A quantitative Gaussian Correlation Inequality and convex influences

Anindya De (Penn)

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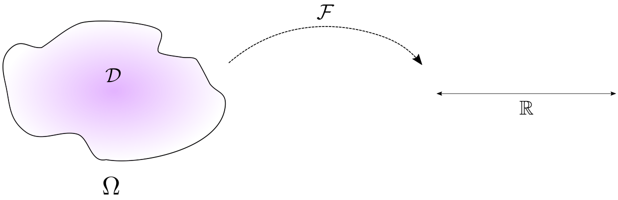
Joint work with

Shivam Nadimpalli
(Columbia)

Rocco Servedio
(Columbia)

Correlation Inequalities

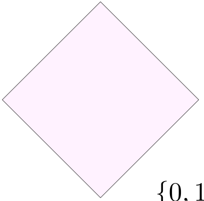


- Consider a family of functions $\mathcal{F}$ over some domain Ω, and a distribution $\mathcal{D}$ over Ω. Informally, a correlation inequality is a statement of the form

$$\mathbf{E}_{\mathcal{D}}[f \cdot g] - \mathbf{E}_{\mathcal{D}}[f] \cdot \mathbf{E}_{\mathcal{D}}[g] \geq 0$$

  for all $f, g \in \mathcal{F}$.
- Ubiquitous in probabilistic combinatorics and statistical physics.

Example 1: Harris–Kleitman Inequality (1966)



$\{0,1\}^n$

Recall $f : \{0,1\}^n \to \{0,1\}$ is monotone if $x \preccurlyeq y \implies f(x) \le f(y)$.

$x_i \le y_i$ for all i

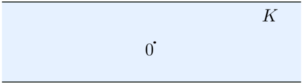
Theorem: For $f, g : \{0,1\}^n \to \{0,1\}$ monotone, we have

$$\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] \ge 0$$

where expectations are with respect to the uniform distribution on $\{0,1\}^n$.

Example 2: Royen's Inequality (2014)

- Also called the Gaussian Correlation Inequality (GCI).
- Open for over 40 years.



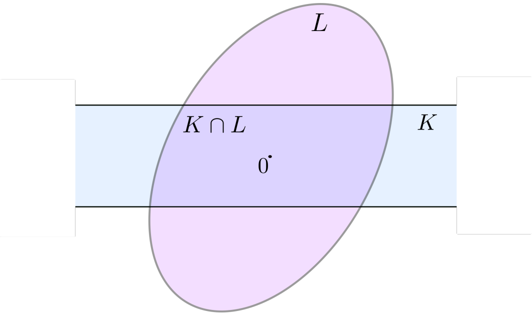
$K \subseteq \mathbb{R}^n$ is symmetric if $x \in K$ implies $-x \in K$.

Theorem: Let $K, L \subseteq \mathbb{R}^n$ be symmetric, convex sets, identified with their $0/1$ indicator functions. Then

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq 0$$

where the expectations are with respect to $\mathcal{N}(0,1)^n$.

Example 2: Royen's Inequality (2014)



Equivalently: If $\gamma_n(\cdot)$ is the n-dimensional, standard Gaussian measure, then for convex, symmetric $K, L \subseteq \mathbb{R}^n$

$$\gamma_n(K \cap L) \geq \gamma_n(K) \cdot \gamma_n(L).$$

And Many More...

	$\mathcal{D}$	Correlation Inequality
Monotone $f, g : \{0,1\}^n \to \{0,1\}$	Uniform	Harris–Kleitman
Symmetric, convex $K, L \subseteq \mathbb{R}^n$	$\mathcal{N}(0,1)^n$	Royen's Inequality
Convex $f, g : \mathbb{R}^n \to \mathbb{R}$	$\mathcal{N}(0,1)^n$	Hu's Inequality
Montone $f, g : [0,1] \to \mathbb{R}$	Uniform	Chebyshev's Inequality
Monotone $f, g : \{0,\dots,m-1\}^n \to \mathbb{R}$	log-supermodular	FKG Inequality

And Many More...

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Towards Quantitative Inequalities

- All inequalities so far are qualitative in nature:

> Given $f, g \in \mathcal{F} \subseteq L^2(\Omega, \mathcal{D})$, we have
>
> $$\mathbf{E}_{\mathcal{D}}[f \cdot g] - \mathbf{E}_{\mathcal{D}}[f] \cdot \mathbf{E}_{\mathcal{D}}[g] \geq 0.$$

- Can we hope to get a better lower bound?

Perhaps in terms of some property of f and g themselves?

Revisiting Harris–Kleitman

Theorem: Given $f, g : \{0,1\}^n \to \{0,1\}$ monotone, we have

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where expectations w.r.t. uniform distribution.

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> where expectations w.r.t. uniform distribution.

- Define $f_0 : \{0,1\}^{n-1} \to \{0,1\}$ as

  $$f_0(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0).$$

  So f_0 is a restriction of f with last bit fixed to 0
- Define g_0 similarly, as well as f_1 and g_1.

Revisiting Harris–Kleitman

Easy to check that

$$\begin{aligned}\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] = &\left(\frac{\mathbf{E}[f_0 \cdot g_0] - \mathbf{E}[f_0] \cdot \mathbf{E}[g_0]}{2} \right) \\ &+ \left(\frac{\mathbf{E}[f_1 \cdot g_1] - \mathbf{E}[f_1] \cdot \mathbf{E}[g_1]}{2} \right) \\ &+ \left(\mathbf{E}\left[\frac{f_1 - f_0}{2} \right] \cdot \mathbf{E}\left[\frac{g_1 - g_0}{2} \right] \right)\end{aligned}$$

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Pause and verify: If f is monotone, then this quantity is

$$\mathbf{Pr}_{\boldsymbol{x} \sim \{0,1\}^n} \left[f(\boldsymbol{x}) \neq f(\boldsymbol{x}^{\oplus n})\right]$$

where $x^{\oplus n} := (x_1, \ldots, x_{n-1}, 1 - x_n)$.

