Gaussian Correlation via Inverse Brascamp-Lieb

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Asymptotic Geometric Analysis
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Gaussian Correlation Inequality

Gaussian Correlation Inequality, Royen '14

$\forall$ convex $K = -K$ and $L = -L$ in $\mathbb{R}^n$, $\gamma^n(K \cap L) \geq \gamma^n(K)\gamma^n(L)$.
I.e., if $Z \sim N(0, \mathrm{Id})$, $P(Z \in K \wedge Z \in L) \geq P(Z \in K)P(Z \in L)$.

Prior partial results:

- Khatri, Šidák '67 - $K = [-a, a] \times \mathbb{R}^{n-1}$.
- Pitt '77 - confirmed $n = 2$, inspired conjecture.
- Folklore - convex K, L are unconditional.
- Schechtman–Schlumprecht–Zinn '98 - K, L centered ellipsoids; convex sets of small measure.
- Hargé '99, Cordero-Erausquin '02 - K centered ellipsoid.
- Royen '14 - resolved conjecture. By origin-symmetry $(X_i^2) \sim \chi^2$ and Laplace transform; more general Γ-distributions.
- Latała–Matlak '17 - detailed exposition.

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Extensions and Conjectures

- Eskenazis–Nayar–Tkocz '18 - extended GCI to Gaussian mixtures $\sim S \cdot Z$. E.g. symmetric p-stable laws, $p \in (0, 2)$.
- Conj (Tehranchi '17) - $\gamma^n(K + L)\gamma^n(K \cap L) \geq \gamma^n(K)\gamma^n(L)$? (True for Lebesgue measure, open for $n \geq 2$).

  Assoulin–Chor–Sadovsky '24 - if $K + L \leftrightarrow \mathrm{Conv}(K \cup L)$:

  - True for $K = [-a, a] \times \mathbb{R}^{n-1}$.
  - False in general.
- Royen '20-'25, Assoulin–Chor–Sadovsky '24 - further improvements over GCI.
- Nakamura–Tsuji '25+ (forthcoming).

Equivaent formulations of GCI

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Let $X \sim N(0, \Sigma)$ in $\mathbb{R}^N = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$. Denote $P_i : \mathbb{R}^N \to \mathbb{R}^{n_i}$, $i = 1, 2$.

- $\forall$ convex $K = -K$, $L = -L$ in $\mathbb{R}^N$,
$$\mathbb{P}(X \in K \wedge X \in L) \geq \mathbb{P}(X \in K)\mathbb{P}(X \in L).$$
- $\forall$ even quasi-concave $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}_+$,
$$\mathbb{E}f_1(X)f_2(X) \geq \mathbb{E}f_1(X)\mathbb{E}f_2(X).$$
- $\forall N = n_1 + n_2$, $\forall$ even quasi-concave $f_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$,
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- $\forall N = n_1 + n_2$,
$$\mathbb{P}(\max_{1 \leq k \leq N} |X_k| \leq 1) \geq \mathbb{P}(\max_{1 \leq k \leq n_1} |X_k| \leq 1)\mathbb{P}(\max_{n_1+1 \leq k \leq N} |X_k| \leq 1).$$

To close equivalence, approximate $K \subset \mathbb{R}^n$ by $\cap_{k=1}^{n_1} \{ |\langle \cdot, u_k \rangle| \leq 1 \}$, $L \subset \mathbb{R}^n$ by $\cap_{k=n_1+1}^{n_1+n_2} \{ |\langle \cdot, u_k \rangle| \leq 1 \}$. Let $T : \mathbb{R}^n \ni x \mapsto (\langle x, u_k \rangle)_{k=1}^N \in \mathbb{R}^N$. Then $T_* \gamma^n \sim N(0, \Sigma')$ degenerate in $\mathbb{R}^N$. Approximate by non-deg. Σ.

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New proof of GCI using Inverse Brascamp-Lieb (M.'25)

Denote $g_A(x) = \exp(-\frac{1}{2}\langle Ax, x\rangle)$, $\mathbb{R}^N = \oplus_{i=1}^m \mathbb{R}^{n_i}$, $x = (x_1, \ldots, x_m)$.
Let $\bar{Q} \in \mathbb{M}^N_{sym}$ arbitrary signature, $Q_i \in \mathbb{M}^{n_i}_{\geq 0}$, $c_i > 0$, $i = 1, \ldots, m$.

