Conclude that if a measure's density is supported on a convex set with non-empty interior and is p-concave, then the measure is $\frac{p}{np+1}$−concave.
c) Deduce that if the density of a measure μ on $\mathbb{R}^n$ is $p-$concave, then the density of a marginal measure $\pi_H(\mu)$ is $\frac{p}{kp+1}$-concave, if H is an $(n-k)$-dimensional subspace (note that this is a generalization of Brunn s principle).

Question 2.14 (2 points)**.** We say that a function f in $\mathbb{R}^n$ is *unconditional* if it is invariant under coordinate reflections. That is, $f(\epsilon_1 x_1, ..., \epsilon_n x_n) = f(x)$ for any choice of $\epsilon_i \in \{-1, 1\}$. A set K is called unconditional if 1_K is an unconditional function.

Suppose K is an unconditional convex body and V is an unconditional convex function in $\mathbb{R}^n$. Denote $d\mu(x) = e^{-V(x)}dx$. Show that $\log \mu(e^t K)$ is a concave function in $t \in \mathbb{R}$.
Hint: pass the integration from $\mathbb{R}^n$ to the set $\{x \in \mathbb{R}^n : \forall i = 1, ..., n,\ x_i \geq 0\}$, and make a change of variables in the Prekopa-Leindler inequality given by $(x_1, ..., x_n) = (e^{t_1}, ..., e^{t_n})$.

Question 2.15 (1 point)**.** a) Prove Minkowski's first inequality: $V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}$ (similar to the isoperimetric inequality which we deduced in class.)
b) Prove Minkowski's quadratic inequality: for convex bodies K and L in $\mathbb{R}^n$,

$$V_2(K, L)|K| \leq V_1(K, L)^2.$$

Hint: use the Brunn-Minkowski inequality to obtain some information about $\frac{d^2}{dt^2}|K + tL|^{\frac{1}{n}}$.

Question 2.16 (1 point)**.** (this question is added upon Alex's request) Give an example of a (rough, non-convex) set K such that $\lim_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon}$ does not exist, and

$$\liminf_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon} < \limsup_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon}.$$

Question 2.17 (1 point)**.** Show that any convex function $V : \mathbb{R}^n \to \bar{\mathbb{R}}$ is
a) continuous on the support of e^{-V} (i.e. on the set where V does not take infinite values)
b) Of class C^2 almost everywhere on the support of e^{-V}.

Question 2.18 (1 point)**.** a) Suppose $V \in C^2(\mathbb{R}^n)$. Show that for all $z_1, z_2 \in \mathbb{R}^n$,

$$V\left(\frac{z_1 + z_2}{2}\right) + \beta(z_1, z_2) = \frac{V(z_1) + V(z_2)}{2}, \tag{1}$$

where, letting $z(t) = \frac{(1-t)z_1 + (1+t)z_2}{2}$, we have

$$\beta(z_1, z_2) = \frac{1}{8} \cdot \int_{-1}^{1} (1 - |t|) \langle \nabla^2 V(z(t))(z_1 - z_2), z_1 - z_2 \rangle dt \geq 0. \tag{2}$$

b) Conclude that convexity of a C^2-smooth function is equivalent to the non-negative definiteness of its Hessian.

Question 2.19 (2 points)**.** a) Show that for any pair of convex bodies K and L the function $|K + tL|$ is a polynomial in t of degree n.
b) Conclude that $|K + tL| = \sum_{k=0}^{n} \binom{n}{k} V_k(K, L) t^k$. This is called the Steiner polynomial.

Question 2.20 (2 points)**.** For a convex set K define the Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ by $\nu_K(x) = \{n_x\}$ (the set of all outer normal vectors to ∂K at x; it is a singleton almost everywhere). Define also a measure S_K on the sphere $\mathbb{S}^{n-1}$ by letting, for every Borel measurable $\Omega \subset \mathbb{S}^{n-1}$:

$$S_K(\Omega) = |\nu_K^{-1}(\Omega)|_{n-1}.$$

Here $|\cdot|_{n-1}$ stands for the $(n-1)-$Hausdorff measure, i.e. for $M \subset \partial K$ we let $|M|_{n-1} = \int_M$ in the sense we usually do it in class. The measure S_K is called the *surface area measure* of K.
a) Show that for a pair of convex bodies K and L,

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(\theta) dS_K(\theta).$$

In particular,

$$|K| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(\theta) dS_K(\theta).$$

b) Use Minkowski's first inequality to prove that the surface area measure determines a convex body uniquely up to shifts (i.e. if $dS_K = dS_L$ then $K = L + v$ for some vector v.)

Question 2.21 (2 points)**.** Recall that the projection of a convex body K onto a hyperplane $\theta^\perp$, for some $\theta \in \mathbb{S}^{n-1}$, is the set defined as

$$K|\theta^\perp = \{x \in \theta^\perp : \exists t \in \mathbb{R} : x + t\theta \in K\}.$$

a) Prove the Cauchy formula for a symmetric convex body K:

$$|K|\theta^\perp|_{n-1} = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle| dS_K(u).$$

Hint: option 1 – use elementary geometry and approximation by polytopes. option 2 – use Questions 2.5 part c) and 2.20 part a).
b) Suppose K and L are symmetric convex bodies such that for every $\theta \in \mathbb{S}^{n-1}$ one has $|K|\theta^\perp|_{n-1} = |L|\theta^\perp|_{n-1}$. Conclude that $K = L + v$ for some vector $v \in \mathbb{R}^n$.
(you don't want to me to add a hint here on which Question(s) to use, right?)

Question 2.22 (1 point)**.** Prove that when $h \in C^2(\mathbb{R}^2)$ is a support function of a strictly convex compact region K in $\mathbb{R}^2$, the surface area measure has a density expressible in the form

$$f_K(u) = h(u) + \ddot{h}(u),$$

for all $u \in \mathbb{S}^1$. Note that $h + \ddot{h}$ is translation invariant.

Question 2.23 (10 points)**.** Prove (perhaps using elementary Harmonic Analysis?) that for every pair of π-*periodic* infinitely smooth functions ψ and h on $[-\pi, \pi]$, such that $h + \ddot{h} > 0$ and $h > 0$, one has

$$\left(\int_{-\pi}^{\pi} (h^2 - \dot{h}^2) du\right)\left(\int_{-\pi}^{\pi} (\psi^2 - \dot{\psi}^2 + \psi^2\frac{h+\ddot{h}}{h}) du\right) \le 2\left(\int_{-\pi}^{\pi} (h\psi - \dot{h}\dot{\psi}) du\right)^2. \tag{3}$$

(I can provide explanation/motivation upon request. Note that the assumption is $\pi-$periodic rather than $2\pi-$periodic.)

Question 2.24 (2 points)**.** Prove the Rogers-Shepherd inequality. For a convex body in $\mathbb{R}^n$, define the *difference body*

$$K - K = \{x - y : x, y \in K\}.$$

Show that

$$|K - K| \le \binom{2n}{n}|K|.$$

Hint: use the Brunn-Minkowski inequality to show that $|K \cap (x + K)|^{\frac{1}{n}}$ is a concave function supported on $K - K$, and therefore it can be estimated from below by $1 - \rho_{K-K}(x)$. Using this estimate (among other considerations) show that

$$|K|^2 = \int_{K-K} |K \cap (x + K)| dx \ge \binom{2n}{n}^{-1} |K| \cdot |K - K|.$$

Question 2.25 (2 points)**.** Prove the Grunbaum inequality: let K be a convex body whose barycenter is at the origin (that is $\int_K x dx = 0$.) Show that for any $\theta \in \mathbb{S}^{n-1}$, one has

$$|\{x \in K : \langle x, \theta\rangle \ge 0\}| \ge \left(\frac{n}{n+1}\right)^n |K| \ge \frac{|K|}{e}.$$

Question 2.26 (3 points)**.** Prove Busemann's theorem: given $x \in \mathbb{R}^n \setminus \{0\}$, the function $\frac{|x|}{|x^\perp \cap K|}$ is convex in $\mathbb{R}^n$. Conclude that it is a norm. The unit ball of this norm is called the *intersection body* of K.