Influences of Variables on Boolean Functions

Definition: Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, we define the influence of coordinate $i \in [n]$ on f as

$$\mathbf{Inf}_i[f] := \mathbf{Pr}_{\boldsymbol{x} \sim \{0,1\}^n} \left[f(\boldsymbol{x}) \neq f(\boldsymbol{x}^{\oplus i}) \right]$$

where $x^{\oplus i} := (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)$.

Kahn–Kalai–Linial, Talagrand, Friedgut, Friedgut–Kalai,
Benjamini–Kalai–Schramm, Russo–Margulis, and friends. . .

Robustifying Harris–Kleitman

$$\begin{aligned}\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] = &\left(\frac{\mathbf{E}[f_0 \cdot g_0] - \mathbf{E}[f_0] \cdot \mathbf{E}[g_0]}{2}\right) \\ &+ \left(\frac{\mathbf{E}[f_1 \cdot g_1] - \mathbf{E}[f_1] \cdot \mathbf{E}[g_1]}{2}\right) \\ &+ \left(\mathbf{E}\left[\frac{f_1 - f_0}{2}\right] \cdot \mathbf{E}\left[\frac{g_1 - g_0}{2}\right]\right)\end{aligned}$$

- Can check that $\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] = 0$ if and only if

$$\forall\ i \in [n] : \mathbf{Inf}_i[f] = 0 \text{ or } \mathbf{Inf}_i[g] = 0.$$

Robustifying Harris–Kleitman

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- Hope: A better lower bound in terms of $\sum_{i=1}^{n} \mathbf{Inf}_i[f] \cdot \mathbf{Inf}_i[g]$.

Talagrand's Correlation Inequality

Theorem: Let $f, g : \{0,1\}^n \to \{0,1\}$ be monotone. Then

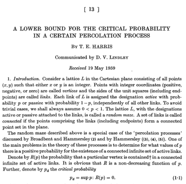
$$\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] \geq \frac{1}{C} \cdot \Psi\left(\sum_{i=1}^{n} \mathbf{Inf}_i[f] \cdot \mathbf{Inf}_i[g]\right)$$

where C is a universal constant and $\Psi(x) := \frac{x}{\log(e/x)}$.

- Key lemma is a decoupled level-2 inequality.
- Several applications of Talagrand's lemma in additive combinatorics, analysis of Boolean functions, etc.

Qualitative Correlation Inequalities

Monotone $f, g : \{0,1\}^n \to \{0,1\}$	Symmetric, convex $K, L \subseteq \mathbb{R}^n$
Uniform distribution	$\mathcal{N}(0,1)^n$



[13]

A LOWER BOUND FOR THE CRITICAL PROBABILITY IN A CERTAIN PERCOLATION PROCESS

By T. E. Harris

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Received 19 May 1959

1. *Introduction.* Consider a lattice L in the Cartesian plane consisting of all points (x, y) such that either x or y is an integer. Points with integer coordinates (positive, negative, or zero) are called *vertices* and the sides of the unit squares (including end-points) are called *links*. Each link of L is assigned the designation *active* with probability p or *passive* with probability $1-p$, independently of all other links. To avoid trivial cases, we shall always assume $0 < p < 1$. The lattice L, with the designations active or passive attached to the links, is called a *random maze*. A set of links is called *connected* if the points comprising the links (including endpoints) form a connected point set in the plane.

The random maze described above is a special case of the 'percolation processes' discussed by Broadbent and Hammersley (2) and by Hammersley ((3), (4), (5)). One of the main problems in the theory of these processes is to determine for what values of p there is a positive probability for the existence of a connected infinite set of active links.

Denote by $R(p)$ the probability that a particular vertex is contained† in a connected infinite set of active links. It is obvious that R is a non-decreasing function of p. Further, denote by p_d the *critical probability*

$$p_d = \sup p : R(p) = 0. \quad (1\cdot 1)$$

H-K: $\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] \geq 0$

Royen: $\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq 0$

Quantitative Correlation Inequalities

Monotone $f, g : \{0,1\}^n \to \{0,1\}$ Uniform distribution	Symmetric, convex $K, L \subseteq \mathbb{R}^n$ $\mathcal{N}(0,1)^n$

?

Talagrand: $\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g]$

$$\geq \frac{1}{C} \cdot \Psi\left(\sum_{i=1}^{n} \mathbf{Inf}_i[f] \cdot \mathbf{Inf}_i[g]\right)$$

where $\Psi(x) := \frac{x}{\log(e/x)}$

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Our Contributions

Theorem: Let $K, L \subseteq \mathbb{R}^n$ be symmetric, convex sets, identified with their indicator functions. Then

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq \frac{1}{C} \cdot \Phi\left(\sum_{\alpha \in \mathbb{N}^n : \|\alpha\|_1 = 2} \widehat{K}(\alpha)\widehat{L}(\alpha)\right)$$

where C is a constant and $\Phi(x) := \frac{x}{\log^2(1/x)}$.

- Term inside $\Phi(\cdot)$ is always non-negative when L and K are symmetric and convex.
- Correlation gap in terms of the inner product of degree-2 part of the Hermite expansion of K and L.

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$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq \frac{1}{C} \cdot \Phi\left(\sum_{i=1}^{n} \mathbf{Inf}_i[K] \cdot \mathbf{Inf}_i[L]\right)$$

where C is a constant and $\Phi(x) := \frac{x}{\log^2(1/x)}$.

- New notion of influence for symmetric, convex sets.
- Strategy generalizes to other domains easily—e.g. essentially recovers Talagrand's result over $\{0,1\}^n$.

Proof Strategy

For symmetric, convex $K, L \subseteq \mathbb{R}^n$, consider

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L]$$

where expectations are w.r.t. $\mathcal{N}(0,1)^n$.

