

Ehrhard's Inequality

The one-dimensional functional form and an L^2 proof by dimensional lifting

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Abstract

We give two separately readable analytic arguments. The first proves the one-dimensional functional Ehrhard inequality directly from the heat equation, using an exact equation for the interpolation deficit and an L^2 estimate for its negative part. The second establishes a sharp conditional Gaussian moment inequality by an independent concavity-preservation argument, and then derives geometric Ehrhard for convex sets by lifting into one additional dimension. The lifting identities are exact, and all approximation and endpoint passages are included. Both L^2 arguments are parabolic; neither takes geometric Ehrhard as an input.

Introduction and notation

Let γ_d be standard Gaussian probability measure on $\mathbb{R}^d$, and write

$$\varphi(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}, \quad \Phi(t) = \int_{-\infty}^t \varphi(s) ds, \quad I(a) = \varphi(\Phi^{-1}(a)), \quad 0 < a < 1.$$

The identities used throughout are

$$\varphi'(t) = -t\varphi(t), \quad I'(a) = -\Phi^{-1}(a), \quad I''(a) = -\frac{1}{I(a)}.$$

In particular, I is concave. The heat semigroup is normalized by

$$P_t^{(d)}F(x) = \int_{\mathbb{R}^d} F(x + \sqrt{t}z) d\gamma_d(z), \quad \partial_t P_t^{(d)}F = \frac{1}{2}\Delta P_t^{(d)}F.$$

We omit the superscript when the dimension is clear.

Ehrhard established the geometric inequality for convex sets by Gaussian symmetrization [1]. The direct functional heat-flow method is due to Borell [2]; see also Barthe–Huet [3] and the explicit deficit equation in Shenfeld–van Handel [4, Lemma 3.1]. The first part below writes that mechanism as a negative-part energy argument. The second part uses only its single-function concavity-preservation version, proved again with the localization needed for unbounded profiles. It then passes through a linear conditional inequality recorded in Livshyts [5, Lemma 7.4]. This organization keeps the independent analytic input separate from the dimensional-lifting step. No claim of novelty is made for the inequalities or the heat-flow mechanism.

I. The one-dimensional functional inequality

1 Statement

For a finite Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$, define

$$\mathcal{M}(f) = \Phi^{-1} \left(\int_{\mathbb{R}} \Phi(f) d\gamma_1 \right).$$

The integral lies strictly between zero and one, so $\mathcal{M}(f)$ is finite.

Theorem 1.1 (One-dimensional functional Ehrhard). *Let $a, b > 0$ with $a + b = 1$, and let $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ be Borel functions. Suppose that*

$$h(ax + by) \geq af(x) + bg(y) \quad \text{for all } x, y \in \mathbb{R}. \quad (1.1)$$

Then

$$\mathcal{M}(h) \geq a\mathcal{M}(f) + b\mathcal{M}(g). \quad (1.2)$$

The functions need not be concave.

Equivalently, for $F, G, H : \mathbb{R} \rightarrow (0, 1)$, the pointwise hypothesis

$$\Phi^{-1}H(ax + by) \geq a\Phi^{-1}F(x) + b\Phi^{-1}G(y)$$

implies

$$\Phi^{-1} \left(\int H d\gamma_1 \right) \geq a\Phi^{-1} \left(\int F d\gamma_1 \right) + b\Phi^{-1} \left(\int G d\gamma_1 \right).$$

We first prove Theorem 1.1 for smooth bounded profiles with bounded derivatives, and then give a premise-preserving approximation.

2 The deficit equation and its negative-part energy

Define

$$u(t, x) = \Phi^{-1}(P_t \Phi(f)(x)), \quad v(t, x) = \Phi^{-1}(P_t \Phi(g)(x)), \quad w(t, x) = \Phi^{-1}(P_t \Phi(h)(x)).$$

Since $P_t \Phi(f) = \Phi(u)$ solves the heat equation, the chain rule gives

$$u_t = \frac{1}{2}u_{xx} - \frac{1}{2}u u_x^2, \quad (2.1)$$

and the same equation holds for v and w . For example,

$$\varphi(u)u_t = \frac{1}{2}\partial_{xx}\Phi(u) = \frac{1}{2}\varphi(u)(u_{xx} - u u_x^2).$$

