Coercive Inequalities and U-Bounds

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Online Asymptotic Geometric Analysis Seminar

Introduction

In 1975, L. Gross obtained the following Logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^n} f^2 log\left(\frac{f^2}{\int_{\mathbb{R}^n} f^2 d\mu}\right) d\mu \leq 2\int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad (1)$$

where ∇ is the standard gradient on $\mathbb{R}^n$ and $d\mu = \frac{e^{-\frac{|x|^2}{2}}}{Z} d\lambda$ is the Gaussian measure.

Plan

Euclidean space: $\mathbb{R}^n$ → Carnot group: $\mathbb{G}$

Euclidean gradient: ∇_{euc} → Subgradient: $\nabla = (X_1, \dots, X_n)$

Gaussian measure → More general probability measures

Introduction

In 1975, L. Gross obtained the following Logarithmic Sobolev inequality ([33]):

$$\int_{\cancel{\mathbb{R}^n}\,\mathbb{G}} f^{\cancel{2}\,q} log \left(\frac{f^{\cancel{2}\,q}}{\int_{\cancel{\mathbb{R}^n}\,\mathbb{G}} f^{\cancel{2}\,q} d\mu} \right) d\mu \leq 2 \int_{\cancel{\mathbb{R}^n}\,\mathbb{G}} |\nabla_{\mathbb{G}} f|^{\cancel{2}\,q} d\mu, \qquad (1)$$

where $\nabla_{\mathbb{G}}$ is the standard gradient on $\overset{\mathbb{G}}{\cancel{\mathbb{R}^n}}$ and $d\mu = \frac{e^{-\cancel{\frac{|x|^2}{2}}\,U(N)}}{Z} d\lambda$ is the Gaussian measure.

Plan

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda.$$

d: Carnot-Carathéodory distance

W. Hebisch and B. Zegarliński (2009): q −LSI for $U(d) = d^p$, for p the finite index conjugate of q.

EBD (2021): q −LSI for $U(d)$, where $U'' \leq \beta U'$ and $\mathrm{U} \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

LSI → Hypercontractivity

LSI → Talagrand inequality

Plan

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

No LSI

(W. Hebisch, B. Zegarliński-2010)

Force LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

(EBD, B. Zegarliński-2021)

Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

(EBD, B. Zegarliński-2021)

U-Bound

Prove ϕ −LSI and q −Poincaré

(EBD, B. Zegarliński-2021)

Extend Poincaré and LSI to higher order.

(EBD, Y. Wang, B. Zegarliński-2022)

U-Bound + Weak Poincaré

Extend Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

(EBD, Y. Qiu, B. Zegarliński, M. Zhang-2022)

Introduction

L.Gross showed that (1) can be extended to infinite dimensions.
He proved that if $\mathcal{L}$ is the non-positive self-adjoint operator on $L^2(\mu)$ such that

$$(-\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

then (1) is equivalent to the fact that the semigroup $P_t = e^{t\mathcal{L}}$ generated by $\mathcal{L}$ is hypercontractive.

Introduction

In 1985, D. Bakry and M. Emery extended the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemaniann manifolds under the Curvature-Dimension condition.

Introduction

The q-Logarithmic Sobolev inequality, in the setting of a metric measure space, was obtained by S. Bobkov and M. Ledoux in 2000, in the form:

$$\int f^q log \frac{f^q}{\int f^q d\mu} d\mu \leq c \int |\nabla f|^q d\mu,$$

where $q \in (1, 2]$.In 2005, S. Bobkov and B. Zegarliński showed that the q-LSI is better than $q = 2$ i in the sense that one gets a stronger decay of tail estimates i.e. if μ satisfies the Logarithmic Sobolev inequality for $q \in (1, 2]$, then for every bounded locally Lipschitz function f such that $|\nabla f| \leq M \; \mu - a.e.$ for $M \in (0, \infty)$, we have

$$\mu(e^{tf}) \leq exp\{\frac{cM^q}{q^q(q-1)} t^q + t\mu(f)\} \qquad \forall t > 0.$$

Introduction

In addition, when the space is finite, and under weak conditions, S. Bobkov and B. Zegarliński proved that the corresponding semigroup P_t is ultracontractive i.e.

$$\| P_t f \|_\infty \leq \| f \|_p$$

for all $t \geq 0$ and $p \in [1, \infty)$.