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  Royen's proof

Outline

1. Harmonic Analysis — Hermite basis, Ornstein–Uhlenbeck operator

$\downarrow$

2. Royen's Proof — Monotonicity

$\downarrow$

3. Quantitative GCI — Influences for convex sets

Outline

1. Harmonic Analysis — Hermite basis, Ornstein–Uhlenbeck operator

$\downarrow$

2. Royen's Proof — Monotonicity

$\downarrow$

3. Quantitative GCI — Influences for convex sets

Harmonic Analysis over Gaussian Space

- Consider $L^2\big(\mathbb{R}^n, \mathcal{N}(0,1)^n\big)$ as an inner-product space endowed with the inner product

$$\langle f, g \rangle := \mathbf{E}_{\boldsymbol{x} \sim \mathcal{N}(0,1)^n} \big[f(\boldsymbol{x}) \cdot g(\boldsymbol{x}) \big].$$

- Define $\|f\| := \sqrt{\langle f, f \rangle}$.

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> Fact: The Hermite polynomials $(h_j)_{j \in \mathbb{N}}$ form a complete, orthonormal basis for $L^2\big(\mathbb{R}, \mathcal{N}(0,1)\big)$.

$$h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = \frac{x^2-1}{\sqrt{2}},\ h_3(x) = \frac{x^3-3x}{\sqrt{6}},\ \ldots$$

The Hermite Basis

Hermite polynomials $(h_j)_{j \in \mathbb{N}}$

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Given a function $f \in L^2\big(\mathbb{R}^n, \mathcal{N}(0,1)^n\big)$, we can write

$$f = \sum_{\alpha\in\mathbb{N}^n} \widetilde{f}(\alpha) h_\alpha$$

$\uparrow$

The Hermite expansion of f

The Hermite Basis

Hermite polynomials $(h_j)_{j\in\mathbb{N}}$

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$$\uparrow$$

The Hermite coefficients of f

Hermite spectrum

Parseval's Formula

Fact: Let $f, g \in L^2\big(\mathbb{R}^n, \mathcal{N}(0,1)^n\big)$, with Hermite expansions

$$f = \sum_{\alpha \in \mathbb{N}^n} \widetilde{f}(\alpha) h_\alpha \qquad \text{and} \qquad g = \sum_{\alpha \in \mathbb{N}^n} \widetilde{g}(\alpha) h_\alpha.$$

Then

$$\langle f, g \rangle = \mathbf{E}_{\boldsymbol{x} \sim \mathcal{N}(0,1)^n} \big[f(\boldsymbol{x}) \cdot g(\boldsymbol{x}) \big] = \sum_{\alpha \in \mathbb{N}^n} \widetilde{f}(\alpha) \widetilde{g}(\alpha).$$

As a special case, note that

$$\|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \widetilde{f}(\alpha)^2$$

The Ornstein–Uhlenbeck Noise Operator

Definition: Given a function $f \in L^2\big(\mathbb{R}^n, \mathcal{N}(0,1)^n\big)$, define the Ornstein–Uhlenbeck noise operator U_ρ as

$$\mathrm{U}_\rho f(x) := \mathbf{E}_{\boldsymbol{g} \sim \mathcal{N}(0,1)^n}\left[f\Big(\rho x + \sqrt{1-\rho^2}\,\boldsymbol{g}\Big)\right].$$

- Analogous to the Bonami–Beckner operator T_ρ over $\{0,1\}^n$.
- Note: $\rho = 1 \implies \mathrm{U}_\rho f(x) = f(x)$, and $\rho = 0 \implies \mathrm{U}_\rho f(x) = \mathbf{E}[f]$.

Hermite Expansion under the OU Operator

Definition: For $\alpha \in \mathbb{N}^n$, the degree or level of α (and $\widetilde{f}(\alpha)$) is

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

Hermite Expansion under the OU Operator

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$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

The Ornstein–Uhlenbeck operator diagonalizes the Hermite basis:

Fact: Let $f \in L^2\big(\mathbb{R}^n, \mathcal{N}(0,1)^n\big)$ with Hermite expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} \widetilde{f}(\alpha) h_\alpha.$$

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Then the Hermite expansion of $\mathrm{U}_\rho f$ is given by

$$\mathrm{U}_\rho f = \sum_{\alpha \in \mathbb{N}^n} \rho^{|\alpha|} \widetilde{f}(\alpha) h_\alpha.$$

Quick Recap

Let $f, g \in L^2\big(\mathbb{R}^n, \mathcal{N}(0,1)^n\big)$. Then

1. Hermite expansion: $f = \sum_{\alpha \in \mathbb{N}^n} \widetilde{f}(\alpha) h_\alpha$.
2. Parseval's formula: $\langle f, g \rangle = \sum_{\alpha \in \mathbb{N}^n} \widetilde{f}(\alpha) \widetilde{g}(\alpha)$.
3. Noise operator: $\mathrm{U}_\rho f(x) := \mathbf{E}_{\boldsymbol{g} \sim \mathcal{N}(0,1)^n} \left[f\left(\rho x + \sqrt{1-\rho^2} \boldsymbol{g} \right) \right]$ which acts on Hermite expansion as follows:

$$\mathrm{U}_\rho f = \sum_{\alpha \in \mathbb{N}^n} \rho^{|\alpha|} \widetilde{f}(\alpha) h_\alpha$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$.

Reference: O'Donnell's *Analysis of Boolean Functions*

Outline

1. Harmonic Analysis — Hermite basis, Ornstein–Uhlenbeck operator

$\downarrow$

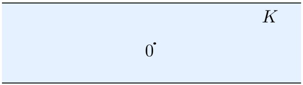
2. Royen's Proof — Monotonicity

$\downarrow$

3. Quantitative GCI — Influences for convex sets

Royen's Inequality (2014)

- Also called the Gaussian Correlation Inequality (GCI).
- Open for over 40 years.



$K \subseteq \mathbb{R}^n$ is symmetric if $x \in K$ implies $-x \in K$.

Theorem: Let $K, L \subseteq \mathbb{R}^n$ be symmetric, convex sets, identified with their $0/1$ indicator functions. Then

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq 0$$

where the expectations are with respect to $\mathcal{N}(0,1)^n$.

Royen's Inequality (2014)

- Note that $\langle \mathrm{U}_1 K, L \rangle = \mathbf{E}[K \cdot L]$ and $\langle \mathrm{U}_0 K, L \rangle = \mathbf{E}[K] \cdot \mathbf{E}[L]$.