Put $z = ax + by$ and

$$C(t, x, y) = w(t, z) - au(t, x) - bv(t, y).$$

The hypothesis is $C(0, \cdot, \cdot) \geq 0$, while the desired conclusion is $C(1, 0, 0) \geq 0$. Write

$$D = \partial_x + \partial_y, \quad p = w_x(t, z), \quad q = u_x(t, x), \quad r = v_x(t, y).$$

Because $a + b = 1$,

$$D^2C = w_{xx}(t, z) - au_{xx}(t, x) - bv_{xx}(t, y), \quad C_x = a(p - q), \quad C_y = b(p - r).$$

Suppressing the evaluation points, the nonlinear terms satisfy

$$\begin{aligned} wp^2 - auq^2 - bvr^2 &= p^2C + au(p^2 - q^2) + bv(p^2 - r^2) \\ &= p^2C + u(p + q)C_x + v(p + r)C_y. \end{aligned}$$

Consequently,

$$\boxed{C_t = \frac{1}{2}D^2C - B \cdot \nabla C - \frac{1}{2}p^2C, \quad B = \frac{1}{2}(u(p + q), v(p + r)).} \quad (2.2)$$

This is a linear equation for C once u, v, w are fixed. The diffusion is degenerate, but its quadratic form is nonnegative.

Let $C_- := \max\{-C, 0\}$ and introduce the integrable weight

$$\rho(x, y) = (1 + x^2 + y^2)^{-2}, \quad E(t) = \int_{\mathbb{R}^2} C_-^2 \rho \, dx \, dy.$$

Multiplying (2.2) by $-2C_- \rho$ and integrating by parts gives

$$\begin{aligned} E'(t) &\leq - \int_{\mathbb{R}^2} |DC_-|^2 \rho \, dx \, dy \\ &\quad + \int_{\mathbb{R}^2} C_-^2 \left(\frac{1}{2}D^2\rho + \operatorname{div}(B\rho) - p^2\rho \right) \, dx \, dy. \end{aligned} \quad (2.3)$$

For clarity, the diffusion term is

$$\int C_- D^2C_- \rho = - \int |DC_-|^2 \rho + \frac{1}{2} \int C_-^2 D^2\rho,$$

and the drift term is $\int C_-^2 \operatorname{div}(B\rho)$. The negative-part chain rule is justified by smooth convex approximations; spatial cutoffs justify the integrations over $\mathbb{R}^2$.

For smooth bounded initial profiles with bounded first two derivatives, u, v, w and their first two spatial derivatives are bounded uniformly on $0 \leq t \leq 1$. Indeed, the heat averages remain in a fixed compact subinterval of $(0, 1)$, on which Φ^{-1} and its derivatives are bounded. Thus B and its first spatial derivatives are bounded. Also

$$\sup_{\mathbb{R}^2} \frac{|\nabla \rho| + |D^2 \rho|}{\rho} < \infty.$$

It follows from (2.3) that $E'(t) \leq KE(t)$ for a finite constant K . Since $E(0) = 0$, Grönwall's inequality gives $E = 0$. Continuity implies $C \geq 0$ everywhere. At $(t, x, y) = (1, 0, 0)$ this proves (1.2) in the smooth bounded case.

3 Approximation and the marginalization consequence

The approximation must retain (1.1); arbitrary separate truncations would not suffice. For an integer $N \geq 1$, set

$$\begin{aligned} f_N &= \max\{-N^2, \min(f, N)\}, & g_N &= \max\{-N^2, \min(g, N)\}, \\ h_N &= \max\{c_N, \min(h, N)\}, & c_N &= \max\{-aN^2 + bN, aN - bN^2\}. \end{aligned} \quad (3.1)$$

These bounded profiles satisfy

$$h_N(ax + by) \geq af_N(x) + bg_N(y). \quad (3.2)$$

To see this, first cap all three original profiles above at N . The hypothesis is preserved because $s \mapsto \min(s, N)$ is increasing and concave. If neither capped input is raised by the lower cutoff, this already proves (3.2). If the first input is raised to $-N^2$, the new weighted sum is at most $-aN^2 + bN \leq c_N$; the second input is handled in the same way.