Thm (Even Inverse Brascamp–Lieb, M. '25 after Nakamura-Tsuji '24)

For all even log-concave $h_i \in L^1(\mathbb{R}^{n_i}, g_{Q_i}(x)dx)$:

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m h_i(x_i)^{c_i}\,dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} h_i(x_i)\,dx_i\right)^{c_i}} \geq \inf_{\substack{A_i \geq 0 \\ A_i + Q_i > 0}} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m g_{A_i}(x_i)^{c_i}\,dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} g_{A_i}(x_i)\,dx_i\right)^{c_i}}.$$

- Case $Q_i = 0$ was proved by Nakamura–Tsuji '24; our proof is a verbatim repetition of theirs.
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Plan for the talk

- Show how this proves GCI.
- Discuss history of Brascamp–Lieb-type inqs.
- Sketch Nakamura–Tsuji proof.
- Concluding Remarks.

Proof of GCI via Inverse BL

Recall $\mathbb{R}^N = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, $P_i : \mathbb{R}^N \to \mathbb{R}^{n_i}$, $i = 1, 2$.
X centered Gaussian in $\mathbb{R}^N$. Denote $\Sigma = \mathrm{Cov}\, X$, $\Sigma_i = \mathrm{Cov}\, P_i X$.
Apply inverse BL w/ $\bar{Q} = \Sigma^{-1}$, $Q_i = \Sigma_i^{-1}$, $c_i = 1$, $h_i = \mathbf{1}_{K_i}$, $K_i = [-1, 1]^{n_i}$.

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$$= \inf_{A_1, A_2 \ge 0} \left(\frac{\det(\mathrm{Id}_{n_1} + A_1 \Sigma_1) \det(\mathrm{Id}_{n_2} + A_2 \Sigma_2)}{\det(\mathrm{Id}_N + (A_1 \oplus A_2) \Sigma)} \right)^{\frac{1}{2}}$$
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where we used

$$\mathbb{E} g_{A_i}(P_i X) = \det(\Sigma_i)^{-\frac{1}{2}} \det(\Sigma_i^{-1} + A_i)^{-\frac{1}{2}} = \det(\mathrm{Id}_{n_i} + A_i \Sigma_i)^{-\frac{1}{2}}.$$

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Apply inverse BL w/ $\bar{Q} = \Sigma^{-1}$, $Q_i = \Sigma_i^{-1}$, $c_i = 1$, $h_i = \mathbf{1}_{K_i}$, $K_i = [-1, 1]^{n_i}$.

$$\frac{\mathbb{P}(\max_{1 \leq j \leq N} |X_j| \leq 1)}{\mathbb{P}(\max_{1 \leq j \leq n_1} |X_j| \leq 1)\, \mathbb{P}(\max_{n_1+1 \leq j \leq N} |X_j| \leq 1)}$$
$$= \frac{\mathbb{E}\mathbf{1}_{K_1}(P_1 X)\mathbf{1}_{K_2}(P_2 X)}{\mathbb{E}\mathbf{1}_{K_1}(P_1 X)\mathbb{E}\mathbf{1}_{K_2}(P_2 X)} \geq \inf_{A_1, A_2 \geq 0} \frac{\mathbb{E} g_{A_1}(P_1 X) g_{A_2}(P_2 X)}{\mathbb{E} g_{A_1}(P_1 X)\mathbb{E} g_{A_2}(P_2 X)}$$
$$= \inf_{A_1, A_2 \geq 0} \left(\frac{\det(\mathrm{Id}_{n_1} + A_1 \Sigma_1)\det(\mathrm{Id}_{n_2} + A_2 \Sigma_2)}{\det(\mathrm{Id}_N + (A_1 \oplus A_2)\Sigma)} \right)^{\frac{1}{2}}$$
$$= \inf_{A_1, A_2 \geq 0} \left(\frac{\det(\mathrm{Id}_N + (A_1 \oplus A_2)(\Sigma_1 \oplus \Sigma_2))}{\det(\mathrm{Id}_N + (A_1 \oplus A_2)\Sigma)} \right)^{\frac{1}{2}},$$

where we used

$$\mathbb{E} g_{A_i}(P_i X) = \det(\Sigma_i)^{-\frac{1}{2}} \det(\Sigma_i^{-1} + A_i)^{-\frac{1}{2}} = \det(\mathrm{Id}_{n_i} + A_i \Sigma_i)^{-\frac{1}{2}}.$$

Royen's lemma

Lemma (Royen '14)

Let $\Sigma^{(t)} := (1-t)(\Sigma_1 \oplus \Sigma_2) + t\Sigma$, $t \in [0,1]$.
Then $\forall A_1, A_2 \geq 0$, $[0,1] \ni t \mapsto \det(\mathrm{Id}_N + (A_1 \oplus A_2)\Sigma^{(t)})$ is $\searrow$.

