The dimensional Brunn–Minkowski inequality in Gauss space

Alexandros Eskenazis
(joint work with G. Moschidis)

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The classical Brunn–Minkowski inequality asserts that for every compact sets A, B in $\mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\big|\lambda A + (1-\lambda)B\big|^{\frac{1}{n}} \geq \lambda|A|^{\frac{1}{n}} + (1-\lambda)|B|^{\frac{1}{n}},$$

where $|\cdot|$ denotes the Lebesgue measure and the Minkowski convex combination of sets is given by

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This inequality captures the optimal concavity of the Lebesgue measure and becomes an equality if A and B are homothetic and convex.

The Brunn–Minkowski inequality (continued)

Broadly speaking, modern Brunn–Minkowski theory tries to relate the *size* of the *sum* of given sets with the *size* of the individual *summands*, where *size* and *sum* are interpreted more loosely than in the classical Brunn–Minkowski inequality.

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In this talk, we will be interested in the case that:

- *Sum* refers to the usual Minkowski addition $+$ of subsets of $\mathbb{R}^n$.
- The *size* of such a set A is given by its standard Gaussian measure,

$$\gamma_n(A) = \frac{1}{(2\pi)^{n/2}} \int_A e^{-|x|^2/2} \, \mathrm{d}x,$$

where $|x|$ is the Euclidean length of a vector x.

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The most profound Brunn–Minkowski-type inequality for the Gaussian measure is Ehrhard's inequality (1983), which asserts that for every Borel sets A, B in $\mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\Phi^{-1}\big(\gamma_n\big(\lambda A + (1-\lambda)B\big)\big) \geq \lambda\Phi^{-1}\big(\gamma_n(A)\big) + (1-\lambda)\Phi^{-1}\big(\gamma_n(B)\big),$$

where Φ^{-1} is the inverse of the distribution function $\Phi(x) = \gamma_1((-\infty, x])$.

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Ehrhard's original proof required both sets A, B to be convex. The general version stated here is due to Borell (2003).

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Then, plugging $A = \{x : \; x_1 < a\}$ and $B = \{x : \; x_1 < b\}$, we deduce that $\zeta_n \circ \Phi$ has to be a concave function. So, the choice $\zeta_n = \Phi^{-1}$ is extremal.

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Is it true that for every convex sets K, L in $\mathbb{R}^n$ which contain the origin and $\lambda \in (0,1)$, the inequality

$$\gamma_n\big(\lambda K + (1-\lambda)L\big)^{\frac{1}{n}} \geq \lambda \gamma_n(K)^{\frac{1}{n}} + (1-\lambda)\gamma_n(L)^{\frac{1}{n}}$$

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holds true?

• Taking $K = [-1, 1]^n$, $L = \{x\}$ and letting $x \to \infty$, it becomes clear that the dimensional Brunn–Minkowski inequality cannot hold for an arbitrary pair of convex sets in $\mathbb{R}^n$.

The Gardner–Zvavitch problem (continued)

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Yes.

Remark. It was already observed by Gardner and Zvavitch that the dimensional Brunn–Minkowski inequality neither trivially implies nor follows from Ehrhard's inequality.

Symmetry in Brunn–Minkowski theory (cf. Karoly Böröczky's talk)

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Recall that the Gaussian measure is log-concave, i.e.

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for every Borel sets A, B in $\mathbb{R}^n$ and $\lambda \in (0,1)$. The Gardner–Zvavitch problem is a strengthening of log-concavity for the smaller class of symmetric convex sets.

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Another example of a symmetric refinement of log-concavity is the deep B-inequality of Cordero, Fradelizi and Maurey (2004), according to which for every symmetric convex set K in $\mathbb{R}^n$ and every $a, b > 0$ and $\lambda \in (0,1)$,

$$\gamma_n\big(a^\lambda b^{1-\lambda} K\big) \geq \gamma_n(aK)^\lambda \gamma_n(bK)^{1-\lambda}.$$

Notice that for every convex set K,

$$\gamma_n\big(\big(\lambda a + (1-\lambda)b\big)K\big) \geq \gamma_n(aK)^\lambda \gamma_n(bK)^{1-\lambda}$$

but symmetry is crucial if one wants to replace the arithmetic mean by the geometric mean.