Now convolve f_N, g_N, h_N with the same nonnegative smooth compactly supported approximate identity η_δ . The resulting smooth bounded profiles still satisfy the hypothesis, since

$$a(x - s) + b(y - s) = ax + by - s.$$

Apply the smooth result. At Lebesgue points the mollified profiles converge to f_N, g_N, h_N as $\delta \downarrow 0$, and dominated convergence applies after composition with Φ . Then let $N \rightarrow \infty$. Since $c_N \rightarrow -\infty$, all three truncated profiles converge pointwise to the originals. A second application of dominated convergence, followed by continuity of Φ^{-1} on $(0, 1)$, finishes the proof of Theorem 1.1.

Remark 3.1 (Profiles taking the value $-\infty$). Set $\Phi(-\infty) = 0$. The same truncation and proof apply to Borel profiles with values in $\mathbb{R} \cup \{-\infty\}$ when the three Gaussian integrals are positive. Their values are automatically less than one. If one input integral vanishes, the conclusion is immediate with $\Phi^{-1}(0) = -\infty$. This extension includes concave profiles supported on a convex domain.

Corollary 3.2 (Integration of one coordinate). *Let $F : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ be jointly concave. Then*

$$s \mapsto \Phi^{-1} \left(\int_{\mathbb{R}} \Phi(F(s, x)) d\gamma_1(x) \right)$$

is concave on $\mathbb{R}^m$.

Proof. Fix $s_0, s_1 \in \mathbb{R}^m$ and $a, b > 0, a + b = 1$. Joint concavity gives

$$F(as_0 + bs_1, ax + by) \geq aF(s_0, x) + bF(s_1, y).$$

Apply Theorem 1.1 to the three sections. □

As a check on the normalization and sharpness, for $\kappa, c \in \mathbb{R}$,

$$\int \Phi(\kappa x + c) d\gamma_1(x) = \Phi \left(\frac{c}{\sqrt{1 + \kappa^2}} \right). \quad (3.3)$$

Indeed, for independent standard Gaussians X, Z , the left side is $\mathbb{P}(Z - \kappa X \leq c)$. Therefore profiles with a common affine slope and intercepts $c_0, c_1, ac_0 + bc_1$ give equality in Theorem 1.1.

II. An L^2 proof by dimensional lifting

The argument in this part is independent of Theorem 1.1. We first prove concavity preservation for a single heat solution in $\mathbb{R}^d$, including the localization required by unbounded concave profiles. Differentiating translation concavity then yields a sharp quadratic inequality for a linear function under a conditioned Gaussian. That inequality, applied in dimension $n + 1$, proves the two-set theorem in dimension n .

4 A single transformed heat solution

Lemma 4.1 (Concavity preservation). *Suppose $H : \mathbb{R}^d \rightarrow \mathbb{R}$ is concave and globally Lipschitz. Then*

$$U(t, x) = \Phi^{-1}(P_t^{(d)}\Phi(H)(x))$$

is concave as a function of x for every $t \geq 0$.

Proof. First assume that H is smooth and that every positive-order derivative is bounded. Let $L = \text{Lip}(H)$. The chain rule gives

$$U_t = \frac{1}{2}\Delta U - \frac{1}{2}U|\nabla U|^2. \quad (4.1)$$

All heat averages lie in $(0, 1)$, and U is smooth locally up to $t = 0$. Indeed, the derivatives of $\Phi(H)$ are bounded, and on a compact space-time set its heat averages stay away from zero and one.

Gradient and growth bounds. Differentiation under the Gaussian integral and concavity of I yield

$$\begin{aligned} |\nabla P_t\Phi(H)| &\leq LP_t\varphi(H) = LP_t(I(\Phi(H))) \\ &\leq LI(P_t\Phi(H)). \end{aligned}$$

Dividing by $I(P_t\Phi(H)) = \varphi(U)$ gives $|\nabla U| \leq L$. Fix $T > 0$ and set $M = \max_{0 \leq t \leq T} |U(t, 0)|$. Then

$$|U(t, x)| \leq M + L|x| \quad (0 \leq t \leq T). \quad (4.2)$$

The concavity deficit. Fix $a, b > 0$ with $a + b = 1$, put $z = ax + by$, and define

$$C(t, x, y) = U(t, z) - aU(t, x) - bU(t, y), \quad D_i = \partial_{x_i} + \partial_{y_i}.$$