Proof:
Changing orthonormal basis, may assume $A = A_1 \oplus A_2 \geq 0$ diagonal. In Royen's proof, $A = \mathrm{diag}(\Lambda_i)$ are Laplace transform parameters; for us, A is the covariance of the saturating Gaussians.

$$|\mathrm{Id}_N + A\Sigma^{(t)}| = 1 + \sum_{\emptyset \neq J \subset [N]} |(A\Sigma^{(t)})_J| = 1 + \sum_{\emptyset \neq J \subset [N]} |A_J||\Sigma^{(t)}_J|.$$

Reduces to showing $[0,1] \ni t \mapsto |\Sigma^{(t)}_J|$ is $\searrow$. Indeed:

$$\Sigma^{(t)}_J = \begin{pmatrix} \Sigma_{J_1} & t\Sigma_{J_1 J_2} \\ t\Sigma_{J_2 J_1} & \Sigma_{J_2} \end{pmatrix}, \; |\Sigma^{(t)}_J| = |\Sigma_{J_1}||\Sigma_{J_2}||\mathrm{Id}_{J_1} - t^2 M|, \; 0 \leq M \leq \mathrm{Id}_{J_1}.$$

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Even Inverse BL — Reformulation

Thm (Even Inverse Brascamp–Lieb)

For all even log-concave $h_i \in L^1(\mathbb{R}^{n_i}, g_{Q_i}(x)dx)$:

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m h_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} h_i(x_i) dx_i\right)^{c_i}} \geq \inf_{\substack{A_i \geq 0 \\ A_i + Q_i > 0}} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m g_{A_i}(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} g_{A_i}(x_i) dx_i\right)^{c_i}}.$$

Change parametrization: $f_i = e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} h_i$.

$Q = \bar{Q} - c_1 Q_1 \oplus \ldots \oplus c_m Q_m$ arbitrary signature, $B_i = A_i + Q_i$.

Thm (Even Inverse Brascamp–Lieb again)

For all even $f_i \in L^1(\mathbb{R}^{n_i})$, $f_i \succcurlyeq g_{Q_i}$

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m f_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i) dx_i\right)^{c_i}} \geq \inf_{\substack{B_i \geq Q_i \\ B_i > 0}} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m g_{B_i}(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_{B_i}(x_i) dx_i\right)^{c_i}}.$$

Write $f_i \succcurlyeq g_{Q_i}$ if $f_i = g_{Q_i} \cdot$ log-concave, $f_i \preccurlyeq g_{Q_i}$ if $f_i = g_{Q_i} \cdot$ log-convex.

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Comparison with other Brascamp–Lieb-type inqs

Thm (Forward Brascamp–Lieb '76, Lieb '90). Combine Bennet–Carbery–Christ–Tao '08, Valdimarsson '07, Kolesnikov '11)

Let $Q \in \mathbb{M}^N_{\geq 0}$. Then for all $f_i \in L^1(\mathbb{R}^{n_i})$, $f_i \ll g_{Q_i}$ (no evenness):

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m f_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i) dx_i\right)^{c_i}} \leq \inf_{0 < B_i \leq Q_i} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m g_{B_i}(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_{B_i}(x_i) dx_i\right)^{c_i}}.$$

- Can replace x_i in numerator with $P_i x$, $P_i : \mathbb{R}^N \to \mathbb{R}^{n_i}$. Generalized Hölder's inq; optimal constant in Young's inq.
- Taking $Q_i = \Lambda \mathrm{Id}_{n_i}$, $\Lambda \nearrow \infty$ (no restriction on f_i, $B_i > 0$) $\rightsquigarrow$ Lieb '90. Brascamp–Lieb ($n_i = 1$) used BL–Luttinger symmetrization. Lieb used $O(2)$ invariance (like proof of B-conj by BL '70s).
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- K $\rightsquigarrow$ $f_i \ll g_{\mathrm{Id}}$, $h_i \gg g_{\mathrm{Id}}$ $\Rightarrow \exists! (\nabla\varphi)_*(f_i dx) = h_i dx$, $0 \leq \nabla^2\varphi \leq \mathrm{Id}$.

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- Observe “dual nature" to (even) inverse BL from previous slide.