Introduction

Definition

A Lie group on $\mathbb{R}^N$, $\mathbb{G}$ =$(\mathbb{R}^N, \circ)$ is a Carnot group if:

(C.1) $\mathbb{R}^N$ can be split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_r}$, and the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, ..., \lambda^r x^{(r)}), \qquad x^{(i)} \in \mathbb{R}^{N_i},$$

is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$.

(C.2) If N_1 is as above, let $X_1, ..., X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_j(0) = \partial/\partial x_j|_0$ for $j = 1, ..., N_1$. Then

$$rank(Lie\{X_1, ..., X_{N_1}\}(x)) = N \qquad \forall x \in \mathbb{R}^N.$$

Definition

The vector valued operator $\nabla := (X_1, X_2, ..., X_{N_1})$ is called the sub-gradient on $\mathbb{G}$, and $\triangle = \sum_{i=1}^{N_1} X_i^2$ is called the sub-Laplacian on $\mathbb{G}$.

Introduction

In the setting of Carnot groups, D. Bakry and M. Emery's Curvature-Dimension condition will no longer hold true. In 2010, W. Hebisch and B. Zegarliński developed a method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space was developed.

their method is based on U-bounds,

$$\int f^q \mathcal{U}(d)\, d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu$$

where $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$ is a probability measure, $U(d)$ and $\mathcal{U}(d)$ are functions having a suitable growth at infinity.

$$\int f^q \mathcal{U}(d)\, d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu;\ d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

$\mathcal{U}(d) \to \infty$

"in all directions"

↓

Hebsich-Zegarlinski Theorem 1 (2009)

↓

q −Poincaré

$\mathcal{U}(d) = U(d) + |\nabla U(d)|^q$

↓

Hebsich-Zegarlinski Theorem 2 (2009)

↓

q −Logarithmic Sobolev

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$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda.$$

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Applications:

LSI → Hypercontractivity

LSI → Talagrand inequality

Carnot-Carathéodory distance

Definition

We say that γ is horizontal if there exist measurable functions $a_1, \ldots, a_{N_1} : [0,1] \to \mathbb{R}$ such that $\gamma'(t) = \sum_{i=1}^{N_1} a_i(t) X_i(\gamma(t))$for almost all $t \in [0,1]$. For such a horizontal curve γ, we define the length of γ to be

$$|\gamma| = \int_0^1 \left(\sum_{i=1}^{N_1} a_i^2(t) \right)^{\frac{1}{2}} dt.$$

Definition

The Carnot-Carathéodory distance or the control distance between two points x and y is defined by

$$d(x,y) = \inf \left\{ t | \gamma : [0,t] \to G, \gamma(0) = x, \gamma(t) = y \; |\gamma'(s)| \le 1 \; \forall s \in [0,t] \right\},$$

where $\gamma : [0,1] \to G$ is an absolutely continuous horizontal path on $[0,1]$.

U-Bound; $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$

Theorem (EBD, 2021)

Assume that outside the open unit ball $B = \{d(x) < 1\}$, the metric d satisfies the following: $|\nabla d|$ is bounded, say $|\nabla d| \leq 1$, and there exist finite positive constants K and c_0 such that

$$\Delta d \leq K + U'(d) \left(|\nabla d|^2 - c_0 \right). \tag{2}$$

(i) If $U'' \leq \beta U'$ for some positive constant β, outside B, then for any $q \in (1, \infty)$, there exist constants C_q, D_q, independent of f, such that

$$\int |f|^q |U'(d)|^q d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

(ii) If, in addition, $U \leq \gamma U'^q$ for some positive constant γ and some $q > 1$, outside B, then

$$\int |f|^q U(d) d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

Take $\mathcal{U}(d) = |U'(d)|^q$, by Hebisch-Zegarlinski Theorem 1, (i)$\rightarrow q-$Poincaré. To apply Theorem 2, we need $\mathcal{U}(d) = U(d) + |\nabla U(d)|^q = U(d) + |U'(d)\nabla d|^q \leq U(d) + |U'(d)|^q$. Using (i) and (ii), we get $q-$LSI.