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• (Böröczky–Kalantzopoulos, 2020) If K and L have n hyperplane symmetries. ✓

The local Gardner–Zvavitch problem

In 2018, Kolesnikov and Livshyts took a different route to attack the Gardner–Zvavitch problem (inspired by important earlier work of Kolesnikov and E. Milman). The main idea is to prove the dimensional Brunn–Minkowski inequality "infinitesimally", that is, when K and L are small perturbations of each other.

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Recall that the generator of the Ornstein–Uhlenbeck semigroup is the elliptic differential operator $\mathscr{L}$ whose action on a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\forall\, x \in \mathbb{R}^n, \qquad \mathscr{L}u(x) = \Delta u(x) - \sum_{i=1}^{n} x_i \partial_i u(x).$$

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We also denote by $\|A\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm of a matrix A, which is given by $\|A\|_{\mathrm{HS}}^2 = \sum_{i,j} a_{ij}^2$.

The local Gardner–Zvavitch problem (continued)

They proved the following local-to-global principle.

Proposition (Kolesnikov–Livshyts, 2018)

Let $\delta \in [0,1]$ be such that for every symmetric convex set K in $\mathbb{R}^n$, every smooth symmetric function $u : K \to \mathbb{R}$ with $\mathscr{L}u = 1$ on K satisfies

$$\mathscr{F}(u) := \frac{1}{\gamma_n(K)} \int_K \|\nabla^2 u\|_{\mathrm{HS}}^2 + |\nabla u|^2 \, \mathrm{d}\gamma_n \geq \frac{\delta}{n}.$$

Then, for every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$,

$$\gamma_n\big(\lambda K + (1-\lambda)L\big)^{\frac{\delta}{n}} \geq \lambda \gamma_n(K)^{\frac{\delta}{n}} + (1-\lambda)\gamma_n(L)^{\frac{\delta}{n}}.$$

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The main result of their paper is that if K contains the origin and u is *any* smooth function with $\mathscr{L} u = 1$ on K, then $\mathscr{F}(u) \geq \frac{1}{2n}$, thus implying Gaussian BM with exponent $\frac{1}{2n}$ for such convex sets.

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Theorem (E.–Moschidis, 2020)

For every $n \in \mathbb{N}$ and every symmetric convex set K in $\mathbb{R}^n$, every smooth symmetric function $u : K \to \mathbb{R}$ with $\mathscr{L} u = 1$ on K, satisfies

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Plan for the rest of the talk. We will first go over the proof of the local-to-global principle and then show the proof of the local inequality above. Towards the end we will also discuss a related open problem.

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Let $\mathrm{d}\mu(x) = e^{-V(x)}\,\mathrm{d}x$ be a log-concave measure. Fix two smooth, symmetric, strictly convex sets K, L in $\mathbb{R}^n$. If $K_\lambda = (1-\lambda)K + \lambda L$, we want to understand whether $M(\lambda) := \mu(K_\lambda)^{\frac{\delta}{n}}$ is concave on $[0,1]$. In other words, we want to understand whether the following inequality holds:

$$\forall\ \lambda \in (0,1), \quad M''(\lambda)M(\lambda) \leq \frac{n-\delta}{n} M'(\lambda)^2.$$

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Notice that if $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ is given by $\psi(\theta) = h_L(\theta) - h_K(\theta)$, then ψ is even and

$$h_{K_\lambda} = h_K + \lambda\psi.$$

The local-to-global principle (continued)

Lemma (Kolesnikov–E. Milman)

For $x \in \partial K$, let n_x be the unit normal of ∂K at x and define $f : \partial K \to \mathbb{R}$ by $f(x) = \psi(n_x)$. Then

$$M'(0) = \int_{\partial K} f(x)\, \mathrm{d}\mu_{\partial K}(x)$$

and

$$M''(0) = \int_{\partial K} H_x f(x)^2 - \langle II^{-1}(x) \nabla_{\partial K} f(x), \nabla_{\partial K} f(x) \rangle \, \mathrm{d}\mu_{\partial K}(x),$$

where $\mu_{\partial K}$ is the restriction of μ on ∂K, II is the second fundamental form of ∂K and H_x is the weighted mean curvature at x, i.e.