Thm (Even Inverse Brascamp–Lieb again)

Let $Q \in \mathbb{M}^N_{sym}$. Then for all even $f_i \in L^1(\mathbb{R}^{n_i})$, $f_i \gg g_{Q_i}$

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m f_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i) dx_i\right)^{c_i}} \geq \inf_{\substack{B_i \geq Q_i \\ B_i > 0}} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m g_{B_i}(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_{B_i}(x_i) dx_i\right)^{c_i}}.$$

More Brascamp–Lieb-type inqs - II

Thm (Reverse Brascamp–Lieb, conjectured by K. Ball, Barthe '97). Generalized formulation by Valdimarsson '07

Let $Q \in \mathbb{M}^N_{\geq 0}$, $Q_i \in \mathbb{M}^{n_i}_{sym}$. Then $\forall f_i \in L^1(\mathbb{R}^{n_i})$, $f_i \gg g_{Q_i}$ (no evenness):

$$\frac{\int^*_{\mathbb{R}^N} \sup_{y=\sum c_i x_i + x} \left(e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m f_i(x_i)^{c_i} \right) dy}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i) dx_i \right)^{c_i}} \geq \inf_{\substack{B_i \geq Q_i \\ B_i > 0}} \{\text{Same as LHS, } f_i = g_{B_i}\}$$

- $Q = \Lambda \mathrm{Id}_N$ and $Q_i = -\Lambda \mathrm{Id}_{n_i}$, $\Lambda \nearrow \infty$ $\rightsquigarrow$ no condition on f_i.
- Barthe pioneered OT proof (forward & reverse in tandem). OT - Valdimarsson; Heat-flow (Barthe–Cordero, Barthe–Huet).
- Reverse BL is an extension of Prekopá-Leindler inq. (Reverse) BL very useful in convex geometry (Ball '89-'91, Barthe '98...)
- Ball: if $\mathrm{Id}_N = \sum_{i=1}^m c_i P_i^* P_i$ (e.g. John's position), then Gaussian RHS in BL and reverse BL is 1 ("geometric case").

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More Brascamp–Lieb-type inqs - III

$B^i_j : E_i \to E^j$ linear, $\{c_i\}_{i=1}^k, \{d_j\}_{j=1}^m > 0$, $\sum_{i=1}^k c_i \dim(E_i) = \sum_{j=1}^m d_j \dim(E^j)$.

Thm (Forward-Reverse BL, Liu–Courtade–Cuff-Verdu '18, CL '19)

Let $D = D(\mathbf{c}, \mathbf{d}, \mathbf{B})$ be the optimal constant in the implication:

$$\prod_{i=1}^k f_i^{c_i}(x_i) \leq \prod_{j=1}^m g_j^{d_j}\left(\sum_{i=1}^k c_i B^i_j x_i\right) \;\; \forall x_i \in E_i \;\Rightarrow \prod_{i=1}^k \left(\int_{E_i} f_i\right)^{c_i} \leq D \prod_{j=1}^m \left(\int_{E^j} g_j\right)^{d_j}.$$

Then D saturated by centered Gaussians & $D(\mathbf{c}, \mathbf{d}, \mathbf{B}) = D(\mathbf{d}, \mathbf{c}, \mathbf{B}^*)$.

- Strictly generalizes forward and reverse BL into single statement, no difference between forward and reverse directions.
- $k = 1, c_1 = 1 \rightsquigarrow$ Forward BL; $m = 1, d_1 = 1 \rightsquigarrow$ Reverse BL. As in Barthe's work, For/Rev optimal constants are reciprocals.
- Proof: information-theory — dual formulation, subadditivity of entropy (Carlen–Lieb–Loss '04, Carlen–Cordero-Erausquin '09).

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More Brascamp–Lieb-type inqs - IV

Thm (Inverse BL, Chen–Dafnis–Paouris '15, Barthe–Wolff '22)

$\bar{Q} \in \mathbb{M}^N_{sym}$, $Q_i \in \mathbb{M}^{n_i}_{\geq 0}$, $\bar{Q} - \oplus_{i=1}^m c_i Q_i \leq 0$. Then $\forall h_i \in L^1(\mathbb{R}^{n_i}, g_{Q_i}(x_i)dx_i)$

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m h_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} h_i(x_i) dx_i\right)^{c_i}} \geq \inf_{\substack{A_i \geq 0 \\ A_i + Q_i > 0}} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m g_{A_i}(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} g_{A_i}(x_i) dx_i\right)^{c_i}}.$$

- As usual, $x_i \leftrightarrow P_i x$.
- CDP - "geometric case" when RHS = 1.
- BW - general version, $c_i \in \mathbb{R}$. Obtained precise condition on $(\bar{Q}, P_i, c_i)$ characterizing when lower-bound is saturated by centered Gaussians.
- Wolff - general formulation of inverse BL equivalent to Forward-Reverse BL (!) (need to use negative c_i's...)
- Proofs: CDP - heat-flow / Gaussian integration; BW - Barthe's OT + BCCT structure analysis.