Write $p_x = \nabla U(t, x)$ and similarly p_y, p_z , and set

$$B = \frac{1}{2}(U(t, x)(p_z + p_x), U(t, y)(p_z + p_y)), \quad c = \frac{1}{2}|p_z|^2.$$

Then

$$\mathcal{Q}C = 0, \quad \mathcal{Q} = \partial_t - \frac{1}{2} \sum_{i=1}^d D_i^2 + B \cdot \nabla_{x,y} + c. \quad (4.3)$$

For completeness, the identity follows from

$$\nabla_x C = a(p_z - p_x), \quad \nabla_y C = b(p_z - p_y)$$

and

$$U(t, z)|p_z|^2 - aU(t, x)|p_x|^2 - bU(t, y)|p_y|^2 = 2cC + 2B \cdot \nabla_{x,y}C.$$

For the second-order terms, D_i translates x, y, z by the same coordinate vector.

Localization and the L^2 estimate. Let $r^2 = |x|^2 + |y|^2$, $R_0(x, y) = 1 + r^2$, and choose

$$A = 1 + 2d + 2LM + 2L^2, \quad \psi(t, x, y) = e^{At} R_0(x, y).$$

Using (4.2) and $|p_x|, |p_y|, |p_z| \leq L$, we obtain

$$\frac{1}{2} \sum_i D_i^2 R_0 = 2d,$$

and

$$\begin{aligned} |B \cdot \nabla R_0| &\leq 2LM(|x| + |y|) + 2L^2 r^2 \\ &\leq (2LM + 2L^2) R_0. \end{aligned}$$

Since $c \geq 0$, it follows that $\mathcal{Q}\psi \geq \psi > 0$. For $\varepsilon > 0$ put $W = C + \varepsilon\psi$. Then $\mathcal{Q}W \geq 0$ and $W(0, x, y) > 0$, because H is concave. Moreover,

$$|C(t, x, y)| \leq 2abL|x - y| \leq L(|x| + |y|).$$

Thus $W \geq 0$ outside a fixed ball in $\mathbb{R}^{2d}$, uniformly on $[0, T]$. Choose a larger ball Ω so that $V := W_-$ vanishes near $\partial\Omega$ for all these times.

Multiply $\mathcal{Q}W \geq 0$ by $-V$ and integrate over Ω . The negative-part chain rule and integration by parts give

$$\boxed{\frac{1}{2} \frac{d}{dt} \int_{\Omega} V^2 + \frac{1}{2} \sum_{i=1}^d \int_{\Omega} |D_i V|^2 + \int_{\Omega} c V^2 \leq \frac{1}{2} \int_{\Omega} (\operatorname{div} B) V^2.} \quad (4.4)$$

There are no spatial boundary terms. The coefficients are smooth on the compact cylinder $[0, T] \times \overline{\Omega}$, so the right side is at most a finite constant times $\int_{\Omega} V^2$. Since $V(0) = 0$, Grönwall's inequality gives $V = 0$. Together with positivity outside Ω , this proves $W \geq 0$ everywhere. Letting $\varepsilon \downarrow 0$ gives $C \geq 0$. The choices of a, b, T were arbitrary.

Finally, for a general concave globally Lipschitz H , let $H_{\delta} = H * \eta_{\delta}$, where η_{δ} is nonnegative, smooth, has integral one, and is supported in the ball of radius δ . Then H_{δ} is concave, $\operatorname{Lip}(H_{\delta}) \leq L$, and all its positive-order derivatives are bounded. Also $|H_{\delta} - H| \leq L\delta$. Apply the smooth case and use dominated convergence to pass to $U(t, x)$ in the concavity inequality. $\square$

The localization is essential to the presentation: a nonconstant concave function on all of $\mathbb{R}^d$ cannot be bounded both above and below. Lemma 4.1 therefore treats the unbounded profiles directly, rather than assuming bounded concave approximants exist.

