Rectifiability in Carnot groups

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1. Carnot groups: basic definitions and properties
2. Rectifiable sets in $\mathbb{R}^n$
3. Rectifiable sets in a Carnot group: C^1 and Lipschitz version
4. Equivalence between C^1_H-regular surfaces and intrinsic Lipschitz graphs
5. Characterization of C^1_H-regular surfaces in terms of suitable weak solutions of PDE system

Carnot groups

Definition

A Carnot group $\mathbb{G}$ of step κ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$, of dimension n, admits a step κ stratification, i.e. a direct sum decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$ such that

$$\begin{cases} [V_1, V_{i-1}] = V_i & \text{if } 2 \leq i \leq \kappa \\ [V_1, V_\kappa] = \{0\} \end{cases}$$

where $[V_1, V_{i-1}]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_{i-1}$.

- V_1 generates all of $\mathfrak{g}$

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Carnot groups

- The exponential map is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Hence any $p \in \mathbb{G}$ can be written in a unique way as $p = \exp(p_1 X_1 + \cdots + p_n X_n)$ and we identify
$$p \quad \longleftrightarrow \quad (p_1, \dots, p_n)$$
and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

- It is useful to know that $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$ where $\mathbb{G}^i = \exp(V_i) = \mathbb{R}^{n_i}$ is the i^{th} layer of $\mathbb{G}$ and $\dim(V_i) = n_i$. We can write
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Carnot groups

Two important families of transformations of $\mathbb{G}$:

- ***Intrinsic left translations*** of $\mathbb{G}$: For any $p \in \mathbb{G}$ the left translation $\tau_p : \mathbb{G} \to \mathbb{G}$ is defined as

$$q \mapsto \tau_p q := p \cdot q.$$

- ***Intrinsic dilations*** of $\mathbb{G}$: for any $\lambda > 0$, the (non isotropic) dilation $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is defined as

$$\delta_\lambda(p^1, \dots, p^\kappa) = (\lambda p^1, \dots, \lambda^i p^i, \dots, \lambda^\kappa p^\kappa)$$

Carnot groups

Definition

A nonnegative function $p \to \|p\|$ on $\mathbb{G}$ is said to be a homogeneous norm if

- $\|p\| = 0$ *if and only if* $p = 0$.
- $\|\delta_\lambda p\| = \lambda \|p\|$ *for all* $p \in \mathbb{G}$ *and* $\lambda > 0$.
- $\|p \cdot q\| \leq \|p\| + \|q\|$ *for all* $p, q \in \mathbb{G}$.

Given any homogeneous norm $\|\cdot\|$, it is possible to introduce a distance in $\mathbb{G}$ given by

$$d(p,q) = \|p^{-1} \cdot q\| \quad \forall p, q \in \mathbb{G}.$$

- $d(\tau_p(q), \tau_p(q')) = d(q,q') \qquad d(\delta_\lambda(q), \delta_\lambda(q')) = \lambda d(q,q')$.
- For any bounded subset Ω of $\mathbb{G}$ there are $c_1, c_2 > 0$ such that

$$c_1|p-q| \leq d(p,q) \leq c_2|p-q|^{1/\kappa} \quad \text{for } p, q \in \Omega.$$

- The topological dimension of $(\mathbb{G}, d)$ is n
- The Hausdorff (or metric) dimension of $(\mathbb{G}, d)$ is $\sum_{i=1}^{\kappa} i \dim V_i > n$.

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Carnot groups: $\mathbb{H}^n$ as model cases

Examples: Heisenberg groups $\mathbb{H}^n$

The n-th Heisenberg group $\mathbb{H}^n$, with $n \geq 1$, is the Carnot group of step 2 with Lie algebra

$$\mathfrak{h}^n := \mathbf{span}\{X_1, \dots, X_{2n}\} \oplus \mathbf{span}\{X_{2n+1}\},$$

the only nontrivial bracket relations being

$$[X_i, X_{i+n}] = X_{2n+1}, \qquad \forall i = 1, \dots, n.$$

- we identify $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$
- $p = (p^1, p^2) \in \mathbb{H}^n$ with $p^1 \in \mathbb{R}^{2n}$ and $p^2 \in \mathbb{R}$
- $\delta_\lambda(p) = (\lambda p^1, \lambda^2 p^2)$
- Topological dimension of $(\mathbb{H}^n, d)$ is $2n+1$
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d-dimensional rectifiable sets in $\mathbb{R}^n$

Definition 1: $E \subset \mathbb{R}^n$ is *d-rectifiable* if

$\mathcal{H}^d(E) < \infty$ and E is the Lipschitz image of a subset of $\mathbb{R}^d$.