Question 2.27. Derive the Santalo formula for the area of a convex region in $\mathbb{R}^2$:

$$|K| = \frac{1}{2}\int_{-\pi}^{\pi} h^2 - \dot{h}^2 dt,$$

where h is the support function of K.
Hint: use Questions 2.22 and 2.20

Question 2.28 (2 points). Using elementary Harmonic Analysis, prove that for every pair of C^1 periodic functions on $[-\pi, \pi]$, one has

$$\int_{-\pi}^{\pi} h^2 - \dot{h}^2 \cdot \int_{-\pi}^{\pi} \psi^2 - \dot{\psi}^2 \leq \left(\int h\psi - \dot{h}\dot{\psi} \right)^2 .$$

Explain why this provides an alternative solution to Question 2.15 b) on the plane (hint: use Questions 2.27 and 2.20 for this explanation).

Question 2.29 (1 point). Prove the general version of Brunn's principle: for a convex body K in $\mathbb{R}^n$ and a $k-$dimensional subspace H, the function $|K \cap (y+H)|^{\frac{1}{k}}$ is concave on its support (inside $H^{\perp}$.) Here $k \in \{1, ..., n-2\}$ (the case $k = n-1$ we did in class.)

Question 2.30. Show that the convolution of log-concave functions is log-concave.
Hint: Use the fact that marginals of log-concave functions are log-concave, in dimension $\mathbb{R}^{2n}$.

3. Linearizations, isoperimetric-type inequalities

Question 3.1 (1 point). Provide an alternative proof (to what was done in class) of the Gaussian Poincare inequality

$$\int_{\mathbb{R}^n} f^2 d\gamma - \left(\int_{\mathbb{R}^n} f d\gamma \right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma$$

using the decomposition of f into the series of Hermite polynomials (the orthonormal system with respect to the Gaussian measure – you can read about them e.g. in Wikipedia.)

Question 3.2 (1 point). As per our discussion in class, prove the following statement using the Borell-Brascamp-Lieb inequality (Question 2.13).

Fix $q \in (-\infty, -n]$. Let $d\mu = e^{-V} dx$ be a probability measure and g be a C^1 function. Suppose $V \in C^2(\mathbb{R}^n)$ and $\nabla^2 V - \frac{\nabla V \otimes \nabla V}{q} \geq 0$ (i.e. V is $q-$concave.) Then, assuming all the integrals below exist,

$$\int \left(g e^{\frac{V}{q}} \right)^2 d\mu - \left(\int g e^{\frac{V}{q}} d\mu \right)^2 \leq \frac{-q}{-q+1} \int \langle e^{-\frac{2V}{-q}} \left(\nabla^2 V + \frac{\nabla V \otimes \nabla V}{-q} \right)^{-1} \nabla g, \nabla g \rangle d\mu.$$

Question 3.3 (1 point). Deduce the Gaussian Poincare inequality from the Gaussian Log-Sobolev inequality via the linearization method (this is sort of a partial case of the argument we discuss in class).

Question 3.4 (1 point). Prove that the (classical) Gaussian Log-Sobolev inequality and the (classical) Lebesgue Log-Sobolev inequality (as stated in class) are indeed equivalent.

Question 3.5 (1 point). By differentiating the infimal convolution directly, prove the Gaussian Log-Sobolev inequality without the convexity assumption on f :

$$Ent_{\gamma}(f^2) \leq 2 \int |\nabla f|^2,$$

for any $f \in C^1(\mathbb{R}^n)$ for which the corresponding integrals converge.

Question 3.6 (1 point). Deduce the Sobolev inequality from the Log-Sobolev inequality for the Lebesgue measure.

Question 3.7 (1 point). Show that the Gaussian Beckner inequality implies the classical Gaussian Log-Sobolev when $p \to 2$.

Question 3.8 (1 point). Deduce Nash's inequality from the (classical) Lebesgue Log-Sobolev inequality: for any non-negative $f \in L^2(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$

$$\left(\int f^2 dx \right)^{1+\frac{2}{n}} \leq \frac{2}{\pi e n} \left(\int |\nabla f|^2 dx \right) \left(\int f dx \right)^{\frac{4}{n}} .$$

Question 3.9 (1 point)**.** Deduce the isoperimetric inequality from the Sobolev inequality for Lebesgue measure.

Question 3.10 (1 point)**.** Prove the following variant of the Generalized Log-Sobolev inequality: given a log-concave measure μ on $\mathbb{R}^n$ with density e^{-V}, and any pair of smooth convex functions f and g with $\int e^{-f}d\mu = \int e^{-g}d\mu$, one has

$$\int g^*(\nabla f)e^{-f}d\mu \geq n\int e^{-f}d\mu - \int \langle \nabla V, x\rangle e^{-f}d\mu - \int fe^{-f}d\mu.$$

Question 3.11 (3 points)**.** Is it possible to obtain Gaussian Beckner inequalities for $p \in [1,2)$ via linearizations of (some) geometric inequalities directly?

Question 3.12 (2 points)**.** Prove the following extension of the Borell-Brascamp-Lieb inequality due to Bolley, Cordero-Erasquin, Fujita, Gentil, Guillin: for convex f and g on $\mathbb{R}^n$ with $n \geq 2$:

$$\int (((1-t)f + tg)^*)^{1-n} \geq (1-t)\int (f^*)^{1-n} + t\int (g^*)^{1-n}.$$

Question 3.13 (Generalized Sobolev, 2 points)**.** Prove the following extension of the Sobolev inequality due to Bolley, Cordero-Erasquin, Fujita, Gentil, Guillin: for convex F and G on $\mathbb{R}^n$ with $n \geq 2$: such that $\int F^{-n} = \int G^{-n} = 1$, and assuming that $\frac{G(x)}{|x|^\gamma} \to_{x\to\infty} 0$, for some $\gamma > \frac{n}{n-1}$, and that all the integrals exist, we have

$$\int G^*(\nabla F)F^{-n} \geq \frac{1}{n-1}\int G^{1-n}.$$

Question 3.14 (Coredero-Erasquin's proof of Colesanti inequality, 4 points)**.** Prove the following inequality: when K is a C^2 convex body, II is its second fundamental form and $f \in C^1(\partial K)$ is an arbitrary function such that $\int_{\partial K} f = 0$, then

$$\int_{\partial K} tr(\mathrm{II})\mathrm{f}^2 - \langle \mathrm{II}^{-1}\nabla_{\partial \mathrm{K}}\mathrm{f}, \nabla_{\partial \mathrm{K}}\mathrm{f}\rangle \leq 0.$$

Here $\nabla_{\partial K} f$ stands for the intrinsic boundary gradient of f. Compare to Question 2.15 part b).

Hint: Use Brascamp-Lieb inequality with $V(x) = \frac{h_K^2(x)}{2}$ and the "body polar coordinates" formula

$$\int_K F(x)dx = \int_0^\infty \int_{\partial K} F(ty)t^{n-1}\langle y, n_y\rangle dtdy,$$

where n_y is the outer unit normal to ∂K at y, and dy stands for the boundary integration.