Examples of Logarithmic Sobolev inequality

Example (EBD, 2021)

The $q-$Poincaré and a $q-$Logarithmic Sobolev inequality are satisfied for the measure

$$d\mu_U = \frac{e^{-(d+1)^p \log(d+1)}}{Z} d\lambda$$

for $q \geq \beta$, where β is the finite index conjugate to p.

Example (EBD, 2021)

The $q-$Poincaré and a $q-$Logarithmic Sobolev inequality are satisfied for the measure

$$d\mu_U = \frac{e^{-\sinh(d)}}{Z} d\lambda$$

for all $q \geq 1$.

Talagrand Inequality

In 2000, F. Otto and C. Villani showed that in the setting of manifolds under D. Bakry and M. Emery's Curvature-Dimension condition, the LSI implies Talagrand's inequality:

$$T_w(\mu,\nu) \leq 2\int \log(f)d\mu, \tag{4}$$

where μ is a measure on $\mathbb{R}^N$ absolutely continuous wrt the Gaussian measure ν, $f = \dfrac{d\mu}{d\nu}$, $w(x,y) = \displaystyle\sum_{i=1}^{N}(x_i - y_i)^2$, and

$$T_w(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^N\times\mathbb{R}^N} w(x,y)d\pi(x,y),$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with μ the first marginal and ν the second marginal.

Talagrand Inequality

We would like to apply the $q-$Logarithmic Sobolev inequality to get hypercontractivity and to obtain the $p-$Talagrand inequality on (X, d, μ) with a constant K :

$$W_p(\mu, \nu)^p \leq \frac{1}{K} \mathrm{Ent}_\mu \left(\frac{d\nu}{d\mu} \right), \tag{6}$$

with p finite index conjugate of q, where

$$W_p(\mu, \nu)^p = \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x, y)^p d\pi(x, y),$$

and

$$\mathrm{Ent}_\mu \left(\frac{d\nu}{d\mu} \right) = \int \frac{d\nu}{d\mu} \log \left(\frac{d\nu}{d\mu} \right) d\mu.$$

Talagrand Inequality

For the quadratic case $p = q = 2$, J. Lott and C. Villani (2007) and for $p \geq 2$, Z. Balogh, A. Engulatov, L. Hunzinker, and O, Maasalo (2011) used the Hamilton-Jacobi infimum convolution operator under the assumption where the space (X, d, μ) supports local Poincaré inequality and the measure μ is a doubling measure.

In our setting, we show hypercontractivity and the $p-$Talagrand inequality using the Hamilton-Jacobi equation in the setting of Carnot groups done by F. Dragoni in 2007 . The advantage of doing so is that the restriction to have μ a doubling measure is no longer required.

Talagrand Inequality and Hypercontractivity

Theorem (EBD, 2021 LSI→Talagrand)

Let $1 < q \leq 2$, and $p \geq 2$ be its finite index conjugate, so that $\frac{1}{p} + \frac{1}{q} = 1$. If (G, d, μ) satisfies the q-Logarithmic Sobolev inequality with constant $c = (q-1)\left(\frac{q}{K}\right)^{q-1}$ for some constant $K > 0$, then it also satisfies the p-Talagrand inequality with the same constant K.