Recall $\langle f, g \rangle = \mathbf{E}[f \cdot g]$ and $\mathrm{U}_\rho f(x) := \mathbf{E}\left[f\left(\rho x + \sqrt{1-\rho^2}\boldsymbol{g}\right)\right]$

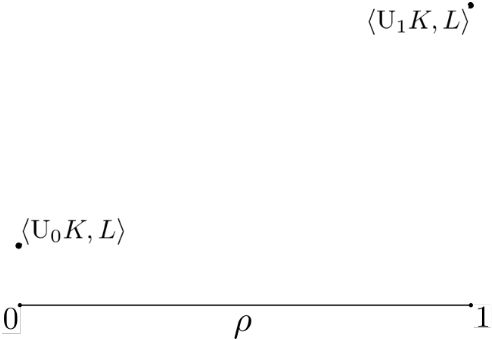
Royen's Inequality (2014)

- Note that $\langle \mathrm{U}_1 K, L\rangle = \mathbf{E}[K \cdot L]$ and $\langle \mathrm{U}_0 K, L\rangle = \mathbf{E}[K] \cdot \mathbf{E}[L]$.

Royen's Inequality: $\langle \mathrm{U}_1 K, L\rangle \geq \langle \mathrm{U}_0 K, L\rangle$.

Royen's Proof

Given $K, L \subseteq \mathbb{R}^n$ convex, symmetric, $\left\langle \mathrm{U}_\rho K, L \right\rangle$ is increasing in ρ.



Recall $\langle f, g\rangle = \mathbf{E}[f \cdot g]$ and $\mathrm{U}_\rho f(x) := \mathbf{E}\left[f\left(\rho x + \sqrt{1-\rho^2}\boldsymbol{g}\right)\right]$

Royen's Proof

- Let $K, L : \mathbb{R}^n \to \{0,1\}$ be the indicator functions of two symmetric, convex sets.
- Note that $\langle \mathrm{U}_1 K, L\rangle = \mathbf{E}[K \cdot L]$ and $\langle \mathrm{U}_0 K, L\rangle = \mathbf{E}[K] \cdot \mathbf{E}[L]$.

Royen's Theorem: $\frac{d}{d\rho}\langle \mathrm{U}_\rho K, L\rangle \geq 0.$

$$\downarrow$$

Corollary: $\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq 0.$

Recall $\langle f, g\rangle = \mathbf{E}[f \cdot g]$ and $\mathrm{U}_\rho f(x) := \mathbf{E}\left[f\left(\rho x + \sqrt{1-\rho^2}\boldsymbol{g}\right)\right]$

Outline

1. Harmonic Analysis — Hermite basis, Ornstein–Uhlenbeck operator

$\downarrow$

2. Royen's Proof — Monotonicity

$\downarrow$

3. Quantitative GCI — Influences for convex sets

A Quantitative GCI

- We have $K, L \subseteq \mathbb{R}^n$ symmetric, convex.
- Want a quantitative lower bound on

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] = \langle \mathrm{U}_1 K, L \rangle - \langle \mathrm{U}_0 K, L \rangle.$$

Proof Sketch

$$\mathrm{U}_\rho K = \sum_{\alpha \in \mathbb{N}^n} \rho^{|\alpha|} \widetilde{K}(\alpha) h_\alpha \qquad \text{and} \qquad L = \sum_{\alpha \in \mathbb{N}^n} \widetilde{L}(\alpha) h_\alpha$$

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$$\boxed{\left\langle \mathrm{U}_\rho K, L\right\rangle = \sum_{\alpha\in\mathbb{N}^n} \rho^{|\alpha|}\widetilde{K}(\alpha)\widetilde{L}(\alpha)}$$

Recall $|\alpha| = \sum_{i=1}^n \alpha_i$.

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$$\big\langle \mathrm{U}_\rho K, L \big\rangle - \langle \mathrm{U}_0 K, L \rangle = \sum_{|\alpha|>0} \rho^{|\alpha|} \widetilde{K}(\alpha) \widetilde{L}(\alpha)$$

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$$\left\langle \mathrm{U}_\rho K, L \right\rangle - \langle \mathrm{U}_0 K, L \rangle = \sum_{|\alpha|>0} \rho^{|\alpha|} \widetilde{K}(\alpha)\widetilde{L}(\alpha)$$
$$= \sum_{i=1}^{\infty} a_i \rho^i$$

where $a_i := \sum_{|\alpha|=i} \widetilde{K}(\alpha)\widetilde{L}(\alpha)$.

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- Cauchy-Schwarz + Parseval $\implies \sum_{i=1}^{\infty} |a_i| \leq 1$.

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- Cauchy-Schwarz + Parseval $\implies \sum_{i=1}^{\infty} |a_i| \leq 1$.
- K, L symmetric $\implies \widetilde{K}(\alpha) = \widetilde{L}(\alpha) = 0$ when $|\alpha|$ odd.

Recall $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Fact: h_α is even if $|\alpha|$ is even, and odd otherwise.

Proof Sketch

$$\left\langle \mathrm{U}_\rho K, L \right\rangle - \langle \mathrm{U}_0 K, L \rangle = \sum_{i=1}^{\infty} a_{2i} \rho^{2i}$$

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$$\left\langle \mathrm{U}_\rho K, L \right\rangle - \langle \mathrm{U}_0 K, L\rangle = \sum_{i=1}^{\infty} a_{2i}\rho^{2i}$$

$$= a_2 \underbrace{\left(t + \sum_{i=2}^{\infty} \frac{a_{2i}}{a_2} t^i \right)}_{=:p(t)}$$

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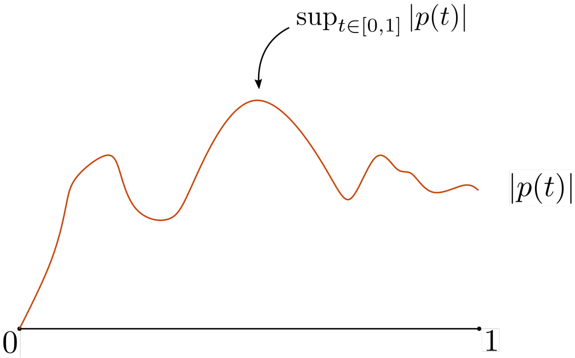
Question: How large must $\sup_{t \in [0,1]} p(t)$ be?