Question 3.15 (1 point)**.** Show that when $\varphi : [-\pi,\pi]$ is C^1, **even** and periodic, then

$$\int_{-\pi}^{\pi} \varphi^2 - \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\varphi\right)^2 \leq \frac{1}{4}\int_{-\pi}^{\pi}\dot{\varphi}^2.$$

Question 3.16 (1 point)**.** Show that when $\varphi : [-\pi,\pi]$ is C^1, periodic, and $\varphi(0) = 0$, then

$$\int_{-\pi}^{\pi} \varphi^2 \leq 4\int_{-\pi}^{\pi}\dot{\varphi}^2.$$

Question 3.17 (1 point)**.** Show that Brascamp-Lieb inequality is "the end of the line" for the linearization method: let $d\mu(x) = e^{-V(x)}dx$ and plug the function $f(x) = \langle \nabla V(x), \theta\rangle + \epsilon\varphi$ into Brascamp-Leib:

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \int \langle (\nabla^2 V)^{-1}\nabla f, \nabla f\rangle d\mu,$$

and observe that while $\langle \nabla V(x), \theta\rangle$ indeed attains equality in the above inequality, and the terms corresponding to ϵ cancel out as well, still, the only inequality that we obtain as a result is again the Brascamp-Lieb inequality.

Question 3.18 (1 point)**.** Let μ be a log-concave probability measure on $\mathbb{R}^n$ with the density e^{-V} for some convex function V and the associated Laplacian $Lu = \Delta u - \langle \nabla u, \nabla V \rangle$. Let $\lambda_1 > 0$ be the first non-trivial eigenvalue of L, that is the smallest number such that there exists a non-zero function f_1 such that

$$Lf_1 = -\lambda_1 f_1.$$

Show that

$$\lambda_1 = \inf_{f \in W^{1,2}(d\mu)} \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu} = \inf \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu - (\int f d\mu)^2}.$$

Hint: use general convexity/compactness considerations to show that the infimum is attained for some function f_1. Then consider $f = f_1 + \epsilon g$ and argue that the derivative in ϵ of that ratio must be zero. Conclude that f has to be an eigenfunction (use general PDE considerations to argue that it exists).

Question 3.19 (1 point)**.** Show that for a positive definite matrix A,

$$det(\mathrm{Id} + \mathrm{tA}) = 1 + \mathrm{t} \cdot \mathrm{tr(A)} + \frac{\mathrm{t}^2}{2} \|\mathrm{A}\|_{\mathrm{HS}}^2 + \mathrm{o(t^2)},$$

where $\|A\|_{HS}^2$ is the square of the Hilbert-Schmidt norm (that is, the sum of the squares of all entries).

Question 3.20 (3 points)**.** Show that one can improve the Gaussian Log-Sobolev inequality to the following: suppose $d\mu = e^{-V - \frac{x^2}{2} - n \log \sqrt{2\pi}} dx = e^{-V} d\gamma$ is a probability measure. Then

$$-\int V d\mu \leq \frac{\int x^2 d\mu - n}{2} + \frac{n}{2} \log \left(2 + \frac{\int |\nabla V|^2 d\mu - \int x^2 d\mu}{n} \right).$$

Question 3.21 (1 point)**.** Prove the following improvement of the Brascamp-Lieb inequality in the unconditional case (recall that a function $f(x)$ is called unconditional if $f(\epsilon_1 x_1, ..., \epsilon_n x_n) = f(x)$, for every $x \in \mathbb{R}^n$ and every choice of signs $\epsilon_i \in \{-1, 1\}$; that is, f is invariant under coordinate reflections).

Suppose f, w are unconditional and w is convex. Then for the probability measure $d\mu = Ce^{-w} dx$ one has

$$\int f^2 d\mu - \left(\int f d\mu \right)^2 \leq \int \langle (\nabla^2 w + T)^{-1} \nabla f, \nabla f \rangle d\mu,$$

where $T = diag[\frac{1}{x_1} \frac{\partial w}{\partial x_1}, ..., \frac{1}{x_n} \frac{\partial w}{\partial x_n}]$.

Hint: use the multiplicative version of Prekopa-Leindler inequality for unconditional functions, as in Question 2.14.

Question 3.22 (2 points, important question)**.** a) Prove the second part of Lemma 7.9 (from the notes) concerning the second derivative of the Legendre of an interpolation: that for a family of convex functions v_t such that $v_t(x) \in C^2(x, t)$, one has

$$\frac{d^2}{dt^2} v_t^*(x) = -\ddot{v}_t(\nabla v_t^*) + \langle (\nabla^2 v_t(x))^{-1} \nabla \dot{v}_t(\nabla v_t^*), \nabla \dot{v}_t(\nabla v_t^*) \rangle.$$

b) Use it to deduce the Brascamp-Lieb inequality from Prekopa-Leindler directly, without going via the Generalized Log-Sobolev. Namely, note that Prekopa-Leindler ineqaulity implies that

$$\frac{d^2}{dt^2} \int e^{-(f+tg)^*} \leq 0,$$

and do the computation which confirms that this is equivalent to the Brascamp-Lieb inequality

$$\int \varphi^2 d\mu - \left(\int \varphi d\mu \right)^2 \leq \int \langle (\nabla^2 V)^{-1} \nabla \varphi, \nabla \varphi \rangle d\mu,$$

with $d\mu = e^{-V} dx$, where $V = f^*$, and $\varphi(x) = g(\nabla f^*(x))$, and we assume that $\int d\mu = 1$.

4. Duality and Blaschke-Santalo type inequalities

Question 4.1 (1 point)**.** Let P be a polytope given by

$$P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le a_i, \, \forall i = 1, ..., N\},$$

for some unit vectors $u_1, ..., u_N$ and positive numbers $a_1, ..., a_N$, and suppose that P is bounded. Show that

$$P^o = co\bar{n}v \left\{ \frac{u_1}{a_1}, ..., \frac{u_N}{a_N} \right\}.$$

Conclude that $(B_1^n)^o = B_\infty^n$.

Question 4.2 (1 point)**.** In this question, K and L stand for convex bodies in $\mathbb{R}^n$ with non-empty interior, containing the origin.

a) Prove that $K^{oo} = K$.

b) Prove that for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ with $\det T \neq 0$,

$$(T^t K)^o = T^{-1} K^o.$$

Conclude that a polar of an ellipsoid is an ellipsoid.

c) Prove that

$$(B_p^n)^o = B_q^n,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for all $p, q > 1$.

d) Prove that

$$(K \cap L)^o = conv(K^o \cup L^o).$$

e) Prove that for every subspace H of $\mathbb{R}^n$

$$(K|H)^o \cap H = K^o \cap H.$$

f) Prove that if $K \subset L$, one has $L^o \subset K^o$.

g) Prove that if K is symmetric then K^o is symmetric.

h) Show that for any (possibly non-convex) set A, we have $A^o = (conv(A))^o$. Conclude that the polar is always a convex set.

Question 4.3 (1 point)**.** Let K be a symmetric convex body. Show that if $K = K^o$ then $K = B_2^n$.

Question 4.4 (1 point)**.** Show that for any symmetric convex body K, we have $h_K(\theta)\rho_{K^o}(\theta) = 1$ for all $\theta \in \mathbb{R}^n$.

Question 4.5 (3 points)**.** Verify Mahler's conjecture in $\mathbb{R}^2$ for symmetric polygons: show that for any symmetric polygon P in $\mathbb{R}^2$,

$$|P| \cdot |P^o| \ge 8.$$

Question 4.6 (1 point)**.** Given a Borel measurable set A in $\mathbb{R}^n$, a function $\alpha : A \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n \setminus 0$, consider the *shadow system*

$$K_t = conv\{x + \alpha(x)v \; : x \in A\},$$

and define the convex body

$$\tilde{K} = conv\{x + t\alpha(x)e_{n+1}\} \subset \mathbb{R}^{n+1}.$$

Show that for $u \in e_{n+1}^\perp$,

$$h_{K_t}(u) = h_{\tilde{K}}(u + t\langle u, v\rangle e_{n+1}).$$

Question 4.7 (2 points)**.** Prove the Blaschke-Santalo inequality using shadow systems.