More general definitions are

Definition 2a: $E \subset \mathbb{R}^n$ is *countably d-rectifiable* if

$$\mathcal{H}^d(E \setminus \bigcup_{i \in \mathbb{N}} S_i) = 0$$

where S_i are d-dimensional C^1 embedded submanifolds

Definition 2b: $E \subset \mathbb{R}^n$ is *countably d-rectifiable* if

$$\mathcal{H}^d(E \setminus \bigcup_{i \in \mathbb{N}} \mathrm{graph}\,(f_i)) = 0$$

where $f_i : \mathbb{R}^d \to \mathbb{R}^{n-d}$ are Lipschitz functions

Equivalence

The Lipschitz definition and C^1 definition are equivalent.
The proof follows from

- Rademacher's Theorem (Differentiability almost everywhere of Lipschitz functions)
- Extension of Lipschitz functions
- Whitney's Extension theorem

Rectifiable sets in $\mathbb{G}$: possible definitions

Definition: $E \subset \mathbb{R}^n$ is *countably d-rectifiable* equivalently if

- $\mathcal{H}^d\left(E \setminus \bigcup_{i\in\mathbb{N}} S_i\right) = 0$ where S_i are d-dimensional C^1 embedded submanifolds or
- $\mathcal{H}^d\left(E \setminus \bigcup_{i\in\mathbb{N}} graph(f_i)\right) = 0$ where $f_i : \mathbb{R}^d \to \mathbb{R}^{n-d}$ are Lipschitz functions

Questions

1. What are *d-dimensional, C^1 submanifolds* in $\mathbb{G}$?
2. What are Lipschitz graphs in $\mathbb{G}$?
3. Which Hausdorff measure do we have to use?

Lipschitz graphs in a Carnot group $\mathbb{G}$

Question

Definition of Lipschitz graphs in a general Carnot group.

Lipschitz graphs in a Carnot group $\mathbb{G}$

Intrinsic Lipschitz graphs were introduced by Franchi, Serapioni, Serra Cassano.

References

- Bigolin, Caravenna, Serra Cassano (2014)
- Citti, Manfredini, Pinamonti, Serra Cassano (2014)
- Fassler, Orponen (2019)
- Monti, Vittone (2012)
- Vittone (2020)
- etc.

Intrinsic graphs in a Carnot group $\mathbb{G}$

Let $\mathbb{W}$, $\mathbb{V}$ be complementary homogeneous subgroups of $\mathbb{G}$, i.e. $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ and $\mathbb{W} \cap \mathbb{V} = \{0\}$.

Definition: *S is a left intrinsic graph* over $\mathbb{W}$ in direction of $\mathbb{V}$

if there is $\varphi : \mathbb{W} \to \mathbb{V}$ s.t.

$$S = \mathrm{graph}\,(\varphi) := \{a \cdot \varphi(a) : a \in \mathbb{W}\}$$

The notion is "intrinsic"

Left translations and intrinsic dilations of graphs are graphs. In particular,

$$p \cdot \mathrm{graph}(\varphi) = \mathrm{graph}(\varphi_p),$$

with $\varphi_p : \mathbb{W} \to \mathbb{V}$.