More Brascamp–Lieb-type inqs - IV

Thm (Inverse BL, Chen–Dafnis–Paouris '15, Barthe–Wolff '22)

$\bar{Q} \in \mathbb{M}^N_{sym}$, $Q_i \in \mathbb{M}^{n_i}_{\geq 0}$, $\bar{Q} - \oplus_{i=1}^m c_i Q_i \leq 0$. Then $\forall h_i \in L^1(\mathbb{R}^{n_i}, g_{Q_i}(x_i)dx_i)$

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m h_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} h_i(x_i) dx_i\right)^{c_i}} \geq \inf_{\substack{A_i \geq 0 \\ A_i + Q_i > 0}} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle \bar{Q}x,x\rangle} \prod_{i=1}^m g_{A_i}(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} e^{-\frac{1}{2}\langle Q_i x_i, x_i\rangle} g_{A_i}(x_i) dx_i\right)^{c_i}}.$$

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- Wolff - general formulation of inverse BL equivalent to Forward-Reverse BL (!) (need to use negative c_i's...)
- Proofs: CDP - heat-flow / Gaussian integration;
  BW - Barthe's OT + BCCT structure analysis.

Nakamura–Tsuji's Even Inverse BL

For $\mathbf{A} = (A_1, \ldots, A_m)$ write $\mathbf{A} \leq \mathbf{B}$ if $A_i \leq B_i$ for all i. Let $0 \leq \mathbf{A} \leq \mathbf{B}$,

$$\mathcal{F}_{\mathbf{A},\mathbf{B}} := \{\mathbf{f} = (f_1, \ldots, f_m) ;\ \text{even } f_i \in L^1(\mathbb{R}^{n_i})\ \ g_{A_i} \lll f_i \lll g_{B_i}\}.$$
$$\mathcal{G}_{\mathbf{A},\mathbf{B}} := \{\mathbf{g} = (g_1, \ldots, g_m) \in \mathcal{F}_{\mathbf{A},\mathbf{B}} ;\ g_i = g_{C_i}\ \ (A_i \leq C_i \leq B_i)\}.$$

If $\mathbf{B}$ is omitted, no restriction of being "more-log-convex".

$$\mathcal{BL}(\mathbf{f}) := \frac{\int e^{-\frac{1}{2}\langle Qx,x\rangle} \prod_{i=1}^m f_i(x_i)^{c_i} dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i) dx_i\right)^{c_i}},$$

Clearly:

$$\mathcal{I}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{f}) \leq I^{\mathcal{G}}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{g} \in \mathcal{G}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{g}).$$

Thm (Even Inverse Brascamp–Lieb, M. '25 after Nakamura-Tsuji '24)

$\forall \mathbf{Q} \geq 0\ \ \mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. In fact $\forall 0 < \mathbf{A} \leq \mathbf{B}\ \ \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}$.

NT - proved for $\mathbf{Q} = 0$, $A_i = \lambda \mathrm{Id}_{n_i}$, $B_i = \Lambda \mathrm{Id}_{n_i}$. We repeat their proof.

Nakamura–Tsuji's Even Inverse BL

For $\mathbf{A} = (A_1, \ldots, A_m)$ write $\mathbf{A} \leq \mathbf{B}$ if $A_i \leq B_i$ for all i. Let $0 \leq \mathbf{A} \leq \mathbf{B}$,

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Nakamura–Tsuji's Even Inverse BL

For $\mathbf{A} = (A_1, \ldots, A_m)$ write $\mathbf{A} \leq \mathbf{B}$ if $A_i \leq B_i$ for all i. Let $0 \leq \mathbf{A} \leq \mathbf{B}$,

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$\forall \mathbf{Q} \geq 0$ $\mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. In fact $\forall 0 < \mathbf{A} \leq \mathbf{B}$ $\mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}$.

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Nakamura–Tsuji's Even Inverse BL

For $\mathbf{A} = (A_1, \ldots, A_m)$ write $\mathbf{A} \le \mathbf{B}$ if $A_i \le B_i$ for all i. Let $0 \le \mathbf{A} \le \mathbf{B}$,

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$\forall \mathbf{Q} \ge 0 \;\; \mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. In fact $\forall 0 < \mathbf{A} \le \mathbf{B} \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}$.

NT - proved for $\mathbf{Q} = 0$, $A_i = \lambda \mathrm{Id}_{n_i}$, $B_i = \Lambda \mathrm{Id}_{n_i}$. We repeat their proof.

Nakamura–Tsuji's Proof

Fix $0 < \mathbf{A} \le \mathbf{B}$.