Theorem (EBD, 2021 LSI→Hypercontractivity)

Assume we have the following 2-Logarithmic Sobolev inequality with the measure $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$, and in the setting of the Carnot group: then, for every bounded measurable function f on $\mathbb{G}$, every $t \geq 0$, and every $a \in \mathbb{R}$,

$$||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a.$$

[redacted]

$$d\mu = \frac{e^{-U(N)}}{Z}\, d\lambda.$$

N: a homogeneous norm

No LSI

(W. Hebisch, B. Zegarliński-2010)

Force LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z}\, d\lambda$$

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Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

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(EBD, B. Zegarliński-2021)

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Setup

We define the step-two Carnot group $\mathbb{G}$, i.e. a group isomorphic to $\mathbb{R}^{n+m}$ with the group law

$$(x,z)\circ(x',z') = \left(x_i + x_i',\ z_j + z_j' + \frac{1}{2} < \Lambda^{(j)} x, x' > \right)_{i=1,..,n; j=1,..,m}$$

for $x, x' \in \mathbb{R}^n, z, z' \in \mathbb{R}^m$, where $< .,. >$ stands for the inner product on $\mathbb{R}^n$, and:

1) The matrices $\Lambda^{(j)}$ are $n \times n$ skew-symmetric
2) The matrices are linearly independent

We are in the setting of Heisenberg group, if in addition:

1) $\Lambda^{(j)}$ are orthogonal
2) $\Lambda^{(k)}\Lambda^{(j)} + \Lambda^{(j)}\Lambda^{(k)} = 0,\ \forall k \neq j.$

Setup; $d\mu = \frac{e^{-U(N)}}{Z} d\lambda$

Heisenberg:

$N \equiv \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$ is the Kaplan norm. In other words, N^{2-Q} is the unique fundamental solution of the sub-Laplacian $\triangle := \sum_{i=1}^{n} X_i^2$,where X_i is the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G} \cong \mathbb{R}^{n+m}$, and $Q = n + 2m$ is the homogeneous dimension.

Step-two:

We consider $N \equiv \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$,where $(x, z) \in \mathbb{G}$ and $a \in (0, \infty)$.

Goal

J. Inglis (2010)

- ☐ Heisenberg-type group
- ☐ $d\mu = \frac{e^{-N^p}}{Z} d\lambda$
- ☐ $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$
- ☐ No U-Bound

[redacted]

Our Setting

- ☐ Step-two Carnot groups
- ☐ $d\mu = \frac{e^{-U(N)}}{Z} d\lambda$
- ☐ $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$, where $a \in (0, \infty)$
- ☐ U-Bound

[redacted]

U-Bound

Theorem (EBD and B. Zegarliński, 2021)

Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ *with* $a \in (0, \infty)$ *be as above and* $g : [0, \infty) \to [0, \infty)$ *be a differentiable increasing function such that* $g''(N) \leq g'(N)^3 N^3$ *on* $\{N \geq 1\}$*. Let* $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$ *be a probability measure, and* Z *the normalization constant. Then, for all locally Lipschitz functions* f*,*

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \tag{2}$$

holds outside the unit ball $\{N < 1\}$ *with* C *and* D *positive constants and* $q \geq 2$*.*

By Hebisch-Zegarlinski Theorem 1, we choose $\mathcal{U}(N) = \frac{g'(N)}{N^2}$, to obtain $q-$Poincaré inequality.

Examples of Poincaré Inequality

Example (EBD and B. Zegarliński, 2021)

The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{exp\left(-cosh\left(N^k\right)\right)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 1$ in the setting of the step-two Carnot group.

Example (J. Inglis, 2010)

The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{exp\left(-N^k\right)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 4$ in the setting of the step-two Carnot group.

Example (EBD and B. Zegarliński, 2021)

The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{exp\left(-N^k log\left(N+1\right)\right)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 3$ in the setting of the step-two Carnot group.