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Intrinsic Lipschitz graphs in a Carnot group $\mathbb{G}$

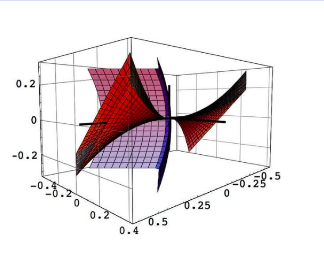
Definition: the cone with vertex p, axis $\mathbb{V}$, opening $s > 0$ is

$$X(p, \mathbb{V}, s) := \{q \in \mathbb{G} : d(q, p) \geq s\, dist(q, \mathbb{V})\}$$

Definition: $\varphi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz

if there is $s > 0$ such that for all $p \in \mathrm{graph}(\varphi)$

$$X(p, \mathbb{V}, s) \cap \mathrm{graph}(\varphi) = \{p\}.$$



Picture by Serra Cassano on researchgate

Intrinsic Lipschitz graphs in a Carnot group $\mathbb{G}$

Intrinsic Lipschitz $\neq$ Lipschitz

It is false even locally that

$$\|\varphi(a)^{-1} \cdot \varphi(a')\| \leq L\|a^{-1} \cdot a'\|,$$

but it is true that φ is locally $1/\kappa$-Hölder continuous

$$\|\varphi(a)^{-1} \cdot \varphi(a')\| \leq L\|a^{-1} \cdot a'\|^{1/\kappa}.$$

Intrinsic Lipschitz graphs in a Carnot group $\mathbb{G}$

Intrinsic Lipschitz graphs are Ahlfors regular (Franchi, Serapioni (2016))

If $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$, if $\varphi : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz
if $\mathbb{W}$ has metric dimension d_m
then there are $0 < c_1 < c_2$ s.t.

$$c_1 r^{d_m} \leq \mathcal{S}^{d_m}\big(\mathrm{graph}\,(\varphi) \cap B(p,r)\big) \leq c_2 r^{d_m} \tag{1}$$

for all $p \in \mathrm{graph}\,(\varphi)$ and $r > 0$, $c_i = c_i(\mathbb{V}, \mathbb{W}, \text{Lipschitz const. of } \varphi)$.

Lipschitz rectifiable sets in $\mathbb{G}$

Lipschitz Definition: E is $(d, d_m, \mathbb{G})_L$-rectifiable

if

- $\mathcal{S}^{d_m}(E) < \infty$,
- there are subgroups $(\mathbb{W}_i, \mathbb{V}_i)$ complementary in $\mathbb{G}$,
- $\mathbb{W}_i$ has topological dimension d and metric dimension d_m,
- there are intrinsic Lipschitz functions $\varphi_i : \mathbb{W}_i \to \mathbb{V}_i$,

and

$$\mathcal{S}^{d_m}\Big(E \setminus \bigcup_{i\in\mathbb{N}} \mathrm{graph}\,(\varphi_i)\Big) = 0$$

$C^1_{\mathbb{H}}$-regular surfaces in $\mathbb{G}$

Questions

1. Definition of C^1_H-regular surfaces.
2. Equivalence between C^1_H-regular surfaces and intrinsic Lipschitz graphs

1. was introduced by Franchi, Serapioni, Serra Cassano and then generalized by Magnani

$C^1_{\mathbb{H}}$-regular surfaces in $\mathbb{G}$

Let $X_1, \dots X_{n_1}$ be a basis of V_1. We define, for $F : \Omega \subset \mathbb{G} \to \mathbb{R}$ for which the partial derivatives $X_i F$ exist, the horizontal gradient of F as $\nabla_H F = (X_1 F, \dots, X_{n_1} F)$.

Definition ($C^1_{\mathbb{H}}$ function)

A continuous function $f : \Omega \subseteq \mathbb{G} \to \mathbb{R}^k$ is of class C^1_H if the distributional derivatives $X_j f_i$ are continuous for every $i = 1, \dots, k$, and $j = 1, \dots, n_1$.

Definition ($C^1_{\mathbb{H}}$-regular surface)

We say that $S \subset \mathbb{G}$ is a $C^1_{\mathbb{H}}$-regular surface of codimension k if $1 \leq k \leq n_1$ and for any $p \in S$, there exist a neighborhood $\mathcal{U}$ of p and a map $f \in C^1_{\mathbb{H}}(\mathcal{U}; \mathbb{R}^k)$ such that

$$S \cap \mathcal{U} = \{q \in \mathcal{U} : f(q) = 0\},$$

and the $k \times n_1$ matrix $(X_j f_i(p))_{ij}$ has maximum rank, then equal to k or, equivalently, the P-differential $d_P f$ is surjective.