Hint 1. Express $|K_t^o|$ combining the formulas from Questions 2.4, 4.4 and 4.6.

Hint 2. pass the integration on $\mathbb{S}^{n-1}$ to the integration on $B_2^{n-1} = \{x \in \mathbb{R}^n : \langle x, v\rangle = 0\}$ with the map $x = \theta - \langle \theta, v\rangle v$.

Hint 3: now extend the integration to $\mathbb{R}^{n-1}$.

Hint 4. Conclude that $|K_t^o|$ is -1-concave in t for any shadow system, using Question 2.13.

Hint 5. Notice that Steiner symmetrization can be realized as a shadow system, and, using the fact

that $|K^o| = |\bar{K}^o|$ for any reflection $\bar{K}$ of K, and the $-1-$concavity of $|K_t^o|$ along any shadow system, conclude that Steiner symmetrization increases $|K^o|$. Conclude the Blaschke-Santalo inequality.
(this proof was discovered by Campi and Gronchi).

Question 4.8 (1 point)**.** a) For any $\varphi : \mathbb{R} \to \bar{R}$ one has φ^* is a convex function.
b) If φ is convex then $\varphi^{**} = \varphi$.
c) If $f \geq g$ then $f^* \leq g^*$.
d) Find $|x_1|^*$.
e) Find $(\frac{\|x\|_p^q}{q})^*$.
f) For a convex body K, one has $(-\log 1_K)^* = h_K$.
g) For an $a \in \mathbb{R}$, find $(a\varphi)^*$ in terms of φ^*.
h) Letting $\varphi_a(x) = \varphi(ax)$ for some $a \in \mathbb{R}$, find φ_a^*.
i) Show that $(\varphi + a)^* = \varphi^* - a$, for any $a \in \mathbb{R}$.
j) Show that

$$(f^* + g^*)^*(z) = \inf_{x,y\in\mathbb{R}^n : x+y=z} (f(x) + g(y)).$$

k) Fix $\alpha > 1$. Show that is f is $\alpha-$homogeneous (i.e. $f(tx) = t^\alpha f(x)$ for all $t \in \mathbb{R}$) then $f^*(\nabla f) = (\alpha - 1)f$.
Hint: use one of the properties we proved in class, combined with the fact that for an $\alpha-$homogeneous function one has $\langle \nabla f, x\rangle = \alpha f$ (verify this).

Question 4.9 (1 point)**.** Find an alternative short proof of the functional Blaschke-Santalo inequality for unconditional functions by passing the integration from $\mathbb{R}^n$ to the set

$$\{x \in \mathbb{R}^n : \forall i = 1, ..., n,\ x_i \geq 0\},$$

and making a change of variables in the Prekopa-Leindler inequality given by $(x_1, ..., x_n) = (e^{t_1}, ..., e^{t_n})$. (see also a similar Question 2.14).

Question 4.10 (1 point)**.** Show that the Santaló point of a convex body exists and is unique.

Question 4.11 (4 points)**.** Find a statement and a proof for the Blaschke-Santalo inequality and the functional Blaschke-Santalo inequality for non-symmetric convex sets and non-even functions (as per our discussion in class).

Question 4.12 (3 points)**.** a) Note that the Blaschke-Santalo inequality on the plane is equivalent to showing that for any even periodic function $h \in C^2([-\pi, \pi])$, such that $h \geq 0$ and $h + \ddot{h} \geq 0$,

$$F(h) = \int_{-\pi}^{\pi} h^{-2} dt \cdot \int_{-\pi}^{\pi} h^2 - \dot{h}^2 dt \leq 4\pi^2.$$

(or equivalently, one may drop the even assumption and restrict to $[0, \pi]$).

Hint: use Questions 4.4 and 2.4 to conclude that

$$|K^o| = \frac{1}{2} \int_{-\pi}^{\pi} h^{-2} dt.$$

Also use Question 2.27.

b) Observe that the equality is attained when h is the support function of an ellipse.

c) Find some way to show that this inequality is true.

Option 1: maybe use basic Harmonic Analysis (I don't know if it is possible and would love to see it if it works)?

Option 2: maybe use variational approach? That is, suppose that a given function h maximizes the functional $F(h)$, argue* that it suffices to assume that $h \in C^1([-\pi, \pi])$ and $h > 0$ and $h + \dot{h} > 0$, then argue that for any $\epsilon > 0$ and any even smooth $\psi > 0$, $\frac{d}{d\epsilon} F(h + \epsilon\psi) = 0$, and conclude some ODE that h must satisfy (in view of the arbitrarity of ψ). Then conclude that the support function of an ellipsoid is the only type of function that satisfies this ODE.

* This “argue” may not be easy and you are encouraged to pursue other steps in this hint even if this step is not clear at first.

Option 3: try whatever you like! :)

Question 4.13 (1 point)**.** Let K be a smooth convex body with $\mathrm{II} > 0$. For $x \in \partial K$ let $x^* \in \partial K^*$ be given by $x^* = \nabla \|x\|_K$. Show that the Gauss curvature at x of ∂K is inverse to the Gauss curvature at x^* of K^o.
Hint: use the properties of Legendre transform of $h_K(x)$.

Question 4.14 (5 points)**.** a) Find an example of a non-symmetric convex body for which the Santaló point and the center of mass do not coincide.
b) How far could they be?
c) For a convex body K in $\mathbb{R}^n$, let $d(K)$ be the distance between the center of mass and the Santaló point. How large could $\frac{d(K)}{diam(K)}$ be?

Question 4.15 (1 point)**.** Let H be a Hanner polytope (as defined inductively in class). Show that indeed

$$|H||H^o| = \frac{4^n}{n!}.$$

Question 4.16 (2 points, Saint-Raimond's theorem via Meyer's proof)**.** Prove the (symmetric) Mahler conjecture in the case when the body K is unconditional (that is, it is symmetric with respect to every coordinate hyperplane).
Hint 1: Note that the result is true in dimension 1 and proceed by induction.
Hint 2: Consider $K^+ = \{x \in K : x_i \geq 0 \ \forall i = 1, ..., n\}$. Given a point $x \in K^+$ consider n cones

$$K_i = conv\{x, K^+ \cap e_i^\perp\}.$$

Note that

$$|K| \geq 2^n \sum_{i=1}^n |K_i|,$$

recall Question 1.3 and write the above out to deduce that the vector with coordinates $(..., \frac{2|K\cap e_i^\perp|}{n|K|}, ...)$ belongs to K^o (use the unconditionality in the process).
Hint 3: Do the same argument for K^o, and then use properties of polarity along with the fact that $K \cap e_i^\perp = K|e_i^\perp$ (which is another place where the fact that K is unconditional is used!!!), to conclude that

$$|K||K^o| \geq \frac{4}{n^2} \sum_{i=1}^n |K \cap e_i^\perp| \cdot |(K \cap e_i^\perp)^o|,$$

and use induction.

Question 4.17 (5 points)**.** Iryeh and Shibata's proof of Mahler's conjecture in $\mathbb{R}^3$ followed the same idea as in Question 4.16, and hinged on the fact that it is possible to bring a symmetric convex body in $\mathbb{R}^3$ into a position where it is possible to split it into 8 parts with coordinate hyperplanes so that each part has the same volume, and each of the three coordinate hyperplane sections of K is split into four equal parts, and also each projection of K onto coordinate hyperplane coincides with a section.
a) verify that this fact ensures the validity of Mahler conjecture (in the same way as above);
b) prove this challenging fact.

Question 4.18 (3 points)**.** Verify the non-symmetric Mahler conjecture in dimension 2.