Let's analyze this question in more generality. . .

An Extremal Bound for Complex Power Series



$$p(t) = t + c_2 t^2 + c_3 t^3 + \dots$$

An Extremal Bound for Complex Power Series

Question: Let $p(t) = \sum_{i=1}^{\infty} c_i t^i$ where $c_i \in \mathbb{C}$, and

- $c_1 = 1$; and
- $\sum_{i=1}^{\infty} |c_i| \leq M$ where $M \geq 3/2$.

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- Proof uses Hadamard's Three Circles Theorem.
- This bound is tight: construction using Chebyshev polynomials.

Proof

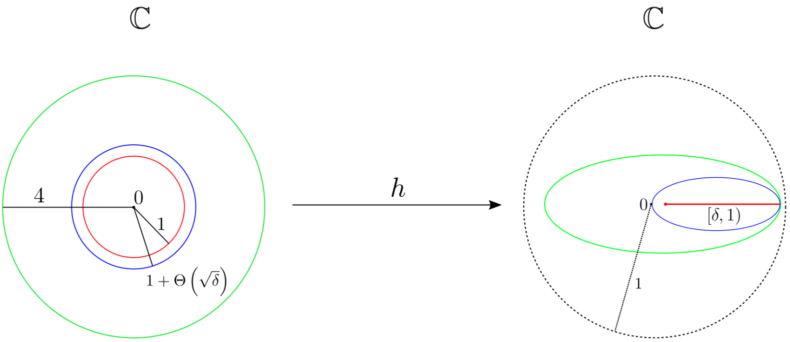
Given $p(t) = t + c_2 t^2 + c_3 t^3 + \dots$
where $1 + \sum_{i=2}^{\infty} |c_i| \leq M$ for $M \geq 3/2$.

Want: Lower bound on $\sup_{t \in [0,1]} |p(t)|$.

- Note that $p(0) = 0$, so must move away from 0.
- Write $p(t) := t \cdot q(t)$, where

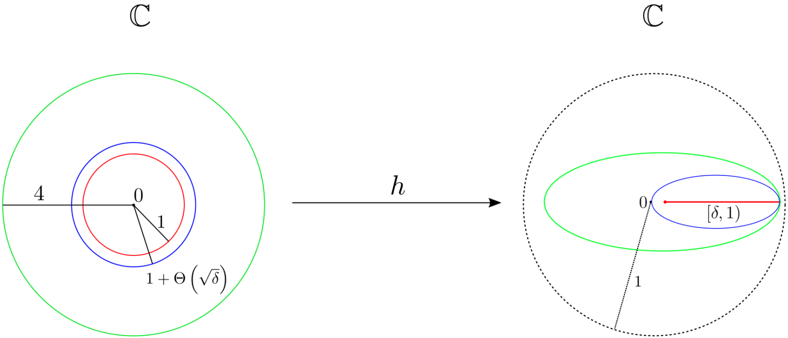
$$q(t) := 1 + c_2 t + c_3 t^2 + \dots$$

and optimize $q(t)$ over $[\delta, 1)$ for some δ that we will fix later.



We define $h : \mathbb{C} \to \mathbb{C}$ (see above) as

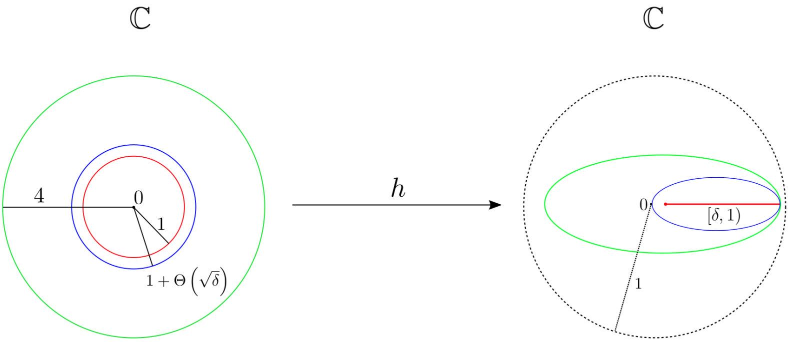
$$h(z) = A\left(z + \frac{1}{z}\right) + B$$



- Given $p(t) = t + c_2t^2 + c_3t^3 + \ldots$ with $1 + |c_2| + \ldots \leq M$.
- We defined $q(t) := \frac{p(t)}{t}$, i.e.

$$q(t) = 1 + c_2t + c_3t^2 + \ldots.$$

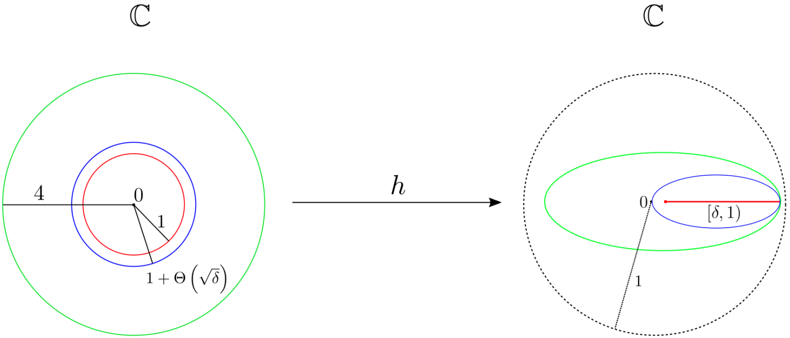
- We will be interested in $\psi := q \circ h$.



- Let $\alpha(r) := \sup_{|z|=r} |\psi(z)|$.
- Then Hadamard's three circles theorem implies that

$$\alpha\left(1+\Theta\left(\sqrt{\delta}\right)\right)^{\log\left(\frac{4}{1}\right)} \leq \alpha(1)^{\log\left(\frac{4}{1+\Theta\left(\sqrt{\delta}\right)}\right)} \alpha(4)^{\log\left(\frac{1+\Theta\left(\sqrt{\delta}\right)}{1}\right)}$$

Recall $q(t) := p(t)/t$ and $\psi(z) := q(h(z))$.