$$H_x = \mathrm{tr}(II(x)) - \langle \nabla V(x), n_x \rangle.$$

The local-to-global principle (continued)

So, we have to show that for every symmetric K and every even function $f : \partial K \to \mathbb{R}$,

$$\int_{\partial K} \underbrace{Hf^2 - \langle II^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f\rangle}_{\Phi(\partial K, V, f, \nabla f)} \,\mathrm{d}\mu_{\partial K} \le \frac{n-\delta}{n\mu(K)} \left(\int_{\partial K} f(x)\,\mathrm{d}\mu_{\partial K}(x)\right)^2.$$

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Remark. This inequality with $\delta = 1$ and μ being the Lebesgue measure, appeared in work of Colesanti (2008).

The local-to-global principle (continued)

Denote by $\mathscr{L}_\mu$ the elliptic operator associated to μ, whose action on a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{L}_\mu u = \Delta u - \langle \nabla V, \nabla u \rangle$.

Theorem (Reilly formula)

For every smooth function $u : K \to \mathbb{R}$,

$$\int_K (\mathscr{L}_\mu u)^2 \, \mathrm{d}\mu = \int_K \|\nabla^2 u\|_{\mathrm{HS}}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \, \mathrm{d}\mu + \int_{\partial K} \Psi \, \mathrm{d}\mu_{\partial K},$$

for some explicit $\Psi = \Psi(\partial K, V, u, \nabla u)$.

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for some explicit $\Psi = \Psi(\partial K, V, u, \nabla u)$.

Crucial observation! If $f : \partial K \to \mathbb{R}$ is the Neumann boundary data of u, i.e. $f(x) = \langle \nabla u(x), n_x \rangle$ for $x \in \partial K$, then

$$\Phi(\partial K, V, f, \nabla f) \leq \Psi(\partial K, V, u, \nabla u).$$

The local-to-global principle (continued)

Conclusion. To derive a dimensional Brunn–Minkowski inequality for μ it suffices to show that *for every* symmetric convex set K, *for every* even function $f : \partial K \to \mathbb{R}$ *there exists* a $u : K \to \mathbb{R}$ with Neumann boundary data f, such that

$$\int_K (\mathscr{L}_\mu u)^2 \,\mathrm{d}\mu - \int_K \|\nabla^2 u\|_{\mathrm{HS}}^2 + \langle \nabla^2 V \nabla u, \nabla u\rangle \,\mathrm{d}\mu \leq \frac{n-\delta}{n\mu(K)} \left(\int_{\partial K} f(x) \,\mathrm{d}\mu_{\partial K}(x) \right)^2 .$$

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If $\int_{\partial K} f \, \mathrm{d}\mu_{\partial K} = 0$, this is trivially true by the log-concavity of μ, so we can rescale to $\int_{\partial K} f \, \mathrm{d}\mu_{\partial K} = \mu(K)$. Then (...), the equation $\mathscr{L}_\mu u = 1$ has a unique solution on K with Neumann boundary condition $\langle \nabla u(x), n_x \rangle = f(x)$ and rearranging we get

$$\frac{1}{\mu(K)} \int_K \|\nabla^2 u\|_{\mathrm{HS}}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \, \mathrm{d}\mu \geq \frac{\delta}{n}.$$

Back to Brunn–Minkowski

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Applying the above reasoning to the Lebesgue measure, we deduce that the classical Brunn–Minkowski inequality is a consequence of the following statement. For every K and every $u : K \to \mathbb{R}$ with $\Delta u = 1$ on K,

$$\frac{1}{|K|}\int_K \|\nabla^2 u(x)\|_{\mathrm{HS}}^2 \,\mathrm{d}x \geq \frac{1}{n}.$$

Back to Brunn–Minkowski

Applying the above reasoning to the Lebesgue measure, we deduce that the classical Brunn–Minkowski inequality is a consequence of the following statement. For every K and every $u : K \to \mathbb{R}$ with $\Delta u = 1$ on K,

$$\frac{1}{|K|}\int_K \|\nabla^2 u(x)\|_{\mathrm{HS}}^2 \, \mathrm{d}x \geq \frac{1}{n}.$$