5 The sharp conditional inequality for linear functions

For a convex Borel set $K \subset \mathbb{R}^d$ with $a = \gamma_d(K) \in (0, 1)$, write $X \sim \gamma_d(\cdot \mid K)$. All expectations in this section are with respect to this conditional probability measure.

Proposition 5.1 (Sharp conditional moment inequality). *For every $\theta \in \mathbb{R}^d$,*

$$\boxed{\operatorname{Var}\langle X, \theta \rangle + \left(1 + \frac{a\Phi^{-1}(a)}{I(a)}\right) (\mathbb{E}\langle X, \theta \rangle)^2 \leq |\theta|^2.} \quad (5.1)$$

The coefficient of the squared mean is sharp for every $a \in (0, 1)$.

Proof. Because K has positive measure, replacing it by its closure changes neither its measure nor its moments. Hence suppose K is closed. The function $d_K(x) = \text{dist}(x, K)$ is convex and 1-Lipschitz. Consequently

$$H_m(x) = m - m^2 d_K(x)$$

is concave and globally Lipschitz. It satisfies

$$\Phi(H_m(x)) \longrightarrow \mathbf{1}_K(x) \quad \text{for every } x \in \mathbb{R}^d. \quad (5.2)$$

On K , $H_m = m \rightarrow +\infty$; off K , $d_K > 0$ and $H_m \rightarrow -\infty$. Apply Lemma 4.1 at time one and pass to the limit by dominated convergence. The resulting finite function

$$x \longmapsto \Phi^{-1}(P_1 \mathbf{1}_K(x)) = \Phi^{-1} \gamma_d(K - x) \quad (5.3)$$

is concave. Every translate of K has measure strictly between zero and one, so passage through Φ^{-1} is legitimate.

Set $F(t) = \gamma_d(K + t\theta)$ and $G(t) = \Phi^{-1}(F(t))$. Differentiating the translated Gaussian density yields

$$F'(0) = -a\mathbb{E}\langle X, \theta \rangle, \quad F''(0) = a(\mathbb{E}\langle X, \theta \rangle^2 - |\theta|^2). \quad (5.4)$$

Differentiation is justified even for unbounded K : on bounded t -intervals, the derivatives are dominated by a polynomial times $e^{c|x|}e^{-|x|^2/2}$, which is integrable. Since

$$(\Phi^{-1})'(a) = \frac{1}{I(a)}, \quad (\Phi^{-1})''(a) = \frac{\Phi^{-1}(a)}{I(a)^2},$$

concavity of (5.3) gives

$$0 \geq G''(0) = \frac{a}{I(a)} \left[\mathbb{E}\langle X, \theta \rangle^2 - |\theta|^2 + \frac{a\Phi^{-1}(a)}{I(a)} (\mathbb{E}\langle X, \theta \rangle)^2 \right]. \quad (5.5)$$

Separating the variance from the second moment proves (5.1).

For sharpness, take $K = \{x : \langle x, e \rangle \leq z\}$ with $|e| = 1$ and $a = \Phi(z)$. Elementary integration by parts gives

$$\mathbb{E}\langle X, e \rangle = -\frac{I(a)}{a}, \quad \mathbb{E}\langle X, e \rangle^2 = 1 - \frac{zI(a)}{a}.$$

These identities give equality in (5.1) for $\theta = e$. The mean is nonzero, so increasing its coefficient would violate the inequality. $\square$

The inequality is the one stated in [5, Lemma 7.4]; the proof above obtains it from the independent energy argument. Its coefficient

$$\beta(a) := 1 + \frac{a\Phi^{-1}(a)}{I(a)}$$

is positive. Indeed, with $a = \Phi(z)$, the numerator $\varphi(z) + z\Phi(z)$ has derivative $\Phi(z) > 0$ and limit zero at $-\infty$. In matrix form, if $b_K = \mathbb{E}X$ and $\Sigma_K = \text{Cov}(X)$, the result reads

$$\Sigma_K + \beta(a)b_K b_K^\top \preceq \text{Id}. \quad (5.6)$$

The extra term records the relation between the covariance deficit and the barycenter. It is exactly the term needed for the lifting calculation.

6 Dimensional lifting and geometric Ehrhard

We now use only Proposition 5.1, in one additional dimension. The heat flow and functional theorem play no further role.