$C^1_{\mathbb{H}}$-regular surfaces in $\mathbb{G}$

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Problem in the model case, i.e. Heisenberg groups $\mathbb{H}^n$

Rectifiable sets in the model case of Carnot groups, i.e. $\mathbb{H}^n$

In Heisenberg groups:

C^1 Definition: E is $(d, \mathbb{H})$-rectifiable if $\mathcal{S}^{d_m}(E) < \infty$ and

$$\mathcal{S}^{d_m}(E \setminus \bigcup_{i \in \mathbb{N}} S_i) = 0$$

where S_i are $C^1_{\mathbb{H}}$-regular surfaces.

Lipschitz Definition: E is $(d, \mathbb{H})_L$-rectifiable if $\mathcal{S}^{d_m}(E) < \infty$ and

$$\mathcal{S}^{d_m}(E \setminus \bigcup_{i \in \mathbb{N}} \Gamma_i) = 0$$

where Γ_i are intrinsic Lipschitz d-graphs.

where $\mathcal{S}^{d_m}$ is the Hausdorff measure w.r.t. the distance in $\mathbb{H}^n$ and

$$\begin{cases} d_m = d, & \text{for } 1 \le d \le n, \\ d_m = d + 1, & \text{for } n + 1 \le d \le 2n. \end{cases}$$

Rectifiable sets in the model case of Carnot groups, i.e. $\mathbb{H}^n$

Warning: in $\mathbb{H}^n$ we have the equivalence

- E is $(d, \mathbb{H})$-rectifiable $\implies$ E is $(d, \mathbb{H})_L$-rectifiable
- Franchi, Serapioni, Serra Cassano (2011)

$$E \text{ is } (2n, \mathbb{H})\text{-rectifiable} \iff E \text{ is } (2n, \mathbb{H})_L\text{-rectifiable}$$

- Vittone (2020)

$$E \text{ is } (d, \mathbb{H})_L\text{-rectifiable} \implies E \text{ is } (d, \mathbb{H})\text{-rectifiable}$$

Equivalence in $\mathbb{G}$

General case

Equivalence between C^1_H-regular surfaces and intrinsic Lipschitz graphs

Intrinsically linear functions in $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$

Definition: $\ell : \mathbb{W} \to \mathbb{V}$ is an *intrinsically linear map*

if ℓ is defined on all of $\mathbb{W}$ and

$$\mathrm{graph}\,(\ell) := \{a \cdot \ell(a) : \ a \in \mathbb{W}\}$$

is a homogeneous subgroup of $\mathbb{G}$.

- Intrinsically linear maps are not homogeneous homomorphisms - in general

Proposition

Let $\mathbb{W}$ and $\mathbb{V}$ be complementary subgroups in $\mathbb{G}$ with $\mathbb{V}$ horizontal of dimension $1 \leq h \leq n_1$. If $\ell : \mathbb{W} \to \mathbb{V}$ is an intrinsically linear map, then there is a $h \times n_1$ matrix $\mathcal{M}_\ell$ s.t.

$$\ell(a) = \mathcal{M}_\ell a^1, \qquad \text{for all } a = (a^1, \dots, a^\kappa) \in \mathbb{W}.$$

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UID functions in $\mathbb{G}$

Definition: uniformly intrinsically differentiable at 0

$\varphi : \mathbb{W} \to \mathbb{V}$ with $\varphi(0) = 0$.
φ is *uniformly intrinsically differentiable* at 0 (φ is *UID* at 0) if there is an intrinsically linear map $d\varphi_0 : \mathbb{W} \to \mathbb{V}$ s.t.

$$\lim_{r\to 0} \sup_{a,a'} \frac{\|d\varphi_0(a^{-1} \cdot a')^{-1} \cdot \varphi(a)^{-1} \cdot \varphi(a')\|}{\|a^{-1} \cdot a'\|} = 0$$

the supremum is for $\|a\| < r$, $0 < \|a^{-1} \cdot a'\| < r$.

