Functional Volume Product Along the Fokker-Planck Heat Flow

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Polarity and the even Santaló inequality

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Santaló's inequality for symmetric convex bodies

Let K be a symmetric convex body. Then,

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Shown by Blaschke and Santaló; see the elegant proof by Meyer and Pajor.

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Proven by Ball in his Ph.D thesis.

Approximation of Polarity

Terence Tao's Blog (paraphrasing): If f is log-concave, then the Laplace transform of $f^{\frac{1}{p}}(x)$ is essentially $(f^{\frac{1}{p}})^{\circ}(\frac{x}{p})$, ignoring lower-order contributions... in which case Klartag's formulation of Santaló's inequality begins to look quite a lot like Beckner's sharp Hausdorff-Young inequality as $p \to 0^{+}$.

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2. We define, for $p \in (0,1)$, the L^p Laplace transform as

$$\mathcal{L}_p(f)(x) := (L(f^{\frac{1}{p}}))(x)^{\frac{p}{p-1}} = \left(\int_{\mathbb{R}^n} f^{\frac{1}{p}}(y)\, e^{x \cdot y}\, dy \right)^{\frac{p}{p-1}} \qquad \forall x \in \mathbb{R}^n.$$

   Note that $\mathcal{L}_p(f)$ is always log-concave.

Laplace Transform Polarity

The L^p Laplace transform converges to polarity

$$\lim_{p\to 0^+} \mathcal{L}_p(f)(x/p) = \lim_{p\to 0^+} \left(\int_{\mathbb{R}^n} \left(e^{x\cdot y} f(y)\right)^{\frac{1}{p}} dy \right)^{\frac{p}{p-1}}$$

$$= f^{\square}(x) = \operatorname{ess}\inf_{y\in\mathbb{R}^n} \frac{e^{-x\cdot y}}{f(y)} \geq \inf_{y\in\mathbb{R}^n} \frac{e^{-x\cdot y}}{f(y)} = f^{\circ}(x).$$

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By considering $f(y) = e^{-|y|^2/2} + \delta(y)$, we see that

$$f^{\circ}(x) = \begin{cases} \frac{1}{2}, & \text{if } |x|^2 \leq 2\ln(2), \\ e^{-|x|^2/2}, & \text{otherwise.} \end{cases}$$

But, $f^{\Box}(x) = (e^{-|y|^2/2})^{\circ}(x) = e^{-|x|^2/2}$.

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Let f be an even function such that $\int_{\mathbb{R}^n} f \in (0, \infty)$. Then,

$$M_p(f) := \int_{\mathbb{R}^n} f \left(\int_{\mathbb{R}^n} \mathcal{L}_p(f) \left(\frac{x}{p} \right) \right)^{1-p} \leq M_p(e^{-|x|^2}),$$

with equality when f is Gaussian.

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Shown very recently by Nakamura-Tsuji. But they showed much more!

Fokker-Planck Heat Flow

For a (nonnegative) integrable function f its Fokker-Planck flow is $P_0 f = f$, and, for $t > 0$,

$$P_t f(x) = e^{nt/2} \int_{\mathbb{R}^n} f(y) e^{\frac{-|e^{t/2}x-y|^2}{2(e^t-1)}} \frac{dy}{(2\pi(e^t-1))^{\frac{n}{2}}}$$

$$= \left(\int_{\mathbb{R}^n} f(y) e^{\frac{e^{t/2}}{e^t-1} x \cdot y - \frac{1}{2(e^t-1)}|y|^2} dy \right) \frac{e^{-\frac{1}{1-e^{-t}}|x|^2/2}}{(2\pi(1-e^{-t}))^{\frac{n}{2}}}.$$

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It verifies the equation $\partial_t P_t f = \mathcal{D}^\star P_t f$, where

$$\mathcal{D}^\star f = \frac{1}{2}\big(\Delta f + \mathrm{div}_x(xf)\big).$$

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Nakamura-Tsuji's inequality for the functional volume product along the Fokker-Planck flow

Let f be even and integrable. Then,

$$t \mapsto M_p(P_t f)$$

is increasing in t.

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Implies, by sending $p \to 0^+$, that $t \mapsto M(P_t f)$ is monotonically increasing when f is even, integrable and regular enough.

What about non-even functions?