Question 4.19 (3 points)**.** Using the ideas from Question 4.16, prove the result of Barthe, Fradelizi: if a convex body K in $\mathbb{R}^n$ has all the symmetries of the regular simplex then it verifies the non-symmetric Mahler conjecture, that is, $|K||K^o| \geq |S_n|^2$ where S_n is the self-dual regular simplex.

Question 4.20 (10 points). Is it possible to use the ideas from Question 4.19 to prove the non-symmetric Mahler conjecture in $\mathbb{R}^3$, that is, to show that for any convex body K in $\mathbb{R}^3$ one has $|K||K^o| \geq |S_3|^2$ where S_3 is the self-dual regular simplex? Maybe one could prove the appropriate non-symmetric version of the fact proved by Iryeh and Shibata about bringing K into a certain position?

Question 4.21 (2 points). Prove the following result of Fradelizi and Meyer: Mahler's conjecture is equivalent to the following functional version. For any convex function φ on $\mathbb{R}^n$ one has

$$\int e^{-\varphi} \cdot \int e^{-\varphi^*} \geq 4^n.$$

Question 4.22 (2 points). Prove the following result of Fradelizi and Meyer which extends the functional Blaschke-Santalo: let $\rho : [0, \infty) \to [0, \infty)$ be a measurable function and suppose f and g are even log-concave functions such that $f(x)g(y) \leq \rho^2(\langle x, y\rangle)$ whenever $\langle x, y\rangle \geq 0$. Then

$$\int f \cdot \int g \leq \left(\int \rho(|x|^2)\right)^2.$$

Question 4.23 (5 points). We saw in class that the $p-$Beckner inequality on the circle for periodic functions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2 - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^p\right)^{\frac{2}{p}} \leq (2-p)\frac{1}{2\pi}\int_{-\pi}^{\pi} \dot{f}^2$$

holds not only for $p \in [1, 2)$ but also for $p = -2$. By any chance, is it possible to argue that there is a range of negative p for which this holds (rather than just one value $p = -2$)? Maybe argue similarly to Question 3.7?

Question 4.24 (2 points). Show that Talagrand's transport-entropy inequality implies the Gaussian Poincare inequality.
Hint: linearize.

Question 4.25 (10 points). Try and make some progress on the question we discussed in class: for any even log-concave measure μ and any symmetric convex body K one has

$$\mu(K)\mu(K^o) \leq \mu(B_2^n)^2.$$

Maybe you can find a proof in some partial case – for some class of measures, for unconditional measures/bodies, in dimension 2, etc?

Question 4.26 (1 point). Prove the symmetric Gaussian Poincare inequality

$$Var_\gamma(f) \leq \frac{1}{2}\mathbb{E}_\gamma|\nabla f|^2$$

for all *even* locally-Lipschitz functions f on $\mathbb{R}^n$ by using the decomposition into Hermite polynomials (rather than by linearizing Blaschke-Santalo inequality like we did in class).

Question 4.27 (1 point). Show that the Blaschke-Santalo inequality and Fathi's inequality are in fact equivalent (in class we only deduced the latter from the former).

Question 4.28 (2 points). Prove the result of Saraglou.
a) See the lecture notes for the definition of the log-addition. Show that the Log-Brunn-Minkowski inequality for Lebesgue measure

$$\left|\frac{K +_0 L}{2}\right| \geq \sqrt{|K| \cdot |L|}$$

(for any symmetric convex bodies K and L in $\mathbb{R}^n$) implies the Log-Brunn-Minkowski inequality for any even log-concave measure μ on $\mathbb{R}^n$ with full support:

$$\mu\left(\frac{K +_0 L}{2}\right) \geq \sqrt{\mu(K)\mu(L)}$$

(for any symmetric convex bodies K and L in $\mathbb{R}^n$). Conclude that the Log-Brunn-Minkowski conjecture implies the B-conjecture.
Hint: use the Prekopa-Leindler inequality.

b) Show the converse implication.
Hint: consider the situation near the origin and use the scale-invariance of the inequality in the Lebesgue case.

Question 4.29 (2 points)**.** Confirm that the validity of the B-conjecture for all rotation-invariant log-concave measures is equivalent to the fact that for any even log-concave measure μ,

$$\mu(RB_2^n)\mu\left(\frac{1}{R}B_2^n\right) \leq \mu(B_2^n)^2.$$

(recall that this corresponds to a very partial case and a sanity check in the Conjecture from Question 4.25.)

Question 4.30 (2 points)**.** Show Klartag's theorem generalizing the functional Brunn-Minkowski inequality: for any even log-concave measure μ,

$$\int e^{-\phi}d\mu \cdot \int e^{-\phi^*}d\mu \leq \left(\int e^{-\frac{x^2}{2}}d\mu\right)^2.$$

Hint: use Cafarelli's contraction theorem.

Question 4.31 (10 points)**.** Attempt to make any progress on the "original B-conjecture": let $z \in \mathbb{R}^n$ and let K be a symmetric convex set in $\mathbb{R}^n$. Then the function

$$\frac{\gamma(tK+z)}{\gamma(tK)}$$

is non-decreasing in $t \geq 1$. Here γ is the standard Gaussian measure.

Question 4.32 (2 points)**.** Show that the B-theorem of Cordero-Erasquin, Fradelizi and Maurey would follow from the confirmation of the conjecture from Question 4.31.

Hint: write the conclusion in terms of a non-negative derivative at $t = 1$; then note that the arising inequality implies that certain function which depends on $z \in \mathbb{R}^n$ is increasing along each ray, and therefore it is convex at the point $z = 0$. Consider the Laplacian in z.

Question 4.33 (2 points)**.** Prove the result of Bobkov: the following are equivalent:

- For a symmetric convex body K of volume 1, the measure with the density
$$\frac{1}{\sqrt{2\pi}^n\gamma(K)}e^{-\frac{x^2}{2}}1_K(x)dx$$
is isotropic.
- For a symmetric convex body K of volume 1 and for any volume-preserving linear transformation T on $\mathbb{R}^n$, $\gamma(K) \geq \gamma(TK)$.

Hint: use the B-theorem.

Question 4.34 (3 points)**.** Prove the improved Log-Sobolev inequality: for any convex function V on $\mathbb{R}^n$ such that $\int e^{-V} = 1$,

$$-\int Ve^{-V} \leq \frac{n}{2}\log\frac{\int \Delta V e^{-V}}{n} - n\log\sqrt{2\pi e}.$$

Question 4.35 (10 points)**.** Is it possible to deduce from the Reverse Log-Sobolev inequality and/or the (generalized) Log-Sobolev inequality the following corollary of the Entropy Power Inequality?

Let X and Y be any two centered random vectors in $\mathbb{R}^n$ and X' and Y' are independent centered Gaussians (whose covariance matrices are scalar), such that $h(X) = h(X')$ and $h(Y) = h(Y')$. Then

$$h(X+Y) \geq h(X'+Y'),$$

where

$$h(X) = -\int f \log f,$$

where f is the density according to which X is distributed.

Question 4.36 (2 points)**.** Find Fathi's original proof for his inequality, which relies on the Reverse Log-Sobolev inequality (which we discussed) as well as the following fact (following from works of Cordero-Erasquin, Klartag and Santambrogio).

Let μ be a centered probability measure whose support has non-empty interior. Then there exists an essentially continuous convex function φ, unique up to translations, such that $\rho = e^{-\varphi}dx$ is a probability measure on $\mathbb{R}^n$ whose push-forward by the map $\nabla\varphi$ is μ. Moreover, it satisfies

$$\rho = argmin\left\{-\frac{1}{2}W_2(\mu,\nu)^2 + Ent_\gamma(\nu)\right\}.$$

Clarification: do not aim to prove this fact, only aim for the implication of Fathi's theorem from this fact combined with the Reverse Log-Sobolev.