1. Infimum in $\mathcal{I}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{f})$ is attained on some $\mathbf{f}_0 \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Here strict log-concavity is used via Arzelà–Ascoli.

2. If $f \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$ then (key computation on next slide)

$$\mathcal{BL}(\mathbf{f})^2 \ge \mathcal{I}_{\mathbf{A},\mathbf{B}} \, \mathcal{BL}(\mathrm{Conv}\, \mathbf{f}),$$

$\mathrm{Conv}\, \mathbf{f} = (\mathrm{Conv}\, f_1, \ldots, \mathrm{Conv}\, f_m)$, $\mathrm{Conv}\, f_i = 2^{n_i/2} (f_i * f_i)(\sqrt{2}\, \cdot)$.

Lemma A: $\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}} \;\Rightarrow\; \mathrm{Conv}\, \mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Hence:

$$\mathcal{I}^2_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathbf{f}_0)^2 \ge \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\mathrm{Conv}\, \mathbf{f}_0) \ge \mathcal{I}^2_{\mathbf{A},\mathbf{B}}$$

$$\rightsquigarrow \; \mathrm{Conv}\, \mathbf{f}_0 \text{ minimizer} \; \rightsquigarrow \; \forall k \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathrm{Conv}^k\, \mathbf{f}_0).$$

3. By CLT, $\mathrm{Conv}^k\, \mathbf{f}_0 \to \mathbf{g} \in \mathcal{G}_{\mathbf{A},\mathbf{B}}$, hence:

$$\mathcal{I}_{\mathbf{A},\mathbf{B}} = \lim_{k \to \infty} \mathcal{BL}(\mathrm{Conv}^k\, \mathbf{f}_0) \ge \mathcal{BL}(\mathbf{g}) \ge \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}} \ge \mathcal{I}_{\mathbf{A},\mathbf{B}} \;\; \rightsquigarrow \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}.$$

4. For all $\mathbf{Q} \ge 0$, $\mathcal{I}_{\mathbf{Q}} = \lim_{\lambda \to 0^+} \lim_{\Lambda \to \infty} \mathcal{I}_{\mathbf{Q}+\lambda \mathrm{Id}, \Lambda \mathrm{Id}} \;\; \rightsquigarrow \;\; \mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. □

Nakamura–Tsuji's Proof

Fix $0 < \mathbf{A} \le \mathbf{B}$.

1. Infimum in $\mathcal{I}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{f})$ is attained on some $\mathbf{f}_0 \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Here strict log-concavity is used via Arzelà–Ascoli.
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$\operatorname{Conv} \mathbf{f} = (\operatorname{Conv} f_1, \ldots, \operatorname{Conv} f_m)$, $\operatorname{Conv} f_i = 2^{n_i/2}(f_i * f_i)(\sqrt{2}\,\cdot)$.

Lemma A: $\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}} \Rightarrow \operatorname{Conv} \mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Hence:

$$\mathcal{I}^2_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathbf{f}_0)^2 \ge \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\operatorname{Conv} \mathbf{f}_0) \ge \mathcal{I}^2_{\mathbf{A},\mathbf{B}}$$

$$\rightsquigarrow \operatorname{Conv} \mathbf{f}_0 \text{ minimizer} \rightsquigarrow \forall k \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\operatorname{Conv}^k \mathbf{f}_0).$$

3. By CLT, $\operatorname{Conv}^k \mathbf{f}_0 \to \mathbf{g} \in \mathcal{G}_{\mathbf{A},\mathbf{B}}$, hence:

$$\mathcal{I}_{\mathbf{A},\mathbf{B}} = \lim_{k \to \infty} \mathcal{BL}(\operatorname{Conv}^k \mathbf{f}_0) \ge \mathcal{BL}(\mathbf{g}) \ge \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}} \ge \mathcal{I}_{\mathbf{A},\mathbf{B}} \rightsquigarrow \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}.$$

4. For all $\mathbf{Q} \ge 0$, $\mathcal{I}_{\mathbf{Q}} = \lim_{\lambda \to 0^+} \lim_{\Lambda \to \infty} \mathcal{I}_{\mathbf{Q}+\lambda \mathrm{Id}, \Lambda \mathrm{Id}} \rightsquigarrow \mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. □

Nakamura–Tsuji's Proof

Fix $0 < \mathbf{A} \leq \mathbf{B}$.