Convex Setting

Need to pick origin in a smart way. Let K be a convex body. If $b(K) = \frac{1}{\mathrm{Vol}_n(K)} \int_K x dx = o$, then

$$\mathrm{Vol}_n(K)\mathrm{Vol}_n(K^\circ) \leq \mathrm{Vol}_n(B_2^n)^2.$$

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Here, $s(K)$, the Santaló point of K, is so that $b((K - s(K))^\circ) = o$. It has the property that

$$s(K) = \mathrm{argmin}_{z \in \mathbb{R}^n} \mathrm{Vol}_n((K - z)^\circ),$$

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2. The infimum is obtained at a unique point, the Santaló point of f.
3. In fact, the Santaló point can be replaced by the barycenter of f.
4. Notice that

$$\int_{\mathbb{R}^n} (\tau_z f)^\circ = L[f^\circ](-z).$$

New results: inequalities

Theorem (L^p Santaló's inequality)

Let f be a nonnegative function, $f \not\equiv 0$. Then,

$$M_p(f) := \int_{\mathbb{R}^n} f \inf_{z \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathcal{L}_p(\tau_z f)\left(\frac{x}{p}\right)\right)^{1-p} \leq M_p(e^{-|x|^2}),$$

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Observe that

$$\mathcal{L}_p(\tau_z f)(x) = \mathcal{L}_p(f)(x)\, e^{\frac{p}{p-1}x\cdot z},$$

and so the second integral is really

$$L(\mathcal{L}_p(f))\left(-\frac{1}{1-p}z\right)^{1-p}.$$

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$$L(\mathcal{L}_p(f)) \left(-\frac{1}{1-p} z \right)^{1-p}.$$

If the infimum is obtained, then it is obtained at a unique point, which is precisely the barycenter of $\mathcal{L}_p(f)$.

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with equality for Gaussians.

Theorem (Laplace-Santaló's inequality)

Let f be a nonnegative function, $f \not\equiv 0$. Suppose either f or $\mathcal{L}_p(f)$ have barycenter at the origin. Then,

$$\int_{\mathbb{R}^n} f \left(\int_{\mathbb{R}^n} \mathcal{L}_p(f) \left(\frac{x}{p} \right) \right)^{1-p} \leq M_p(e^{-|x|^2}),$$

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Monotonicity of the functional L^p Volume product under heat flow

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Motonicity of the functional volume product

Let f be a nonnegative function, $f \not\equiv 0$ **with nice regularity e.g. continuous or log-concave**. Then

$$t \to M(P_t f)$$

is increasing in t.

Important Facts Concerning the Laplace Transform

Mathematics is Consistent

Let f be a nonnegative function on $\mathbb{R}^n$ and $p \in [0,1)$. Then we have

$$\int_{\mathbb{R}^n} f = \infty \Longrightarrow \inf_z \int_{\mathbb{R}^n} \mathcal{L}_p(\tau_z f)\left(\frac{x}{p}\right) = 0.$$

If f is log-concave, the converse is also true.

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If f is log-concave, the converse is also true.

Domain of $L\mathcal{L}_p(f)$

Let $f : \mathbb{R}^n \to [0,\infty)$ be a nonnegative function with $f \not\equiv 0$. Let $p \in [0,1)$.

$$\mathrm{dom}\left(L\mathcal{L}_p(f)\left(\frac{\cdot}{p}\right)\right) \supseteq \left(\frac{1}{1-p}\right) \mathrm{int}\,\mathrm{co}\,(\mathrm{supp}(f)).$$

If f is log-concave, there is equality in the previous assumption.
Anyway, we have in particular that

$$\mathrm{dom}(L(\mathcal{L}_p(f))) \neq \emptyset.$$

The infimum

Theorem (The Laplace-Santaló Point)

Let $f : \mathbb{R}^n \to \mathbb{R}^+$ such that $f \not\equiv 0$. Fix $p \in [0,1)$. Then, the following assertions are equivalent:

1. $\inf_z L(\mathcal{L}_p(f))(z) > 0$.
2. *o lies in the interior of the support of $\mathcal{L}_p(f)$.*
3. *$L(\mathcal{L}_p(f))$ tends to ∞ on the boundary of its domain.*
4. *$\mathcal{L}_p(f) \not\equiv 0$ and $\inf_z L\left(\mathcal{L}_p(f)\left(\frac{\cdot}{p}\right)\right)\left(\left(\frac{1}{p-1}\right)z\right)$ is attained at a (unique) point z_0 in the domain of $L\left(\mathcal{L}_p f\left(\frac{\cdot}{p}\right)\right)$.*
5. *$\mathcal{L}_p(f) \not\equiv 0$ and there exists $s_0 \in \mathbb{R}^n$ such that $\left(\frac{1}{p-1}\right) s_0$ is in the domain of $L\left(\mathcal{L}_p(f)\left(\frac{\cdot}{p}\right)\right)$ and*

$$\mathrm{bar}\left(\mathcal{L}_p\left(f\left(\frac{\cdot}{p}\right)\right)(x)e^{-\left(\frac{1}{1-p}\right)x\cdot s_0}\right) = 0.$$

Moreover, the points z_0 in (4) and s_0 in (5) are equal (and denoted by $s_p(f)$).

Examples $(q = \frac{p}{p-1} < 0)$

1. If $f = 1_{[a,b]}$ on $\mathbb{R}$ where $a < 0 < b$. Then we have $\mathcal{L}_p f(x) = \left(\int_a^b e^{xy}\,dy\right)^q = \left(\frac{e^{bx}-e^{ax}}{x}\right)^q$, and so the support of $\mathcal{L}_p(f)$ is equal to $\mathbb{R}$, while

$$L(\mathcal{L}_p f)(z) = \int_{\mathbb{R}} e^{xz} \left(\frac{e^{bx} - e^{ax}}{x}\right)^q dx \in [0,\infty]$$

has as domain $\operatorname{dom}(L(\mathcal{L}_p(f)) = \big((-q)a, (-q)b\big)$, and $L\mathcal{L}_p(f)$ tends to ∞ at the points $-qa$ and $-qb$. In particular $L(\mathcal{L}_p(f))$ attains its minimum.

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1. If $f = 1_{[a,b]}$ on $\mathbb{R}$ where $a < 0 < b$. Then we have $\mathcal{L}_p f(x) = \left(\int_a^b e^{xy}\,dy\right)^q = \left(\frac{e^{bx}-e^{ax}}{x}\right)^q$, and so the support of $\mathcal{L}_p(f)$ is equal to $\mathbb{R}$, while

$$L(\mathcal{L}_p f)(z) = \int_{\mathbb{R}} e^{xz} \left(\frac{e^{bx} - e^{ax}}{x}\right)^q dx \in [0, \infty]$$

   has as domain $\mathrm{dom}(L(\mathcal{L}_p(f)) = ((-q)a, (-q)b)$, and $L\mathcal{L}_p(f)$ tends to ∞ at the points $-qa$ and $-qb$. In particular $L(\mathcal{L}_p(f))$ attains its minimum.

2. If $f = 1_{[a,\infty)}$, then $\mathcal{L}_p(f)(x) = 1_{]-\infty,0]}(-x)^{-q} e^{qax}$ and

$$L(\mathcal{L}_p(f))(z) = \frac{\Gamma(1-q)}{(z+qa)^{1-q}} \quad \text{if } z > -qa \quad \text{and} \quad \infty \quad \text{otherwise,}$$

   has domain $(-qa, \infty)$. Note that 0 is not in the interior of (the convex hull of) the support of $\mathcal{L}_p f$ and that we have that $\inf L(\mathcal{L}_p f) = 0$, by letting $z \to \infty$.

Examples ($q = \frac{p}{p-1} < 0$)

1. If $f(x) = e^{-\alpha|x|^2}$ for some $\alpha > 0$, then $\mathcal{L}_p(f)(x) = c\, e^{-\beta|x|^2}$ and $L\mathcal{L}_p(f)(z) = C\, e^{\eta|z|^2}$ for some constants $c, C, \beta, \eta > 0$. In particular $L\mathcal{L}_p(f)$ attains its minimum at the origin.

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2. If $f(x) = \left(\frac{1}{1+x^2}\right)^\alpha$ on $\mathbb{R}$, $\alpha > 0$, then, $\mathcal{L}_p(f)(0) = \left(\int f^{1/p}\right)^q$ and $\mathcal{L}_p(f)(x) = 0$ otherwise. Thus, $\mathcal{L}_p(f) \equiv 0$ and so $L\mathcal{L}_p(f)(z) = 0$ for every z.

Proof Strategy I

First, justify that you can take limits.