[redacted]

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

No LSI

(W. Hebisch, B. Zegarliński-2010)

~~**Force LSI by introducing a singularity through an additive or multiplicative term.**~~

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(EBD, B. Zegarliński-2021)

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[redacted]

$\mathcal{U}(d) \to \infty$ "in all directions" → Hebsich-Zegarlinski Theorem 1 (2009) → q −Poincaré

$\mathcal{U}(d) = U(d) + |\nabla U(d)|^q$ → Hebsich-Zegarlinski Theorem 2 (2009) → q −Logarithmic Sobolev

$\mathcal{U}(d) = \phi(U(d)) + |\nabla U(d)|^q$ → EBD-Zegarlinski (2021) → ϕ −Logarithmic Sobolev

$\phi-$Logarithmic Sobolev Inequality

Theorem (EBD and B. Zegarliński, 2021)

Let U be a locally lipschitz function on $\mathbb{R}^N$ which is bounded below such that $Z = \int e^{-U} d\lambda < \infty$, and $d\mu = \frac{e^{-U}}{Z} d\lambda$. Let $\phi : [0, \infty) \to \mathbb{R}^+$ be a non-negative, non-decreasing, concave function such that $\phi(0) > 0$, and $\phi'(0) > 0$. Assume the following classical Sobolev inequality is satisfied:

$$\left(\int |f|^{q+\epsilon} d\lambda\right)^{\frac{q}{q+\epsilon}} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda$$

for some $a, b \in [0, \infty)$, and $\epsilon > 0$. Moreover, if for some $A, B \in [0, \infty)$, we have:

$$\mu \left(|f|^q (\phi(U) + |\nabla U|^q)\right) \leq A\mu|\nabla f|^q + B\mu|f|^q, \tag{3}$$

Then, there exists constants $C, D \in [0, \infty)$ such that:

$$\mu \left(|f|^q \phi \left(\left| log \frac{|f|^q}{\mu|f|^q} \right|\right)\right) \leq C\mu|\nabla f|^q + D\mu|f|^q,$$

for all locally Lipschitz functions f.

Higher order LSI

Choose $\phi(x) = (1+x)^\beta$, for $\beta \in (0,1)$. Then, ϕ satisfies the conditions of the theorem above and we have:

$$\mu\left(|f|^q \left|log \frac{|f|^q}{\mu|f|^q}\right|^\beta\right) \leq \mu\left(|f|^q \phi\left(\left|log \frac{|f|^q}{\mu|f|^q}\right|\right)\right) \leq C\mu|\nabla f|^q + D\mu|f|^q.$$

Theorem (EBD, Y. Wang, and B. Zegarliński, 2022)