Theorem 6.1 (Ehrhard for convex sets). *Let $K, L \subset \mathbb{R}^n$ be convex Borel sets, each of Gaussian measure strictly between zero and one. For every $\lambda \in [0, 1]$,*

$$\Phi^{-1}\gamma_n((1-\lambda)K + \lambda L) \geq (1-\lambda)\Phi^{-1}\gamma_n(K) + \lambda\Phi^{-1}\gamma_n(L). \quad (6.1)$$

The convention $\Phi^{-1}(1) = +\infty$ is allowed on the left.

Proof. First let K, L be compact convex bodies. Write $K_s = (1-s)K + sL$ for $s \in [0, 1]$, and form the convex lift

$$\mathcal{C} = \text{conv}((K \times \{0\}) \cup (L \times \{1\})) = \{(x, s) : 0 \leq s \leq 1, x \in K_s\} \subset \mathbb{R}^{n+1}. \quad (6.2)$$

The section formula follows by grouping together the points from each of the two convex sets in a convex combination. Define $f(s) = \gamma_n(K_s)$ on $[0, 1]$ and extend it by zero outside that interval. For $\varepsilon > 0$ and $t \in \mathbb{R}$, consider the convex body

$$A_{\varepsilon, t} = \{(x, r) : (x, t + \varepsilon r) \in \mathcal{C}\}.$$

Fubini's theorem gives

$$\begin{aligned} F_\varepsilon(t) &:= \gamma_{n+1}(A_{\varepsilon, t}) = \int_{\mathbb{R}} f(t + \varepsilon r) d\gamma_1(r) \\ &= \int_{\mathbb{R}} f(s) \frac{1}{\varepsilon} \varphi\left(\frac{s-t}{\varepsilon}\right) ds. \end{aligned} \quad (6.3)$$

It satisfies $0 < F_\varepsilon(t) < 1$ for every finite t .

Fix ε, t and abbreviate the unnormalized Gaussian moments by

$$a = F_\varepsilon(t), \quad M = \int_{A_{\varepsilon, t}} r d\gamma_{n+1}, \quad S = \int_{A_{\varepsilon, t}} r^2 d\gamma_{n+1}.$$

Differentiating the Gaussian kernel in (6.3) gives the exact identities

$$\boxed{F'_\varepsilon(t) = \frac{M}{\varepsilon}, \quad F''_\varepsilon(t) = \frac{S-a}{\varepsilon^2}.} \quad (6.4)$$

Indeed, the first derivative of the kernel multiplies it by $(s-t)/\varepsilon^2 = r/\varepsilon$, and the second by $((s-t)^2/\varepsilon^4) - 1/\varepsilon^2 = (r^2-1)/\varepsilon^2$. No differentiability of f is needed.

Apply (5.1) to the Gaussian conditioned on $A_{\varepsilon, t}$ and to the last coordinate r . Its mean and second moment are M/a and S/a , respectively. Hence

$$\frac{S}{a} - \frac{M^2}{a^2} + \left(1 + \frac{a\Phi^{-1}(a)}{I(a)}\right) \frac{M^2}{a^2} \leq 1,$$

or equivalently

$$S - a + \frac{\Phi^{-1}(a)}{I(a)} M^2 \leq 0. \quad (6.5)$$

Combining (6.4) and (6.5), we obtain

$$\boxed{(\Phi^{-1} \circ F_\varepsilon)''(t) = \frac{S - a + \Phi^{-1}(a)M^2/I(a)}{\varepsilon^2 I(a)} \leq 0.} \quad (6.6)$$

Thus the quadratic inequality for one linear coordinate on the lifted body becomes concavity of the smoothed interpolation.

Removing the smoothing. The map $s \mapsto K_s$ is continuous in Hausdorff distance. Since these are convex bodies with Gaussian-null boundaries, $s \mapsto \gamma_n(K_s)$ is continuous on $[0, 1]$. At every $t \in (0, 1)$, dominated convergence in (6.3) therefore gives

$$F_\varepsilon(t) \longrightarrow \gamma_n(K_t), \quad \Phi^{-1}(F_\varepsilon(t)) \longrightarrow \Phi^{-1}\gamma_n(K_t).$$

Passing to the limit in the concavity inequality at interior points shows that $t \mapsto \Phi^{-1}\gamma_n(K_t)$ is concave on $(0, 1)$. Continuity of this original function then extends its concavity to $[0, 1]$, proving (6.1) for compact convex bodies.