Question 4.37 (1 point)**.** Suppose u, v on $\mathbb{R}^n$ are 2-homogeneous convex functions. Prove that

$$\int e^{-\frac{u+v}{2}} det\left(\frac{\nabla^2 u + \nabla^2 v}{2}\right) \geq \sqrt{\int e^{-u} det(\nabla^2 u) \cdot \int e^{-v} det(\nabla^2 v)}.$$

Hint: use the fact that for a 2-homogeneous function, $2u = \langle \nabla u, x\rangle$ and the change of variables that we used when proving the Reverse Log-Sobolev inequality, together with the Prekopa-Leindler inequality.

Question 4.38 (1 point)**.** Prove the conclusion of Question 4.25 under the assumption that both K and μ are unconditional.

5. Concentration of measure: the soft approach

Question 5.1 (3 points)**.** Recall that a spherical cap is a non-empty set of the form

$$\mathbb{S}^{n-1} \cap \{\langle x, v\rangle \geq t\},$$

for some $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Prove the isoperimetric inequality on the sphere $\mathbb{S}^{n-1}$: given $A \subset \mathbb{S}^{n-1}$ with $\sigma(A) = \alpha \in (0,1)$ (where σ is the Haar measure on the sphere), prove that the perimeter of A (which we defined in class) is larger that that of a spherical cap of measure α.

Hint: use an analogue of Steiner symmetrizations, for example, or some other approach.

Question 5.2 (1 point)**.** Using the Question 5.1, deduce the sharp concentration inequality on the sphere (which we stated in class).

Hint: use an approach similar to how we deduce the Gaussian sharp concentration from the Gaussian isoperimetry (we will do it in a few weeks).

Question 5.3 (Rahul's question, 2 points)**.** Could you prove a concentration result on the sphere of the type

$$\sigma(A_t) \geq C_1 e^{-c_2 n^2 t^4}$$

for some range of t and some constants? Use the same ideas as what we discussed in class.

Question 5.4 (2 points)**.** a) Prove the Efron-Stein inequality: for any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and any random vector $X = (X_1, ..., X_n)$, one has

$$Var f(X) \leq \mathbb{E}\sum_{i=1}^n Var_i f(X),$$

where Var stands for the variance with respect to the distribution of X, $\mathbb{E}$ stands for the expectation with respect to the distribution of X, and Var_i is the variance with respect to X_i (so $Var_i f(X)$ is a random variable.)

b) Prove the tensorisation property of the Poincare inequality: let μ_1,..., μ_m be a collection of measures on $\mathbb{R}^{k_1}, ..., \mathbb{R}^{k_m}$ respectively, so that $k_1+...+k_m = n$. Let the measure $\mu = \mu_1 \times ... \times \mu_m$ on $\mathbb{R}^n$. Then the Poincaré constant of μ equals the maximum of the Poincaré constants of $\mu_1, ..., \mu_m$.

Question 5.5 (1 point)**.** a) Recall that for a random vector X distributed according to the measure μ and a function f, we denote $Entf(X) = \int f(x)\log f(x)d\mu(x) - \int fd\mu \cdot \log\left(\int fd\mu\right)$. Prove that for any measurable function $f: \mathbb{R}^n \to \mathbb{R}$ and any random vector $X = (X_1, ..., X_n)$, one has

$$Entf(X) \leq \mathbb{E}\sum_{i=1}^{n} Ent_i f(X),$$

where Ent stands for the entropy with respect to the distribution of X, $\mathbb{E}$ stands for the expectation with respect to the distribution of X, and Ent_i is the entropy with respect to X_i (so $Ent_i f(X)$ is a random variable.)

Hint: use the variational characterization of entropy that we proved in class.

b) Recall that we say that a measure μ on $\mathbb{R}^n$ satisfies the Log-Sobolev inequality with constant β if for any locally Lipschitz function f one has $Entf^2(X) \leq 2\beta\mathbb{E}|\nabla f(X)|^2$, and $\beta > 0$ is the smallest number that works here.

Prove the tensorisation property of the Log-Sobolev inequality: let μ_1,..., μ_m be a collection of measures on $\mathbb{R}^{k_1}, ..., \mathbb{R}^{k_m}$ respectively, so that $k_1+...+k_m = n$. Let the measure $\mu = \mu_1 \times ... \times \mu_m$ on $\mathbb{R}^n$. Then the Log-Sobolev constant of μ equals the maximum of the Log-Sobolev constants of $\mu_1, ..., \mu_m$.

Question 5.6 (5 points)**.** (Intentionally vague question, allowing for some freedom). Find any interesting extension or generalization of the Herbst argument in the situation of, say, Generalized Log-Sobolev inequality, or in some other more general situation.

Question 5.7 (1 point)**.** Confirm that if ψ is a convex function on $(\mathbb{R}^n)^+$ then the function $\psi(x_1^2, ..., x_n^2)$ is convex as well. (this concludes the proof of the Thin Shell inequality in the unconditional case, as we discussed in class).

Question 5.8 (1 point)**.** Confirm that for any Borel probability measure μ on a metric space X, one has $\alpha_\mu(t) \to_{t\to\infty} 0$.

Question 5.9 (1 point)**.** Confirm that if the function f on $\mathbb{R}^n$ is $p-$Lipschitz (that is, $|f(x)-f(y)| \leq p|x-y|$ for all $x, y \in \mathbb{R}^n$) then one has $|\nabla f| \leq p$.

Question 5.10 (3 points)**.** a) For a log-concave probability measure μ with density f on $\mathbb{R}^n$, define $b_p : \mathbb{S}^{n-1} \to \mathbb{R}$ via

$$b_p(\theta) = \int_0^\infty f(t\theta)t^p dt.$$

Prove that $B_p(x) = |x|b_p(\frac{x}{|x|})$ is convex for $p \geq 1$, and therefore is a Minkowski functional on $\mathbb{R}^n$ of some convex body K_μ. This body is called Ball's body as it was introduced by Keith Ball.

b) Suppose μ is isotropic. For which p is K_μ isotropic (after being normalized to have volume one)?

c) Prove that verifying the thin shell conjecture for all log-concave measures is equivalent to verifying it only for uniform measures on convex bodies.

d) Prove that verifying the KLS conjecture for all log-concave measures is equivalent to verifying it only for uniform measures on convex bodies.

Question 5.11 (3 points)**.** Confirm that the KLS conjecture is equivalent to showing that for any isotropic convex body K in $\mathbb{R}^n$, cutting K into two parts of equal volume is achieved, up to a multiple of an absolute constant, by an affine hyperplane cut. (For 1 point out of 3, show that the KLS conjecture implies this fact.)

Hint: Explain that (in some sense) for a (nice) closed connected region M one has

$$|\partial M| = \int_{\partial M} |\nabla 1_M|.$$

Question 5.12 (1 point)**.** Let $f : S^{n-1} \to \mathbb{R}$ be a seminorm. Then,
$$\left(\int_{S^{n-1}} |f(\theta)|^q \, d\sigma(\theta)\right)^{1/q} \leq \frac{cq}{p}\sqrt{\frac{n+p}{n+q}} \left(\int_{S^{n-1}} |f(\theta)|^p \, d\sigma(\theta)\right)^{1/p}.$$
Hint: use the reverse Hölder inequality that we proved and integration by parts.

Question 5.13 (1 point)**.** Let μ be any even log-concave measure on $\mathbb{R}^n$. Show that for any symmetric measurable set A in $\mathbb{R}^n$ and any $t > 0$ one has
$$\mu\left(x \in \mathbb{R}^n : \exists y \in A : \langle x, y\rangle > -t\right) \geq 1 - \frac{1}{\mu(A)} e^{-t}.$$
Hint: Use Klartag's extension of the functional Blaschke-Santalo inequality.