This is true by the pointwise inequality

$$\|\nabla^2 u\|_{\mathrm{HS}}^2 = \sum_{i=1}^n \lambda_i(\nabla^2 u)^2 \geq \frac{1}{n}\left(\sum_{i=1}^n \lambda_i(\nabla^2 u)\right)^2$$

$$= \frac{\big(\mathrm{tr}(\nabla^2 u)\big)^2}{n} = \frac{(\Delta u)^2}{n} = \frac{1}{n}.$$

The approach of Kolesnikov and Livshyts

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Fix K and $u : K \to \mathbb{R}$ that satisfies $\mathscr{L}u(x) = \Delta u(x) - \langle x, \nabla u(x) \rangle = 1$. Also, denote by γ_K the measure given by $\gamma_K(A) = \frac{\gamma_n(A \cap K)}{\gamma_n(K)}$.

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Step 1. As in the case of Lebesgue measure,

$$\begin{aligned} \int \|\nabla^2 u\|_{\mathrm{HS}}^2 + |\nabla u|^2 \, \mathrm{d}\gamma_K &\geq \int \frac{(\Delta u)^2}{n} + |\nabla u|^2 \, \mathrm{d}\gamma_K \\ &= \int \frac{\big(1 + \langle x, \nabla u(x)\rangle\big)^2}{n} + |\nabla u(x)|^2 \, \mathrm{d}\gamma_K(x). \end{aligned}$$

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Step 2. Given $x \in \mathbb{R}^n$,

$$\min_{V \in \mathbb{R}^n} \frac{\big(1 + \langle x, V \rangle\big)^2}{n} + |V|^2 = \frac{1}{|x|^2 + n}.$$

The approach of Kolesnikov and Livshyts (continued)

Step 3. It is simple to show that for every star-shaped K,

$$\int \frac{1}{|x|^2 + n} \,\mathrm{d}\gamma_K(x) \geq \int_{\mathbb{R}^n} \frac{1}{|x|^2 + n} \,\mathrm{d}\gamma_n(x) = \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

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Remark. Step 2 is evidently not sharp, since it becomes an equality only for $u_0(x) = -\frac{1}{2}\log(|x|^2+n)$ which satisfies

$$\mathscr{L}u_0(x) = \frac{|x|^2-n}{|x|^2+n} + \frac{2|x|^2}{(|x|^2+n)^2} \neq 1.$$

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However...

Proposition

There exists K and $u : K \to \mathbb{R}$ with $\mathscr{L}u = 1$ such that

$$\int \frac{\big(1+\langle x, \nabla u(x)\rangle\big)^2}{n} + |\nabla u(x)|^2\,\mathrm{d}\gamma_K(x) = \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

Proof of the main theorem

Theorem (E.–Moschidis, 2020)

For every $n \in \mathbb{N}$ and every symmetric convex set K in $\mathbb{R}^n$, every smooth symmetric function $u : K \to \mathbb{R}$ with $\mathscr{L}u = 1$ on K, satisfies

$$\int \|\nabla^2 u\|_{\mathrm{HS}}^2 + |\nabla u|^2 \, \mathrm{d}\gamma_K \geq \frac{1}{n}.$$

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For a matrix A, denote by $\widehat{A}$ its traceless part, $\widehat{A} = A - \frac{\mathrm{tr}(A)}{n}\mathrm{Id}$. Then,

$$\|A\|_{\mathrm{HS}}^2 = \|\widehat{A}\|_{\mathrm{HS}}^2 + \frac{(\mathrm{tr}A)^2}{n}.$$

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In particular, if $\widehat{\nabla}^2 u$ is the traceless part of $\nabla^2 u$, we have

$$\|\nabla^2 u\|_{\mathrm{HS}}^2 = \|\widehat{\nabla}^2 u\|_{\mathrm{HS}}^2 + \frac{(\Delta u)^2}{n}.$$

Proof of the main theorem (continued)

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Notice that

$$\|\widehat{\nabla}^2 u\|_{\mathrm{HS}}^2 = \|\widehat{\nabla}^2 (u - r)\|_{\mathrm{HS}}^2,$$

for every $r \in \mathrm{Ker}(\widehat{\nabla}^2)$, in particular $r(x) = \frac{|x|^2}{2n}$.