Definition: UID at a_0

$\varphi : \mathbb{W} \to \mathbb{V}$ and $p_0 := a_0 \cdot \varphi(a_0)$.

$$\varphi \text{ is } \textit{UID} \text{ at } a_0 \quad \text{if and only if} \quad \varphi_{p_0^{-1}} \text{ is } \textit{UID} \text{ at } 0$$

where $\varphi_{p_0^{-1}} : \mathbb{W} \to \mathbb{V}$ is s.t. $p_0^{-1} \cdot \mathrm{graph}(\varphi) = \mathrm{graph}(\varphi_{p_0^{-1}})$ and $\varphi_{p_0^{-1}}(0) = 0$.

On $\mathbb{R}^n$ this is equivalent to being C^1

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On $\mathbb{R}^n$ this is equivalent to being C^1

UID functions in $\mathbb{G}$

1. If $d\varphi_a : \mathbb{W} \to \mathbb{V}$ exists, it is unique and it is called the intrinsic differential of φ at a.
2. When $\mathbb{V}$ is horizontal, we denote $D^\varphi \varphi(a)$ the matrix associated to $d\varphi_a$ and we call it the intrinsic gradient of φ at a.

Theorem (DD, Potential analysis, 2021)

$\mathbb{V}, \mathbb{W}$ are complementary in $\mathbb{G}$ of step κ and $\mathbb{V}$ is horizontal, if $\varphi : \mathbb{W} \to \mathbb{V}$ is UID in $\mathbb{W}$, then

1. *φ is, locally, intrinsic Lipschitz continuous in $\mathbb{W}$;*
2. *φ is, locally, little $1/\kappa$-Hölder continuous, that is $\varphi \in C(\mathbb{W})$ and for all $\mathcal{F} \Subset \mathbb{W}$*

$$\lim_{r \to 0^+} \sup \left\{ \frac{\|\varphi(a) - \varphi(a')\|}{\|a^{-1}a'\|^{1/\kappa}} \right\} = 0$$

   for all $a, a' \in \mathcal{F}$ with $0 < \|a^{-1}a'\| < r$;
3. *the function $a \mapsto d\varphi_a$ is continuous.*

UID functions in $\mathbb{G}$

Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a Carnot group, with $\mathbb{V}$ horizontal and $\varphi \colon U \subseteq \mathbb{W} \to \mathbb{V}$ be a continuous function.

Theorem (DD, Potential analysis, 2021)

The following are equivalent

1. *graph(φ) is a C^1_H regular surface*
2. *φ is UID on U.*

Proof. (1) $\Rightarrow$ (2). Implicit Function Theorem (Franchi, Serapioni, Serra Cassano (2001), Magnani (2013))
Proof. (2) $\Rightarrow$ (1). Whitney's Extension Theorem (Franchi, Serapioni, Serra Cassano (2003))

Corollary

If graph(φ) is a C^1_H-regular surface, then φ has continuous intrinsic gradient.

UID functions in $\mathbb{G}$

Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a Carnot group, with $\mathbb{V}$ horizontal and $\varphi \colon U \subseteq \mathbb{W} \to \mathbb{V}$ be a continuous function.

Theorem (DD, Potential analysis, 2021)

The following are equivalent

1. *graph(φ) is a C^1_H regular surface*
2. *φ is UID on U.*

Proof. (1) $\Rightarrow$ (2). Implicit Function Theorem (Franchi, Serapioni, Serra Cassano (2001), Magnani (2013))
Proof. (2) $\Rightarrow$ (1). Whitney's Extension Theorem (Franchi, Serapioni, Serra Cassano (2003))

Corollary

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Surfaces in Euclidean spaces and in Carnot groups

Euclidean spaces	Carnot groups
$S = \{p : f(p) = 0\} \subset \mathbb{R}^n$ $f \in C^1(\mathbb{R}^n, \mathbb{R}^k)$ ∇f has rank k	$S = \{p : f(p) = 0\} \subset \mathbb{G}$ $f \in C^1_H(\mathbb{G}, \mathbb{R}^k)$ $\nabla_H f$ has rank k
$S = \mathrm{graph}\,(\varphi)$ $= \{(a, \varphi(a)) : a \in W\}$ $\varphi : W \to V$ $V = \mathbb{R}^k$ and $W = \mathbb{R}^{n-k}$ V and W are complementary linear subspaces φ and $\nabla\varphi$ are continuous	$S = \mathrm{graph}\,(\varphi)$ $= \{a \cdot \varphi(a) : a \in \mathbb{W}\}$ $\varphi : \mathbb{W} \to \mathbb{V}$ $\mathbb{V}$ and $\mathbb{W}$ are complementary homogeneous subgroups φ and $D^{\varphi}\varphi$ are continuous