The order of these limits matters. Because f was extended by zero,

$$F_\varepsilon(0) \longrightarrow \tfrac{1}{2}\gamma_n(K), \quad F_\varepsilon(1) \longrightarrow \tfrac{1}{2}\gamma_n(L).$$

One must therefore first pass to the limit in the interior, and only then recover the endpoint values by continuity of $\gamma_n(K_t)$.

General convex sets. For closed, possibly unbounded convex sets of positive measure, put $K_R = K \cap \overline{B}_R$ and $L_R = L \cap \overline{B}_R$. For large R these are compact convex bodies. Moreover,

$$\bigcup_R ((1 - \lambda)K_R + \lambda L_R) = (1 - \lambda)K + \lambda L.$$

Apply the compact result and use continuity of measure from below. This also allows the left side of (6.1) to tend to $+\infty$.

Finally, a convex Borel set of positive Gaussian measure has nonempty interior and differs from its closure by a null set. A Minkowski combination of the original sets and the corresponding combination of their closures have the same closure, and hence the same Gaussian measure: both are full-dimensional convex sets with null boundary. This proves the Borel version. $\square$

7 Logical dependence and the meaning of the L^2 proof

Denote by (P_d) the conditional inequality (5.1) in dimension d , and by (E_d) geometric Ehrhard for convex sets in that dimension. The lifting calculation proves

$$(P_{d+1}) \implies (E_d), \quad (E_d) \implies (P_d). \quad (7.1)$$

For the second implication, apply Ehrhard to two translates of a fixed convex set and differentiate as in (5.4)–(5.5). Thus the two dimension-independent families are equivalent. The first implication uses one extra dimension. There is no circularity in the proof presented here: (P_d) was obtained from (4.4), before geometric Ehrhard was invoked.

Both proofs in this note are *parabolic* L^2 proofs: the energy is the squared negative part of a deficit evolving under a linear equation. This specifies the sense in which L^2 is used. The argument does not establish the distinct elliptic estimate

$$\mathbb{E}_K(\|D^2u\|_{\text{HS}}^2 + |\nabla u|^2) \geq \left(1 + \frac{a\Phi^{-1}(a)}{I(a)}\right) c^2, \quad (\Delta - x \cdot \nabla)u = c, \quad a = \gamma_d(K),$$

which arises in the Neumann–Reilly route; see Kolesnikov–Milman [6, Section 3]. That estimate is not an input anywhere in this note.

Preparation note. Prepared with ChatGPT at Galyna V. Livshyts's request, combining a direct write-up of the one-dimensional functional inequality with a separate presentation of the dimensional-lifting proof. The analytic and geometric steps have been written with their approximation requirements explicit.

References

- [1] A. Ehrhard, *Symétrisation dans l'espace de Gauss*, Math. Scand. **53** (1983), 281–301. doi:10.7146/math.scand.a-12035.
- [2] C. Borell, *The Ehrhard inequality*, C. R. Math. Acad. Sci. Paris **337** (2003), 663–666. doi:10.1016/j.crma.2003.09.031.
- [3] F. Barthe and N. Huet, *On Gaussian Brunn–Minkowski inequalities*, Studia Math. **191** (2009), 283–304. arXiv:0804.0886.
- [4] Y. Shenfeld and R. van Handel, *The equality cases of the Ehrhard–Borell inequality*, 2017 preprint, arXiv:1709.00061. See Lemma 3.1 for the deficit equation.
- [5] G. V. Livshyts, *On a conjectural symmetric version of Ehrhard's inequality*, 2022 preprint, arXiv:2103.11433v3. The reference to Lemma 7.4 is to this version.
- [6] A. V. Kolesnikov and E. Milman, *Sharp Poincaré-type inequality for the Gaussian measure on the boundary of convex sets*, in *Geometric Aspects of Functional Analysis*, Lecture Notes in Math. **2169**, Springer, 2017, 221–234. arXiv:1601.02925.