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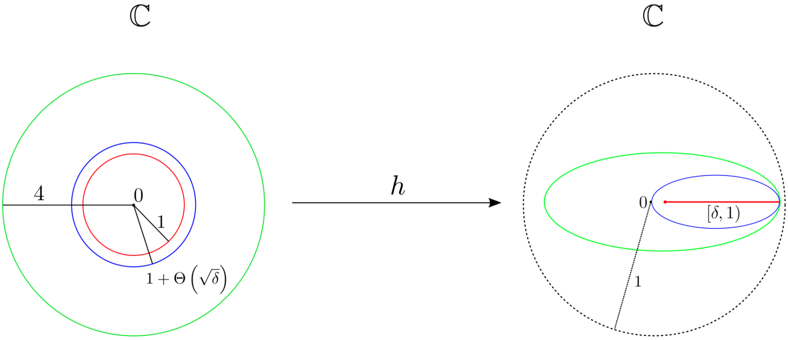

- Let $\alpha(r) := \sup_{|z|=r} |\psi(z)|$.
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$$1 \leq \alpha(1)^{\log\left(\frac{4}{1+\Theta\left(\sqrt{\delta}\right)}\right)} \alpha(4)^{\log\left(\frac{1+\Theta\left(\sqrt{\delta}\right)}{1}\right)}$$

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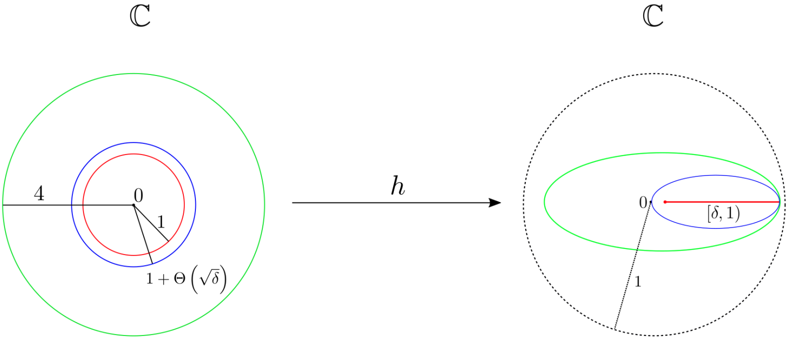

So we have

$$1 \le \left(\sup_{|z|=1} |\psi(z)|\right)^{\log\left(\frac{4}{1+\Theta\left(\sqrt{\delta}\right)}\right)} \left(\sup_{|z|=4} |\psi(z)|\right)^{\log\left(\frac{1+\Theta\left(\sqrt{\delta}\right)}{1}\right)}$$

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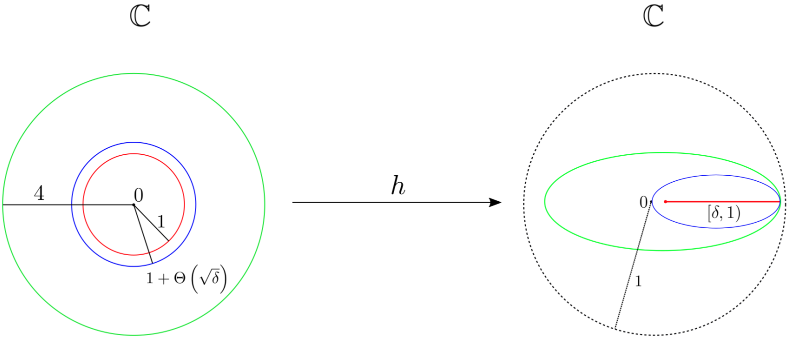

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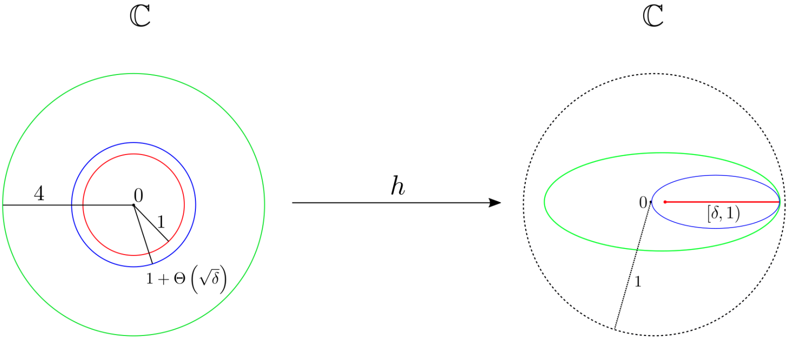
$= \Theta(1)$ $\qquad$ $\approx \Theta\left(\sqrt{\delta}\right)$



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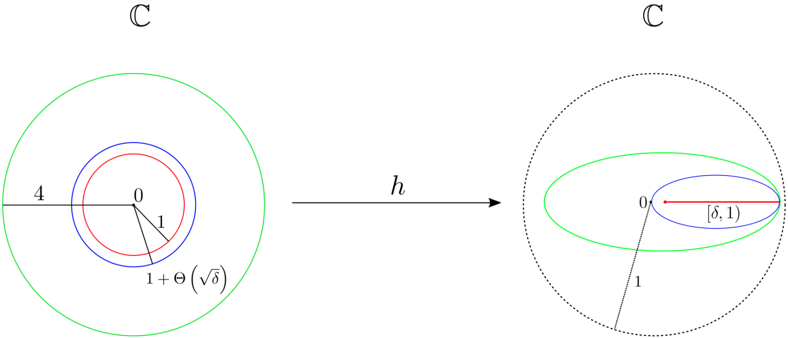