Characterization of C^1_H-regular surfaces

Problem

Characterize uniformly intrinsically differentiable functions
in terms of
existence and continuity of derivatives of φ

Characterization of C^1_H-regular surfaces

The first result is given by Ambrosio, Serra Cassano, Vittone (2006)

References

- Antonelli, DD, Don, Le Donne (2022)
- Antonelli, DD, Don (2022)
- Bigolin, Serra Cassano (2010)
- Corni (2019)
- Kozhevnikov (2015)

Projected vector fields

Definition (Projected vector fields, Kozhevnikov (2015))

Given $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$, a continuous $\varphi : \mathbb{W} \to \mathbb{V}$ and $W \in \mathrm{Lie}(\mathbb{W})$, we define the φ-projected vector field on $\mathbb{W}$ along W as follows

$$D_W^{\varphi}(a) := (d\pi_{\mathbb{W}})_{a\cdot\varphi(a)} W_{a\cdot\varphi(a)}, \qquad \forall a \in \mathbb{W},$$

where $\pi_{\mathbb{W}}$ is the projection on $\mathbb{W}$ given the splitting $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$.

What's the intrinsic gradient of φ?

$\mathfrak{g} = \exp(\mathrm{span}\{X_1, \dots, X_n\})$,
where $X_1, \dots X_{n_1}$ is a basis of V_1,
$\mathbb{W} := \exp(\mathrm{span}\{X_{k+1}, \dots, X_{n_1}, \dots, X_n\})$, $\mathbb{V} := \exp(\mathrm{span}\{X_1, \dots X_k\})$
The intrinsic gradient of $\varphi : \mathbb{W} \to \mathbb{V}$ is

$$D^{\varphi} := (D^{\varphi}_{X_{k+1}}, \dots, D^{\varphi}_{X_{n_1}})$$

Characterization of C^1_H-regular surfaces

Problem in $\mathbb{H}^1$

- $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$
- $\mathbb{W} := \exp(\mathrm{span}\{X_2, X_3\}), \quad \mathbb{V} := \exp(\mathrm{span}\{X_1\})$
  with $[X_1, X_2] = X_3$
- we identify $\mathbb{H}^1 \equiv \mathbb{R}^3 = \{(x_1, x_2, x_3)\}$
- $\varphi : \mathbb{W} \to \mathbb{V}$
- $$D^{\varphi}_{X_2} := \partial_{x_2} + \varphi \partial_{x_3}, \quad D^{\varphi}_{X_3} = \partial_{x_3}.$$
- **Problem: $D^{\varphi}_{X_2}\varphi = \omega$ in a suitable weak sense iff φ is UID.**

Intrinsic gradient $D^{\varphi}_{X_2}$ in $\mathbb{H}^1$ is the Burgers' operator

Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a Carnot group, with $\mathbb{V}$ horizontal and $\varphi \colon U \subseteq \mathbb{W} \to \mathbb{V}$ be a continuous function.
Let $\omega \colon U \subseteq \mathbb{W} \to \mathrm{Lin}(\mathrm{Lie}(\mathbb{W}) \cap V_1; \mathbb{V})$ be a continuous function with values in the space of linear maps.

Horizontal regularity. We say that $D^{\varphi}\varphi = \omega$ in broad* sense if for every $W \in \mathrm{Lie}(\mathbb{W}) \cap V_1$ and every point $a \in U$, there exists a C^1 integral curve of D^{φ}_W starting from a for which the Fundamental Theorem of Calculus with derivative ω holds.

Vertical regularity. φ is **vertically broad* hölder**: φ along the integral curves of D^{φ}_W for $W \in \mathrm{Lie}(\mathbb{W}) \cap V_d$ with $d > 1$ is little $1/d$-Hölder continuous.