Question 5.14 (2 points)**.** Let μ be a measure on a metric space X with Laplace functional $E_\mu(\lambda)$. Suppose the diameter of X is bounded from above by $D < \infty$. Then for any $\lambda \geq 0$ one has
$$E_\mu(\lambda) \leq e^{\frac{D^2\lambda^2}{2}}.$$

Question 5.15 (1 point)**.** Show that whenever for a measure μ one has for all $\lambda \geq 0$ that $E_\mu \leq e^{\frac{\lambda^2}{2c}}$ for some constant $c > 0$ then for every $t > 0$ one has $\alpha_\mu(t) \leq e^{-\frac{ct^2}{8}}$.

Hint: use the result we proved in class.

Remark. This shows that using Payne-Weinberger inequality as means of obtaining concentration bounds is sub-optimal.

Question 5.16 (1 point)**.** Prove the (close to optimal) sub-Gaussian concentration bound for the discrete cube $\{-1, 1\}^n$ equipped with the uniform measure and the Hamming distance (as defined in class): show
$$\alpha_\mu(t) \leq e^{-\frac{t^2 n}{8}}.$$
Hint: use questions 5.14 and 5.15.

Question 5.17 (2 points)**.** Show that Paouris's inequality (that we stated in class) follows from the following result of Guedon and Milman: for an isotropic log-concave random vector X on $\mathbb{R}^n$ and any $p \in \mathbb{R}$ such that $1 \leq |p-2| \leq c_1 n^{\frac{1}{6}}$ one has
$$1 - C\frac{|p-2|}{n^{\frac{1}{3}}} \leq n^{-\frac{1}{2}} \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \leq 1 + C\frac{|p-2|}{n^{\frac{1}{3}}},$$
and for any $p \in \mathbb{R}$ such that $c_1 n^{\frac{1}{6}} \leq |p-2| \leq c_2 n^{\frac{1}{2}}$ one has
$$1 - C\frac{|p-2|^{\frac{1}{2}}}{n^{\frac{1}{4}}} \leq n^{-\frac{1}{2}} \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \leq 1 + C\frac{|p-2|^{\frac{1}{2}}}{n^{\frac{1}{4}}}.$$
Remark. Note that this estimates include negative values of p, unlike the reverse Hölder inequality that we proved in class.

Question 5.18 (1 point)**.** Find an example of a non-Lipschitz function f on the sphere $\mathbb{S}^{n-1}$ which violates the concentration around the median inequality.

Question 5.19 (1 point)**.** Confirm the following fact: suppose for all Lipschitz functions on the space (X, d, μ) one has
$$\mu(f \geq \mathbb{E}f + t) \leq \alpha(t)$$
for some function α on $\mathbb{R}^+$. Then $\alpha_\mu(t) \leq \alpha(\frac{t}{2})$ (where as usual α_μ denotes the concentration function).

Question 5.20 (2 points)**.**

- Show that if the diameter of the metric space (X, d) is bounded by $R > 0$ then for any probability measure μ on X one has $E_\mu(\lambda) \leq e^{cD^2\lambda^2}$. Here E_μ stands for the Laplace functional, as defined in class.
- Deduce (using also a result we proved in class) the nearly sharp sub-Gaussian concentration on the Hamming cube.

Question 5.21 (2 points)**.** Prove that $d(\|\cdot\|_\infty) \to \infty$ when the dimension tends to infinity. Here $d(\|\cdot\|)$ is the Klartag-Vershynin dimension of a norm, as defined in class.

Question 5.22 (1 point)**.** Estimate from below the Poincaré constant of the domain consisting of two unit balls in $\mathbb{R}^n$ connected with a neck of width ϵ.

Question 5.23 (1 point)**.** Prove that in dimension 1, for any log-concave measure μ on $\mathbb{R}$, the isoperimetric sets are rays.

6. Gaussian measures

Question 6.1 (3 points)**.** Find an alternative proof of Bobkov's inequality by approximating the Gaussian measure by the uniform measure on the Hamming cube.

Question 6.2 (1 point)**.** Verify that

$$\frac{\phi^{-1}(\gamma(tB_2^n))}{t} \to_{t\to\infty} 1.$$

(recall that we used this fact to deduce the Gaussian Isoperimetric Inequality from Ehrhard's inequality).

Question 6.3 (1 point)**.** Deduce the Gaussian Isoperimetric Inequality directly from Bobkov's inequality.

Question 6.4 (2 points)**.** Prove Kahane's inequality: let $g_1, ..., g_k, ...$ be a sequence of i.i.d. $N(0,1)$ random variables. For any $q \geq p > 0$, any $n \geq 1$ and any $z_1, ..., z_n \in \mathbb{R}^n$ we have

$$\left(\mathbb{E}\|\sum_{i=1}^n g_i z_i\|^q\right)^{\frac{1}{q}} \leq \frac{\alpha_q}{\alpha_p}\left(\mathbb{E}\|\sum_{i=1}^n g_i z_i\|^p\right)^{\frac{1}{p}},$$

where

$$\alpha_p = (\mathbb{E}|g_i|^p)^{\frac{1}{p}}.$$

Question 6.5 (2 points)**.** Prove the following properties of Ehrhard symmetrizations. Let $S = S(L,e)$ be a Gaussian symmetrization and A and B be arbitrary closed sets. Then

- $\gamma(S(A)) = \gamma(A)$ provided that A is Borel measurable
- If $A \subset B$ then $S(A) \subset S(B)$
- For a vector v, $S(A+v) = S(A) + v$
- If $A_1 \subset A_2 \subset ...$ are open sets and $A = \cup_{i=1}^\infty A_i$ then $S(A) = \cup_{i=1}^n S(A_i)$

Question 6.6 (1 point)**.** Let L_1 and L_2 be two sub-spaces in $\mathbb{R}^n$ such that $(L_1 \cap L_2)^\perp \cap L_1$ and $(L_1 \cap L_2)^\perp \cap L_2$ are orthogonal. Then

$$S(L_1,e) \circ S(L_2,e) = S(L_2,e) \circ S(L_1,e) = S(L_1 \cap L_2, e).$$

Question 6.7 (1 point)**.** Let $n \geq 3$ and $k \geq 2$. Show that for every $k-$symmetrization S there exist 2-symmetrizations $S_1, ..., S_{k-1}$ such that $S = S_1 \circ ... \circ S_{k=1}$. Hint: use Question 6.6.

Question 6.8 (1 point)**.** In dimension 2, show that there is a sequence $\theta_1, ..., \theta_k, ... \in \mathbb{S}^{n-1}$ such that letting $S_i = S(\theta_i^\perp, \theta_i) \circ ... \circ S(\theta_1^\perp, \theta_1)$, one has for every set A, that $S_i(A)$ converges to a half-space of the same Gaussian measure as A.

Question 6.9 (2 points)**.** Prove, for any $\epsilon > 0$, any Gaussian symmetrization S and any set A :

$$S(A) + \epsilon B_2^n \subset S(A + \epsilon B_2^n).$$

Conclude that the Ehrhard symmetrization decreases the Gaussian Perimeter. Using Questions 6.8 and 6.7, conclude the Gaussian Isoperimetric Inequality (directly without passing via the Ehrhard inequality).

Question 6.10 (2 points)**.** Prove that the Gaussian symmetrization of any convex set is also convex. (recall that this was a crucial step in proving Ehrhard's inequality.)

Question 6.11 (2 points). Find lower estimates on the isoperimetric profile of some product measures of your choice (beyond the uniform measure on the cube and the Gaussian).