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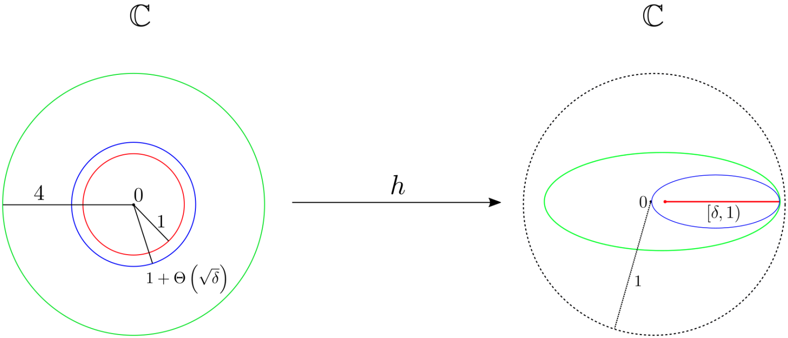

$$1 \leq \left(\sup_{t \in [\delta, 1)} |q(t)| \right)^{\Theta(1)} \left(\sup_{\substack{t \in h(z) \\ |z| = 4}} |q(t)| \right)^{\Theta\left(\sqrt{\delta}\right)}$$

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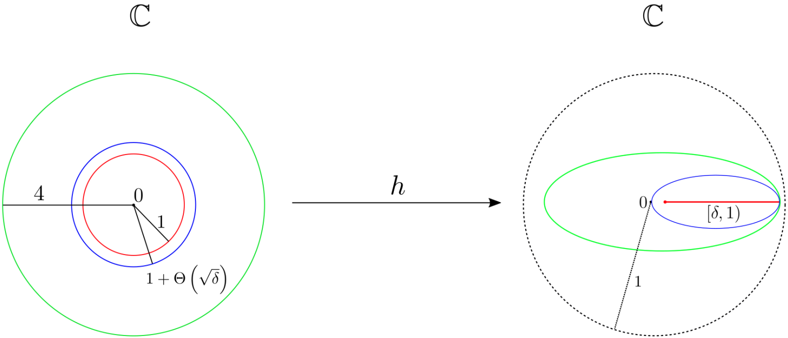

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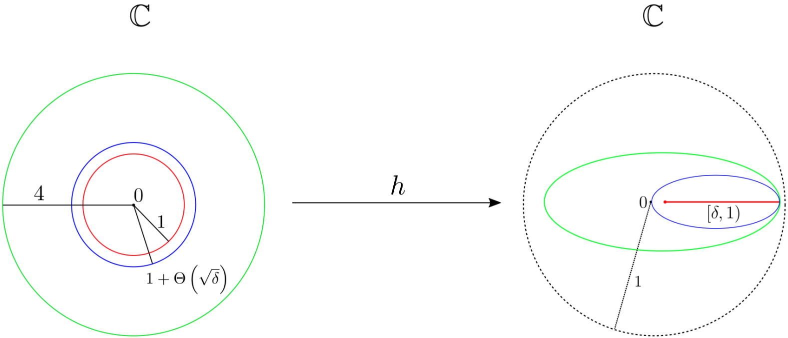
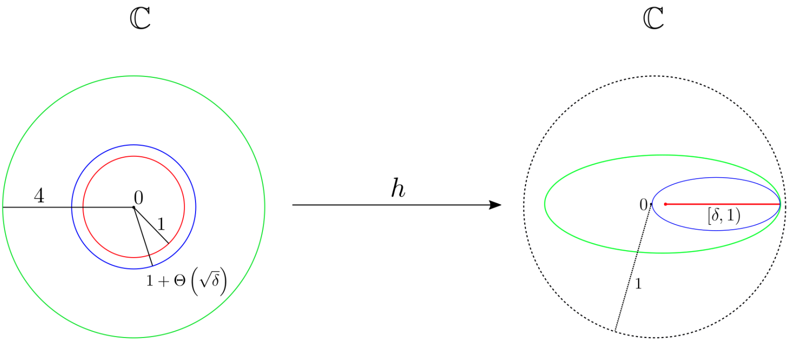
- Recall $q(t) = 1 + c_2 t + c_3 t^2 + \ldots$ where $1 + |c_2| + |c_3| + \ldots \le M$.
- Consequently, if $|z| \le 1$ then $|q(z)| \le M$.



We can rearrange and use $q(t) := p(t)/t$ to get

$$M^{-\Theta\left(\sqrt{\delta}\right)} \leq \sup_{t \in [\delta, 1)} |q(t)| \implies \sup_{\delta \in [0,1]} \delta M^{-\Theta\left(\sqrt{\delta}\right)} \leq \sup_{t \in [0,1]} |p(t)|.$$



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Taking $\delta = \frac{\Theta(1)}{\log^2(M)}$ gives

$$\sup_{t\in[0,1]} |p(t)| \geq \frac{\Theta(1)}{\log^2(M)}. \quad \square$$

Returning to the Quantitative GCI...

Proof Sketch

$$\left\langle \mathrm{U}_\rho K, L \right\rangle - \langle \mathrm{U}_0 K, L\rangle = a_2 \underbrace{\left(t + \sum_{i=2}^{\infty} \frac{a_{2i}}{a_2} t^i \right)}_{=: p(t)}$$

where $a_i := \sum_{|\alpha|=i} \widetilde{K}(\alpha)\widetilde{L}(\alpha)$, $\sum_{i=1}^{\infty} |a_i| \leq 1$, and $t := \rho^2$.

Question: How large must $\sup_{t\in[0,1]} p(t)$ be?

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$\big\uparrow$

Royen: $\left\langle \mathrm{U}_\rho K, L \right\rangle$ increasing in ρ.

$\big\downarrow$

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$$\langle \mathrm{U}_1 K, L\rangle - \langle \mathrm{U}_0 K, L\rangle \geq \Theta\left(\frac{a_2}{\log^2(1/a_2)}\right) = \frac{1}{C}\cdot\Phi(a_2)$$

where C is a constant and $\Phi(x) := \frac{x}{\log^2(1/x)}$.

Recall $a_2 := \sum_{|\alpha|=2} \widetilde{K}(\alpha)\widetilde{L}(\alpha)$.

Looking at a_2 More Closely...

Given $K, L \subseteq \mathbb{R}^n$ symmetric, convex we defined

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- If $|\alpha| = 2$, then either
  - $\alpha = e_i + e_j$ for $i \neq j$.
  - $\alpha = 2e_i$ for some $i \in [n]$.

Looking at a_2 More Closely...