Example in $\mathbb{H}^1$

- $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$
- $\mathbb{W} := \exp(\mathrm{span}\{X_2, X_3\}), \quad \mathbb{V} := \exp(\mathrm{span}\{X_1\})$
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- we identify $\mathbb{H}^1 \equiv \mathbb{R}^3 = \{(x_1, x_2, x_3)\}$
- $\varphi : \mathbb{W} \to \mathbb{V}$
- $$D^{\varphi}_{X_2} := \partial_{x_2} + \varphi \partial_{x_3}, \quad D^{\varphi}_{X_3} = \partial_{x_3}.$$
- Horizontal regularity: $D^{\varphi}_{X_2}\varphi = \omega$ **in the broad* sense** iff locally around every point of $\mathbb{W}$ there exists a family of integral curves γ of $D^{\varphi}_{X_2}$ s.t. $(\varphi \circ \gamma)' = \omega \circ \gamma$
- Vertical regularity: φ is **vertically broad* hölder** iff φ is locally little $1/2$-Hölder continuous along x_3.

General result

Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a Carnot group of step κ, with $\mathbb{V}$ horizontal and $\varphi \colon U \subseteq \mathbb{W} \to \mathbb{V}$ be a continuous function.

Theorem (Antonelli, DD, Don, Le Donne, Annales de l'Institut Fourier, 2022)

The following facts are equivalent.

(a) $\mathrm{graph}(\varphi)$ *is a* C^1_H *regular surface*

(b) φ *is UID on* U.

(c) φ *is vertically broad* hölder on* U *and there exists a continuous function* $\omega \colon U \to \mathrm{Lin}(\mathrm{Lie}(\mathbb{W}) \cap V_1; \mathbb{V})$ *s.t.* $D^{\varphi}\varphi = \omega$ *in the broad* sense on* U.

Question

Can we drop the vertically broad* hölder regularity in (c)?

General result

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Vertically broad* hölder regularity

On the positive side in $\mathbb{H}^n$:

- in $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V}$ 1-dimensional (and so horizontal), vertically broad* hölder regularity can be dropped - Ambrosio, Serra Cassano, Vittone (2006), Bigolin, Serra Cassano (2010)
- in $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\mathbb{V}$ horizontal, vertically broad* hölder regularity can be dropped - Corni (2019)

On the negative side: one cannot drop the assumption on the vertically broad* hölder regularity in arbitrary Carnot groups

In a Carnot group of step 3 with $\mathbb{V}$ 1-dimensional, vertically broad* hölder regularity cannot be dropped - Kozhevnikov (2015)

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Result in Carnot groups of step 2

Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a Carnot group of step 2, with $\mathbb{V}$ 1-dimensional (and so horizontal) and $\varphi \colon U \subseteq \mathbb{W} \to \mathbb{V}$ be a continuous function.

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The following facts are equivalent.

(a) $\mathrm{graph}(\varphi)$ *is a* C^1_H *regular surface*

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(c) *there exists a continuous function* $\omega \colon U \to \mathrm{Lin}(\mathrm{Lie}(\mathbb{W}) \cap V_1; \mathbb{V})$ *s.t.* $D^\varphi \varphi = \omega$ *in the broad* sense on* U.

Idea of the proof in Carnot groups of step 2

Key statement

Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ be a free Carnot group of step 2, with $\mathbb{V}$ 1-dimensional and $\varphi \colon U \subseteq \mathbb{W} \to \mathbb{V}$ be a continuous function.
$D^{\varphi}\varphi = \omega$ **in the broad* sense** $\Rightarrow$ φ **is vertically broad* hölder**

Broad* solution on $\mathbb{G}$

$\Downarrow$

Broad* solution on free Carnot groups of step 2

$\Downarrow$

Vertically broad* hölder regularity on free Carnot groups of step 2

$\Downarrow$

Vertically broad* hölder regularity on $\mathbb{G}$

Idea of the proof in Carnot groups of step 2

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Broad* solution on $\mathbb{G}$

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Broad* solution on free Carnot groups of step 2

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Vertically broad* hölder regularity on free Carnot groups of step 2

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Vertically broad* hölder regularity on $\mathbb{G}$

Thank you for the attention !!!