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Limits of Laplace Transforms

Let $g_1, g_2, \ldots g_\infty : \mathbb{R}^n \to [0, \infty)$ be log-concave functions $\not\equiv 0$ such for almost all $x \in \mathbb{R}^n$,

$$g_k(x) \longrightarrow g_\infty(x).$$

This implies that $Lg_k(z) \longrightarrow Lg_\infty(z)$ in $\mathbb{R} \cup \{\infty\}$ at every $z \in \mathbb{R}^n$. We assume the following two properties of non-degeneracy and domain monotony, respectively:

- $0 < \inf_z Lg_k(z) < \infty$ for every $k \in \mathbb{N} \cup \{\infty\}$,
- either $\mathrm{dom}(Lg_k) \subseteq \mathrm{dom}(Lg_{k+1})$ for every $k \in \mathbb{N}$ or $\mathrm{dom}(Lg_\infty) \subseteq \bigcap_k \mathrm{dom}(Lg_k)$.

Then we have

$$\inf_z Lg_k(z) \longrightarrow \inf_z Lg_\infty(z).$$

Furthermore, for each k, there exists a unique point $z(g_k)$ where the infimum $\inf_z Lg_k(z)$ is attained and

$$z(g_k) \longrightarrow z(g_\infty).$$

Proof Strategy II

Then, truncate and prove the following (as in Nakamura-Tsuji).

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Truncated Theorem

Let f be a non-negative function bounded and compactly supported with $f \not\equiv 0$. Let f_t be its evolution along the Fokker-Planck or the heat semigroup. For $p \in (0,1)$, $q = \frac{p}{p-1}$, define the function $Q(t,z)$ for $t > 0$ and $z \in \mathbb{R}^n$ by

$$Q(t,z) := \log \int_{\mathbb{R}^n} \mathcal{L}_p(\tau_z(f_t)) = \log \int_{\mathbb{R}^n} \mathcal{L}_p(f_t)(x)\, e^{qz\cdot x}\, dx = \log L\mathcal{L}_p(f)(qz).$$

Then it holds that

$$\partial_t Q + \frac{1}{2}\frac{p}{-q}|\nabla_z Q|^2 \geq 0 \qquad \text{on } (0,\infty)\times\mathbb{R}^n.$$

It follows that

$$t \mapsto \int_{\mathbb{R}^n} \mathcal{L}_p(\tau_{s_p(f_t)} f_t)(x) dx = \inf_z \int_{\mathbb{R}^n} \mathcal{L}_p(\tau_z(f_t)) = \inf L\mathcal{L}_p(f_t)$$

is monotonically increasing in $t > 0$.

Tools

1. Variance of a function with respect to a measure μ:

$$\mathrm{Var}_\mu g := \int_{\mathbb{R}^n} |g|^2 d\mu(x) - \left(\int_{\mathbb{R}^n} g d\mu(x)\right)^2.$$

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2. Brascamp-Lieb inequality: Let $h \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ be non-negative, and strictly log-concave. Then, for any locally Lipschitz $g \in L^2(\mathbb{R}^n, \mu)$, with $d\mu_h(x) = \frac{h(x)dx}{\int_{\mathbb{R}^n} h(x)dx}$, one has

$$\mathrm{Var}_{\mu_h} g \leq \int_{\mathbb{R}^n} \left(\nabla g \cdot (\nabla^2(-\log h))^{-1} \nabla g \right) d\mu_h(x).$$

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3. The Cramér-Rao inequality: Let h be a non-negative, integrable function on $\mathbb{R}^n$. Let X be a random vector distributed with respect to μ_h. Then, the following matrix inequality holds:

$$I(X) \geq \mathrm{cov}(h)^{-1},$$

where $I(X)$ is the (Fisher) information matrix of X.