Given $K, L \subseteq \mathbb{R}^n$ symmetric, convex we defined

$$a_2 := \sum_{|\alpha|=2} \widetilde{K}(\alpha)\widetilde{L}(\alpha).$$

- Let $e_i \in \mathbb{N}^n$ be $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, i.e. $(e_i)_j := \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$.
- If $|\alpha| = 2$, then either
  - $\alpha = e_i + e_j$ for $i \neq j$.
  - $\alpha = 2e_i$ for some $i \in [n]$.
- There exists suitable rotation such that $\widetilde{K}(e_i + e_j) = 0$ for all $i \neq j$.

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After suitable rotation, we have $a_2 := \sum_{i=1}^{n} \widetilde{K}(2e_i)\widetilde{L}(2e_i)$.

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So we have $a_2 = \sum_{i=1}^{n} \mathbf{Inf}_i[K] \cdot \mathbf{Inf}_i[L]$.

A Quantitative GCI

Theorem: Let $K, L \subseteq \mathbb{R}^n$ be symmetric, convex sets, identified with their indicator functions. Then

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq \frac{1}{C} \cdot \Phi\left(\sum_{i=1}^{n} \mathbf{Inf}_i[K] \cdot \mathbf{Inf}_i[L]\right)$$

where C is a constant and $\Phi(x) := \frac{x}{\log^2(1/x)}$.

Recall $\mathbf{Inf}_i[K] := -\widetilde{K}(2e_i) = \mathbf{E}\left[K(\boldsymbol{x}) \cdot \frac{(1-\boldsymbol{x}_i^2)}{\sqrt{2}}\right]$.

Generalizing to Other Domains

Probability space $(\Omega, \mathcal{D})$, consider

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for $f, g \in \mathcal{F} \subseteq L^2(\Omega, \mathcal{D})$.

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  Implies qualitative correlation inequalities

A new notion of influences

For a symmetric convex set K and direction v, $\mathbf{Inf}_v[K] := \mathbf{E}\left[K(\boldsymbol{x}) \cdot \frac{(1-\langle \boldsymbol{x}, v\rangle^2)}{\sqrt{2}}\right]$

A new notion of influences

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Questions:

- Is $\mathbf{Inf}_v[K]$ always non-negative?
- Suppose $\mathbf{Inf}_v[K] = 0$. Does it mean v is *irrelevant*?
- Other properties of $\mathbf{Inf}_v[K]$?

Non-negativity of influences

Proposition: For a symmetric convex set K and direction v, $\mathbf{Inf}_v[K] \geq 0$.

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An alternate characterization of $\mathbf{Inf}_v[K]$: For $\mathbf{\Lambda} \in \mathbb{R}_n^+$,

$$\mathbf{\Lambda} \cdot K \coloneqq \{(\mathbf{\Lambda}_1 x_1, \ldots, \mathbf{\Lambda}_n x_n) : (x_1, \ldots, x_n) \in K\}.$$

Let $v = e_1$. Define $\mathbf{\Delta} = (1-\delta, 1, \ldots, 1)$. Then,

$$\mathbf{Inf} e_1[K] = \frac{1}{\sqrt{2}} \lim_{\delta \to 0} \frac{\gamma_n(K) - \gamma_n(\mathbf{\Delta} \cdot K)}{\delta}$$

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It is not hard to prove that $\gamma_n(\mathbf{\Delta} \cdot K) \leq \gamma_n(K)$. Thus, $\mathbf{Inf} e_1[K] \geq 0$.

Characterization of zero influence

Proposition: If $\mathbf{Inf}_{e_1}[K] = 0$, then $K = e_1 \times K_{-e_1}$ where K_{-e_1} is in span$(e_2, \ldots, e_n)$.

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Proof: Let $K_t = \{x \in \mathbb{R}^{n-1} : (t, x) \in K\}$.

If $\mathbf{Inf}_{e_1}[K] = 0$, then $\gamma_{n-1}(K_t)$ is independent of t (not difficult to show).

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However, $\frac{K_t + (-K_t)}{2} \subseteq K_0$ (by convexity). Ehrhard-Borell implies that

$$\Phi^{-1}(\gamma_{n-1}(K_0)) \geq \Phi^{-1}\left(\gamma_{n-1}\left(\frac{K_t + (-K_t)}{2}\right)\right) \geq \Phi^{-1}(\gamma_{n-1}(K_t)).$$

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Proof (contd.): Thus, Ehrhard-Borell holds with equality. [Shenfeld-van Handel, 2018] show that this implies $K_t = K_0$. $\square$

Open question: Small influence = almost cylinder?

Conjecture: Suppose $\mathbf{Inf}_{e_1}[K] \leq \varepsilon$. Then, there exists $K' \subseteq \mathrm{span}(e_2, \ldots, e_n)$ and δ such that $\gamma_n(K \Delta (e_1 \times K')) \leq \delta$ (where $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$).

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Current status: $\delta =$ poly(n, ε) is also unknown. Main issue: Stability of Ehrhard-Borell inequality is completely open.

Some favorites from Boolean function analysis

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Proof of KKL-type inequality

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Note that $\sum_{i=1}^{n} \mathbf{Inf}_{v_i}[K]$ is independent of the choice of basis.

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Proof based on hyperplane separation theorem + some calculation.

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$$\sum_{i=1}^{n} \mathbf{Inf}_{v_i}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(K) - \gamma_n(\mathbf{\Delta} \cdot K)}{\delta} = \Omega(w_{\mathsf{in}} \cdot \mathbf{Var}[K]).$$

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Balancing w_{in}, we obtain $\max_v \mathbf{Inf}_v[K] = \Omega(\sqrt{\log n}/n)$.

Other contributions

- Weak form of Friedgut's junta theorem.
- Sharp threshold results – à la Friedgut-Kalai (ours is quantitatively stronger).
- Kruskal-Katona type result for convex sets (improvement à la O'Donnell-Wimmer).

Summary

- Bootstrapping qualitative correlation inequalities to obtain quantitative correlation inequalities:
    1. Interpolation via noise operators.
    2. Extremal bound for power series of bounded length.
- Influences for symmetric, convex sets over Gaussian space.

Thanks!