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$$\|\widehat{\nabla}^2(u-r)\|_{\mathrm{HS}}^2 = \|\nabla^2(u-r)\|_{\mathrm{HS}}^2 - \frac{(\Delta(u-r))^2}{n} = \|\nabla^2(u-r)\|_{\mathrm{HS}}^2 - \frac{(\Delta u - 1)^2}{n}.$$

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Combining these identities and using the equation $\mathscr{L} u = 1$,

$$\begin{aligned} \|\nabla^2 u(x)\|_{\mathrm{HS}}^2 &= \|\nabla^2 (u-r)(x)\|_{\mathrm{HS}}^2 + \frac{2}{n}\Delta u(x) - \frac{1}{n} \\ &= \|\nabla^2 (u-r)(x)\|_{\mathrm{HS}}^2 + \frac{2}{n}\sum_{i=1}^n x_i \partial_i u(x) + \frac{1}{n}. \end{aligned}$$

The Brascamp–Lieb inequality

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Theorem (Brascamp–Lieb, 1976)

Let $\beta \in (0, \infty)$ and $V : \mathbb{R}^n \to \mathbb{R}$ be such that $\nabla^2 V \geq \beta \mathsf{Id}$. Then, if $\mathrm{d}\mu(x) = e^{-V(x)}\,\mathrm{d}x$, every smooth function $h : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\mathrm{Var}_\mu h := \int h^2 \,\mathrm{d}\mu - \left(\int h \,\mathrm{d}\mu \right)^2 \leq \frac{1}{\beta} \int |\nabla h|^2 \,\mathrm{d}\mu.$$

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In particular, since each $\partial_i(u-r)$ is odd and K is symmetric, we have

$$\sum_{j=1}^n \int \big(\partial_i\partial_j(u-r)\big)^2\,\mathrm{d}\gamma_K \geq \mathrm{Var}_{\gamma_K}\big(\partial_i(u-r)\big) = \int \big(\partial_i(u-r)\big)^2\,\mathrm{d}\gamma_K.$$

Proof of the main theorem (continued)

Adding up, we get

$$\int \|\nabla^2(u-r)\|_{\mathrm{HS}}^2 \,\mathrm{d}\gamma_K \geq \int_K |\nabla(u-r)|^2 \,\mathrm{d}\gamma_K$$
$$= \int_K |\nabla u(x)|^2 - \frac{2}{n}\sum_{i=1}^n x_i \partial_i u(x) + \frac{|x|^2}{n^2} \,\mathrm{d}\gamma_K(x).$$

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Putting everything together,

$$\int \|\nabla^2 u\|_{\mathrm{HS}}^2 + |\nabla u|^2 \,\mathrm{d}\gamma_K \geq \int 2|\nabla u(x)|^2 + \frac{|x|^2}{n^2} + \frac{1}{n} \,\mathrm{d}\gamma_K(x)$$

and the proof is complete. □

Equality cases

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The proof above in fact implies a genuinely stronger statement. There exists a function $\sigma_n : [0,1] \to \mathbb{R}$ such that $x \mapsto \sigma_n^{-1}(x)^{\frac{1}{n}}$ is *strictly* increasing and *strictly* concave such that

$$\sigma_n\big(\gamma_n\big(\lambda K + (1-\lambda)L\big)\big) \geq \lambda\sigma_n\big(\gamma_n(K)\big) + (1-\lambda)\sigma_n\big(\gamma_n(L)\big)$$

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holds for every symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$.

Corollary

Let K, L be two symmetric convex sets in $\mathbb{R}^n$ and $\lambda \in (0,1)$ be such that

$$\gamma_n\big(\lambda K + (1-\lambda)L\big)^{\frac{1}{n}} = \lambda \gamma_n(K)^{\frac{1}{n}} + (1-\lambda)\gamma_n(L)^{\frac{1}{n}}.$$

Then $K = L$.

An Ehrhard-type inequality for symmetric convex sets?

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Fix $n \in \mathbb{N}$. Is there an "optimal" increasing function $\xi_n : [0,1] \to \mathbb{R}$ such that for every origin symmetric convex sets K, L in $\mathbb{R}^n$ and every $\lambda \in (0,1)$, the inequality

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Remark. For every symmetric convex set K, if $\Xi_n(r) = \gamma_n(rK)$, then $\xi_n := \Xi_n^{-1}$ does not satisfy the inequality.

Thank you!