1. Infimum in $\mathcal{I}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{f})$ is attained on some $\mathbf{f}_0 \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Here strict log-concavity is used via Arzelà–Ascoli.
2. If $f \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$ then (key computation on next slide)

$$\mathcal{BL}(\mathbf{f})^2 \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \, \mathcal{BL}(\operatorname{Conv} \mathbf{f}),$$

$\operatorname{Conv} \mathbf{f} = (\operatorname{Conv} f_1, \ldots, \operatorname{Conv} f_m)$, $\operatorname{Conv} f_i = 2^{n_i/2} (f_i * f_i)(\sqrt{2} \, \cdot)$.

Lemma A: $\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}} \;\; \Rightarrow \;\; \operatorname{Conv} \mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Hence:

$$\mathcal{I}^2_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathbf{f}_0)^2 \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\operatorname{Conv} \mathbf{f}_0) \geq \mathcal{I}^2_{\mathbf{A},\mathbf{B}}$$

$$\rightsquigarrow \;\; \operatorname{Conv} \mathbf{f}_0 \text{ minimizer} \;\; \rightsquigarrow \;\; \forall k \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\operatorname{Conv}^k \mathbf{f}_0).$$

3. By CLT, $\operatorname{Conv}^k \mathbf{f}_0 \to \mathbf{g} \in \mathcal{G}_{\mathbf{A},\mathbf{B}}$, hence:

$$\mathcal{I}_{\mathbf{A},\mathbf{B}} = \lim_{k \to \infty} \mathcal{BL}(\operatorname{Conv}^k \mathbf{f}_0) \geq \mathcal{BL}(\mathbf{g}) \geq \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}} \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \;\; \rightsquigarrow \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}.$$

4. For all $\mathbf{Q} \geq 0$, $\mathcal{I}_{\mathbf{Q}} = \lim_{\lambda \to 0^+} \lim_{\Lambda \to \infty} \mathcal{I}_{\mathbf{Q}+\lambda \mathrm{Id}, \Lambda \mathrm{Id}} \;\; \rightsquigarrow \;\; \mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. □

Nakamura–Tsuji's Proof

Fix $0 < \mathbf{A} \leq \mathbf{B}$.

1. Infimum in $\mathcal{I}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{f})$ is attained on some $\mathbf{f}_0 \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Here strict log-concavity is used via Arzelà–Ascoli.
2. If $f \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$ then (key computation on next slide)
$$\mathcal{BL}(\mathbf{f})^2 \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\mathrm{Conv}\, \mathbf{f}),$$
$\mathrm{Conv}\, \mathbf{f} = (\mathrm{Conv}\, f_1, \ldots, \mathrm{Conv}\, f_m)$, $\mathrm{Conv}\, f_i = 2^{n_i/2}(f_i * f_i)(\sqrt{2}\,\cdot)$.
Lemma A: $\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}} \;\Rightarrow\; \mathrm{Conv}\, \mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Hence:
$$\mathcal{I}^2_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathbf{f}_0)^2 \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\mathrm{Conv}\, \mathbf{f}_0) \geq \mathcal{I}^2_{\mathbf{A},\mathbf{B}}$$
$$\rightsquigarrow \; \mathrm{Conv}\, \mathbf{f}_0 \text{ minimizer } \rightsquigarrow \; \forall k \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathrm{Conv}^k\, \mathbf{f}_0).$$
3. By CLT, $\mathrm{Conv}^k\, \mathbf{f}_0 \to \mathbf{g} \in \mathcal{G}_{\mathbf{A},\mathbf{B}}$, hence:
$$\mathcal{I}_{\mathbf{A},\mathbf{B}} = \lim_{k \to \infty} \mathcal{BL}(\mathrm{Conv}^k\, \mathbf{f}_0) \geq \mathcal{BL}(\mathbf{g}) \geq \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}} \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \;\; \rightsquigarrow \;\; \mathcal{I}_{\mathbf{A},\mathbf{B}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{A},\mathbf{B}}.$$
4. For all $\mathbf{Q} \geq 0$, $\mathcal{I}_{\mathbf{Q}} = \lim_{\lambda \to 0^+} \lim_{\Lambda \to \infty} \mathcal{I}_{\mathbf{Q}+\lambda \mathrm{Id}, \Lambda \mathrm{Id}} \;\; \rightsquigarrow \;\; \mathcal{I}_{\mathbf{Q}} = \mathcal{I}^{\mathcal{G}}_{\mathbf{Q}}$. □

Nakamura–Tsuji's Proof

Fix $0 < \mathbf{A} \leq \mathbf{B}$.