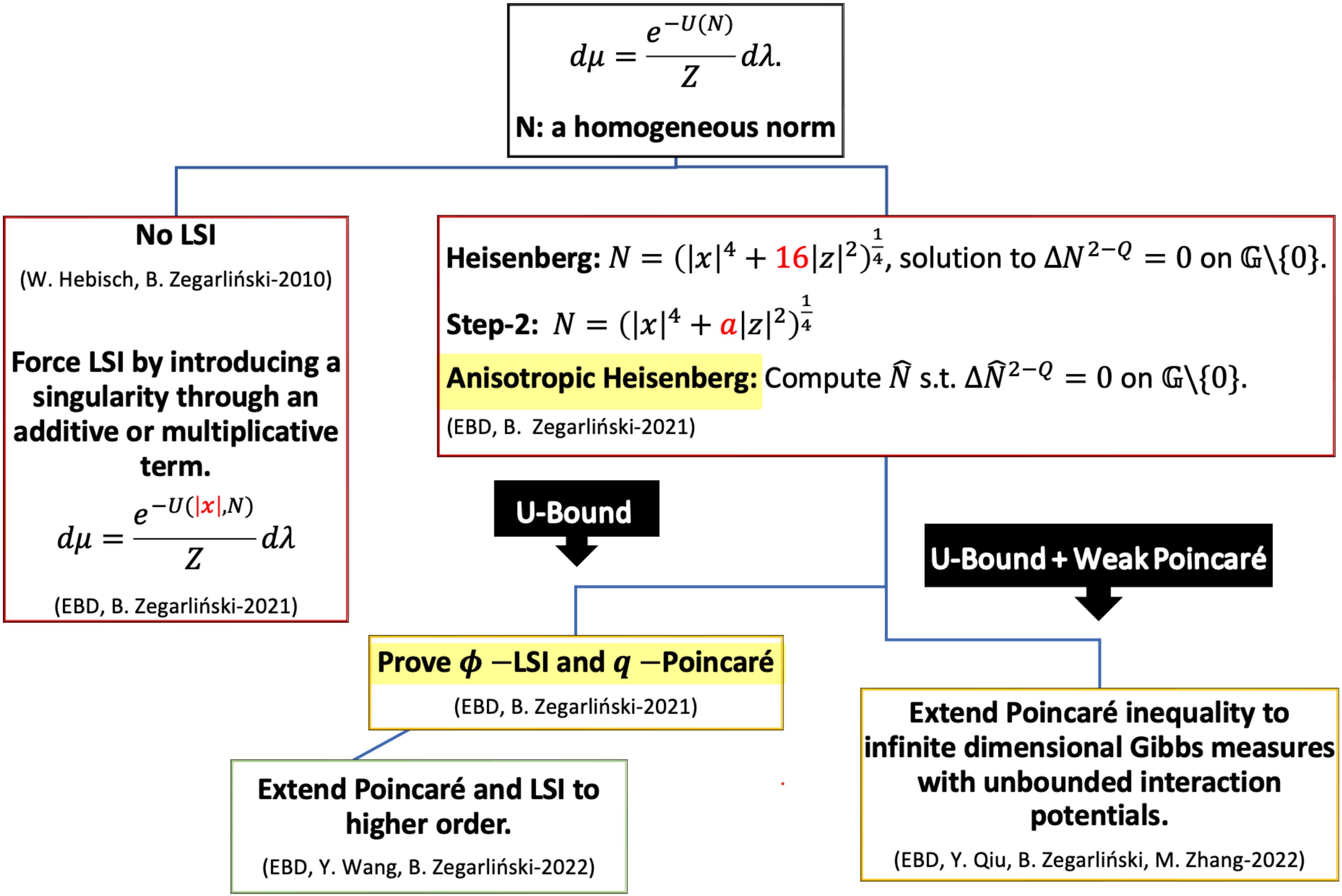
Given the following Logarithmic-Sobolev inequality

$$\int |f|^2 \left|log\left(\frac{|f|^2}{\mu|f|^2}\right)\right|^\beta d\mu \leq C\mu|\nabla f|^2, \tag{4}$$

for $\beta \in (0,1]$. *Then, for all* $m \in \mathbb{N}$,

$$\int |f|^2 \left|log\left(\frac{|f|^2}{\mu|f|^2}\right)\right|^{\beta m} d\mu \leq D \sum_{|\alpha|=0}^{m} \int |\nabla^\alpha f|^2 d\mu, \tag{5}$$

where $\nabla^\alpha f = (X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n} f)$ *such that* $|\alpha| = \sum_{i=1}^{n} \alpha_i$, *and* $C, D \in (0,\infty)$.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What is a polarizable Carnot group?

The Carnot group $\mathbb{G}$ is said to be polarizable if N, where N^{2-Q} is the fundamental solution to the sub-Laplacian, is $\infty-$harmonic in $\mathbb{G}\backslash\{0\}$, i.e. for $\nabla := (X_i)_{1\leq i\leq n}$,

$$\triangle_\infty N := \frac{1}{2} < \nabla\left(|\nabla N|^2\right), \nabla N >= 0 \qquad in\ \mathbb{G}\backslash\{0\}. \tag{6}$$

The concept of polarizable Carnot groups was first introduced by Z. Balogh and J. Tyson in 2002, where they used the $\infty-$harmonicity of N to create a procedure to construct polar coordinates. Moreover, they showed in that the fundamental solution of the $p-$sub-Laplacian can be expressed as the fundamental solution N^{2-Q} of the sub-Laplacian, proved capacity formulas, and produced sharp constants for the Moser-Trudinger inequality.