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$$E_t f(y) = \frac{1}{(2\pi t)^{n/2}} (f * \gamma_t)(y) = \int_{\mathbb{R}^n} f(u)\, e^{-|y-u|^2/(2t)} \frac{du}{(2\pi t)^{n/2}}, \quad \forall y \in \mathbb{R}^n,$$

and, under suitable integrability assumptions, follows the heat equation

$$\partial_t E_t f(x) = \frac{1}{2} \Delta_x E_t f(x).$$

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It is related to the Fokker-Planck heat semi-group via

$$E_t f(x) = (1+t)^{-n/2} P_{\log(1+t)} f\big((1+t)^{-1/2} x\big).$$

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then we have $Q(t,z) = \widetilde{Q}(\log(1+t), (1+t)^{-1/2} z)$ and therefore, for any constant c

$$(\partial_t Q + c|\nabla Q|^2)(t,z) = \frac{1}{1+t}(\partial_t \widetilde{Q} + c|\nabla \widetilde{Q}|^2)(\log(1+t), (1+t)^{-1/2} z).$$

Consequently, proving the truncated theorem along the Fokker-Planck or heat semi-group is totally equivalent.

Proof Strategy IV

To prove the (non)-truncated version, take limits.

Infimum-time relation

Let f be a non-negative function, $f \not\equiv 0$, and let $f_t = E_t f$ be its heat flow evolution. Then for $0 \leq s \leq t$,

$$\mathrm{supp}(\mathcal{L}_p(f_s)) \subseteq \mathrm{supp}(\mathcal{L}_p(f_t)),$$

and consequently

$$\inf L\mathcal{L}_p(f_t) = 0 \Longrightarrow \forall s \in [0, t],\ \inf L\mathcal{L}_p(f_s) = 0.$$

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Let f be a nonnegative function, $f \not\equiv 0$, and let $s \in [0, \infty)$. If $\inf L\mathcal{L}_p(f_s) > 0$ then, as $k \to \infty$, we have

$$\inf L\mathcal{L}_p((f^{(k)})_s) \to \inf L\mathcal{L}_p(f_s).$$

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$$\|U_s(f)\|_{L^{p_2}(\gamma)} \leq \|f\|_{L^{p_1}(\gamma)},$$

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   when $1 < p_1, p_2 < \infty$ and $s > 0$ satisfy $\frac{p_2-1}{p_1-1} \leq e^{2s}$.
5. Borell's reverse hypercontractivity is then the fact that the above reverses when $p_1, p_2 \in (-\infty, 1) \setminus \{0\}$ satisfy the same relation.

Application to Hypercontractivity II

1. For any nonnegative f, we have the point-wise equality, with $q = \frac{p}{p-1}$ and $s = -\frac{1}{2}\log(1-p)$,

$$(U_s f(x))^q \times (2\pi p)^{\frac{nq}{2}} \gamma_1(x) = \mathcal{L}_p(f^p \gamma_1)\left(\frac{x}{\sqrt{p|q|}}\right).$$

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2. Thus, for an explicit constant $\tilde{c}_{n,p} > 0$,

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From Jensen's inequality, we obtain the following.

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Extended hypercontractivity

Let $p \in (0,1), q = p/(p-1)$. Define $s > 0$ via the relation $p = 1 - e^{-2s}$, so that $q = 1 - e^{2s}$. Suppose f is a nonnegative function such that either

$$\int x\, f^p(x)\, d\gamma = 0 \text{ or } \int x\, U_s(f)^q(x)\, d\gamma = 0.$$

Then, for every $p_2 \geq q$ and $p_1 \leq p$, it follows that

$$\|U_s f\|_{L^{p_2}(\gamma)} \geq \|f\|_{L^{p_1}(\gamma)}.$$

Open Question

We proving the truncated theorem, we had

$$\begin{aligned} \frac{d}{dt}Q(t,z) &= \partial_t Q(t, s_p(f_t)) + \nabla_z Q(t, s_p(f_t)) \cdot \partial_t s_p(f_t) \\ &\geq \frac{p}{2q}|\nabla_z Q(t, s_p(f_t))|^2 + \nabla_z Q(t, s_p(f_t)) \cdot \partial_t s_p(f_t) = 0, \end{aligned}$$

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Our choice of $z = s_p(f_t)$ yielded

$$|\nabla_z Q(t,s_p(f_t))|^2 = 0 = \nabla_z Q(t,s_p(f_t))\cdot\partial_t s_p(f_t).$$

Open Question II

p-Santaló region along the Fokker-Planck flow

For a fixed nonnegative, smooth function f and a vector field $t \to u_p(t,f) \in \mathbb{R}^n$, the monotonicity of

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for $t \in (0,\infty)$ still holds if

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Question: does this time-dependent gradient flow equation have at least one (smooth enough) solution on $\mathbb{R}^+$ with $\lim_{t\to\infty} u_p(t,f) = 0$?