1. Infimum in $\mathcal{I}_{\mathbf{A},\mathbf{B}} := \inf_{\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}} \mathcal{BL}(\mathbf{f})$ is attained on some $\mathbf{f}_0 \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Here strict log-concavity is used via Arzelà–Ascoli.
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$$\mathcal{BL}(\mathbf{f})^2 \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\mathrm{Conv}\, \mathbf{f}),$$

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Lemma A: $\mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}} \;\; \Rightarrow \;\; \mathrm{Conv}\, \mathbf{f} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. Hence:

$$\mathcal{I}^2_{\mathbf{A},\mathbf{B}} = \mathcal{BL}(\mathbf{f}_0)^2 \geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \mathcal{BL}(\mathrm{Conv}\, \mathbf{f}_0) \geq \mathcal{I}^2_{\mathbf{A},\mathbf{B}}$$

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Convolution computation

Assume w.l.o.g. $\int_{\mathbb{R}^{n_i}} f_i(x_i)dx_i = 1$, and set $F(x) := e^{-\frac{1}{2}\langle Qx,x\rangle}\prod_{i=1}^m f_i(x_i)$.

$$\mathcal{BL}(\mathbf{f})^2 = \left(\int_{\mathbb{R}^N} F(x)dx\right)^2 = \int_{\mathbb{R}^N} F * F(x)dx$$

$$= \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qy,y\rangle}\prod_{i=1}^m \left(f_i\left(\frac{x_i+y_i}{\sqrt{2}}\right) f_i\left(\frac{x_i-y_i}{\sqrt{2}}\right)\right)^{c_i} dydx.$$

$O(2)$ trick: Brascamp–Lieb $\sim$'75, Lieb '90, Ball's lemma in Barthe '97.

Lemma B: $\forall x \in \mathbb{R}^N$, $\mathbf{f}_x := \left(f_i\left(\frac{x_i+\cdot}{\sqrt{2}}\right) f_i\left(\frac{x_i-\cdot}{\sqrt{2}}\right)\right)_{i=1,\ldots,m} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$, hence:

$$\geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle}\prod_{i=1}^m \left(\int_{\mathbb{R}^N} f_i\left(\frac{x_i+y_i}{\sqrt{2}}\right) f_i\left(\frac{x_i-y_i}{\sqrt{2}}\right) dy\right)^{c_i} dx$$

$$= \mathcal{I}_{\mathbf{A},\mathbf{B}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle}\prod_{i=1}^m \mathrm{Conv}\, f_i(x_i)^{c_i} dx = \mathcal{I}_{\mathbf{A},\mathbf{B}}\, \mathcal{BL}(\mathrm{Conv}\,\mathbf{f}). \quad \square$$

Convolution computation

Assume w.l.o.g. $\int_{\mathbb{R}^{n_i}} f_i(x_i)dx_i = 1$, and set $F(x) := e^{-\frac{1}{2}\langle Qx,x\rangle}\prod_{i=1}^m f_i(x_i)$.

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$O(2)$ trick: Brascamp–Lieb ~'75, Lieb '90, Ball's lemma in Barthe '97.

Lemma B: $\forall x \in \mathbb{R}^N$, $\mathbf{f}_x := \left(f_i\left(\frac{x_i+\cdot}{\sqrt{2}}\right) f_i\left(\frac{x_i-\cdot}{\sqrt{2}}\right)\right)_{i=1,\ldots,m} \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$, hence:

$$\geq \mathcal{I}_{\mathbf{A},\mathbf{B}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle}\prod_{i=1}^m \left(\int_{\mathbb{R}^N} f_i\left(\frac{x_i+y_i}{\sqrt{2}}\right) f_i\left(\frac{x_i-y_i}{\sqrt{2}}\right) dy\right)^{c_i} dx$$
$$= \mathcal{I}_{\mathbf{A},\mathbf{B}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle Qx,x\rangle}\prod_{i=1}^m \mathrm{Conv}\, f_i(x_i)^{c_i} dx = \mathcal{I}_{\mathbf{A},\mathbf{B}}\, \mathcal{BL}(\mathrm{Conv}\,\mathbf{f}). \qquad \square$$

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For CLT, just need $\int f_i(x_i)x_i dx_i = 0$.

Rolling back, $\int h_i(x_i)g_{Q_i}(x_i)x_i dx_i = 0$.

Thm (Nakamura–Tsuji '25+ forthcoming)

For all convex $K_i \subset \mathbb{R}^n$ with $\text{bar}_\gamma(K_i) = 0$,

$$\gamma^n(K_1 \cap K_2) \geq \gamma^n(K_1)\gamma^n(K_2).$$

where $\text{bar}_\gamma(C) := \frac{1}{\gamma^n(C)} \int_C \vec{x} \, d\gamma^n(x)$. Moreover,

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Thanks for you attention!