Question 6.12 (4 points). Solve the isoperimetric problem on the square in dimension 2: prove that if $|A \cap [0,1]^2| = a \in [0,1]$ then $|\partial A \cap [0,1]^2|$ is bounded from below by the case of A being either an appropriately shifted ball, or a half-space.

Question 6.13 (3 points). Let L be a convex body. Find a lower estimate for the anisotropic Gaussian perimeter of a set A with $\gamma(A) = a$, that is

$$\liminf_{\epsilon \to 0} \frac{\gamma(A + \epsilon L) - \gamma(A)}{\epsilon}.$$

For which L is it sharp?

Question 6.14 (2 points). Prove the simple case of the Gaussian Correlation Inequality (called the Sidak Lemma): let K and L be a pair of symmetric strips. Then $\gamma(K \cap L) \geq \gamma(K)\gamma(L)$.

Hint: use the Prekopa-Leindler inequality.

Question 6.15 (1 point). Prove the Gaussian Log-Sobolev inequality by linearizing Bobkov's inequality.

Question 6.16 (1 point). Show that the functional Ehrhard inequality tensorizes, i.e. that from knowing it in dimensions k and m one can deduce it in the dimension $k + m$.

Question 6.17 (5 points). Try and find the proof of Functional Ehrhard Inequality in dimension one, without using the geometric Ehrhard.

Question 6.18 (1 point). Verify that for $a \in [0,1]$,

$$\eta(a) = \sqrt{2\pi} a \Phi^{-1}(a) e^{\Phi^{-1}(a)^2/2} \geq -1. \tag{4}$$

Question 6.19 (3 points). In class we showed that if K is any convex set, $\gamma(K) = a \in [0,1]$, then letting $\eta(a)$ as in (4) we have

$$\frac{1}{\gamma(K)} \int_K \langle x, \theta \rangle^2 \, d\gamma + \frac{\eta(a)}{\gamma(K)^2} \left(\int_K \langle x, \theta \rangle \, d\gamma \right)^2 \leq 1.$$

Find an alternative proof of this fact using Ehrhard's inequality, or perhaps the consequences of Ehrhard's inequality – the generalized Bobkov inequality or the Ehrhard-Brascamp-Lieb inequality which we deduced in class.

7. THE $L2$ METHOD

Question 7.1 (1 point). Outline a second proof of Bochner's identity, via the change of variables $x = y + t\nabla u(y)$, and taking the second derivative, that we briefly discussed in class.

Question 7.2 (1 point). a) Consider the Banach space $X = W^{1,2}(\mu) \times ... \times W^{1,2}(\mu)$ (n times), let $F = (f_1, ..., f_n) \in X$ and consider

$$\|F\| = \sqrt{\int \langle AF, F \rangle d\mu},$$

where $A = A(x)$ is a positive definite matrix of functions in $W^{1,2}(\mu)$. Show that $\|F\|$ is a norm.
b) Show that its dual norm is

$$\|F\|_* = \sqrt{\int \langle A^{-1}F, F \rangle d\mu}.$$

Question 7.3 (2 points). In class, we showed that for μ, a finite Borel measure, and a bounded measurable function h with $\int h d\mu = 0$, $\varepsilon > 0$, and μ_ε such that $d\mu_\varepsilon = (1 + \varepsilon h)d\mu$, we have

$$\|h\|_{H^{-1}(\mu)} = \liminf_{\varepsilon \to 0} \leq \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon}.$$

Show that the $\geq$ inequality also holds.

Question 7.4 (5 points)**.** Suppose μ is an isotropic unconditional log-concave probability measure and ψ – an unconditional $W^{1,2}(\mu)$ function. Construct a transport map T from μ to $(1+\epsilon\partial_i\psi)\mu$ such that

$$\int |Tx-x|^2 d\mu(x) \leq C\log n \int |\partial_i\psi|^2 d\mu.$$

Conclude Klartag's $C\log n$ bound for the KLS conjecture in the case of unconditional log-concave measures, using also several tools that we discussed the class.

Question 7.5 (1 point)**.** Explain why the conclusion of the Question implies the B-conjecture for unconditional log-concave measures.

Question 7.6 (1 point)**.** Prove the reverse implication in the Lemma of Hörmander (reverse to the one we proved in class.)

Question 7.7 (2 points)**.** Deduce (using the generalized Bochner formula that we discussed in class) the following extension of Brascamp-Lieb inequality: let K be a convex set, μ – log-concave measure in $\mathbb{R}^n$ with potential V such that $\mu(K)=1$, and let g be a concave function on K. Then

$$\int gf^2 d\mu - \left(\int gf d\mu\right)^2 \leq \int g\langle(\nabla^2 V)^{-1}\nabla f, \nabla f\rangle d\mu,$$

Question 7.8 (1 point)**.** Let μ be an even log-concave measure on $\mathbb{R}^n$. Show that the fact that $\mu(e^tK)$ is log-concave in $t>0$ for any symmetric convex K is equivalent to the fact that for any symmetric convex K one has

$$\frac{1}{\mu(K)}\int_K \langle\nabla V, x\rangle^2 d\mu - \left(\frac{1}{\mu(K)}\int_K \langle\nabla V, x\rangle d\mu\right)^2 \leq \frac{1}{\mu(K)}\int_K \langle\nabla^2 Vx, x\rangle + \langle\nabla V, x\rangle d\mu.$$

Question 7.9 (2 points)**.** Show that the fact that $\mu(e^tK)$ is log-concave in t for any symmetric convex K and any rotation-invariant log-concave measure μ is equivalent to the fact that for any even log-concave measure ν one has

$$\nu(tB_2^n)\nu(t^{-1}B_2^n) \leq \nu^2(B_2^n).$$

In other words, the result of Cordero-Erasquin and Rotem implies the conjectured log-concave Blaschke-Santalo inequality in a very partial case.

Question 7.10 (4 points)**.** As discussed in class, prove Lemma 2 (from November 29) using the Brascamp-Lieb inequality.

Question 7.11 (1 point)**.** Let μ be a log-concave probability measure on $\mathbb{R}^n$ with density e^{-V} such that $\nabla^2 V \geq t\cdot Id$. Suppose (for simplicity) that $\int x d\mu = 0$ (the barycenter is at the origin). Recall that the covariance matrix is then $Cov(\mu) = (\mathbb{E}_\mu X_i X_j)$. Prove that $\|Cov(\mu)\| \leq \frac{1}{t}$.

Hint: Recall that the operator norm of the covariance matrix is $\sup_{\theta\in\mathbb{S}^{n-1}}\int\langle x,\theta\rangle^2 d\mu$ and use similar ideas to the ones we used when showing that the Gaussian measure minimizes the Poincaré constant among the isotropic measures.

Question 7.12 (2 points)**.** Show that there are log-concave probability measures for which the first eigenfunction does not exist.
Hint: consider $d\mu = \frac{1}{2}e^{-|x|}dx$ on $\mathbb{R}$.

Question 7.13 (1 point)**.** Recall Klartag's "preferred direction" conjecture that we discussed in class: there exists an absolute constant $c>0$ such that if μ is an isotropic log-concave probability measure such that its first eigenfunction exists and is C^2, then

$$\left(\int \nabla f d\mu\right)^2 \geq c\int |\nabla f|^2 d\mu.$$

Show that this conjecture implies the KLS conjecture (which states that the Poincaré constant of an isotropic log-concave probability measure is bounded from above by an absolute constant.)

Question 7.14 (4 points)**.** Prove Klartag's "preferred direction" conjecture from Question 7.13 in dimension 1.

Question 7.15 (8 points)**.** Try to find a lower bound for the $c > 0$ in Klartag's "preferred direction" conjecture from Question 7.13 in all dimensions; it is OK if it depends on n – what is the largest bound that you can get?