Higher-Dimensional Anisotropic Heisenberg Group

For the time being, there is no classification of polarizable Carnot groups and the only examples till now are Euclidean spaces and Heisenberg-type groups. Consider the generators of the Lie algebra:

$$\begin{cases} X = \frac{\partial}{\partial x} + 2ay \cdot \frac{\partial}{\partial t} \\ Y = \frac{\partial}{\partial y} - 2ax \cdot \frac{\partial}{\partial t} \\ Z = \frac{\partial}{\partial z} - 2w \cdot \frac{\partial}{\partial t} \\ W = \frac{\partial}{\partial w} - 2z \cdot \frac{\partial}{\partial t}, \end{cases}$$

Heisenberg: $a = 1$.

Anisotropic Heisenberg: $a = \frac{1}{2}$.

Heisenberg $\mathbb{R}^{n+m}$

- $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$
- $|\nabla \mathrm{N}|^2 = \frac{|\mathrm{x}|^2}{\mathrm{N}^2}$
- $|\Delta N| = (Q-1)|\nabla \mathrm{N}|^2 = (n+2m-1)\frac{|x|^2}{N^2}$
- $\frac{x}{|x|} \cdot \nabla \mathrm{N} = \frac{|\mathrm{x}|^3}{\mathrm{N}^3}$

Step-2 Carnot $\mathbb{R}^{n+m}$

- $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$
- 1) $A\frac{|\mathrm{x}|^2}{\mathrm{N}^2} \leq |\nabla \mathrm{N}|^2 \leq C\frac{|\mathrm{x}|^2}{\mathrm{N}^2}$
- 2) $|\Delta N| \leq B\frac{|x|^2}{N^2}$
- 3) $\frac{x}{|x|} \cdot \nabla \mathrm{N} = \frac{|\mathrm{x}|^3}{\mathrm{N}^3}$
- $a, A, B, C \in (0, \infty)$

Anisotropic Heisenberg $\mathbb{R}^{2n+1}$

- $N = \frac{(B^2+t^2)^{\frac{1}{4n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2}-\frac{1}{4n}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{2}}}$

  where

  $$A = \frac{x_1^2}{2} + \frac{x_{n+1}^2}{2} + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$$

  and

  $$B = \frac{x_1^2}{4} + \frac{x_{n+1}^2}{4} + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$$

- $\frac{|\mathrm{x}|^2}{2^{5+\frac{2}{n}}\mathrm{N}^2} \leq |\nabla \mathrm{N}|^2 \leq \frac{(2n+1)^2|\mathrm{x}|^2}{8n^2\mathrm{N}^2}$
- $|\Delta N| = (Q-1)|\nabla \mathrm{N}|^2 = (2n+1)\frac{|x|^2}{N^2}$
- $\frac{x}{|x|} \cdot \nabla \mathrm{N} \geq -\frac{|\mathrm{x}|^2}{4\mathrm{nN}}$. Problem: this term could be negative, and so we need the dimension $n > 5$.

[redacted]

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

No LSI

(W. Hebisch, B. Zegarliński-2010)

Force LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

(EBD, B. Zegarliński-2021)

Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

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(EBD, B. Zegarliński-2021)

U-Bound

Prove ϕ −LSI and q −Poincaré

(EBD, B. Zegarliński-2021)

Extend Poincaré and LSI to higher order.

(EBD, Y. Wang, B. Zegarliński-2022)

U-Bound + Weak Poincaré

Extend Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

(EBD, Y. Qiu, B. Zegarliński, M. Zhang-2022)

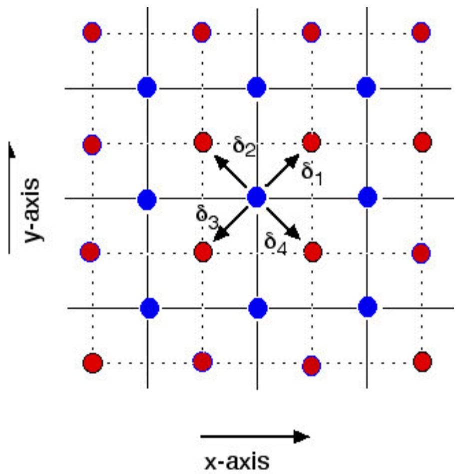
Gibbs measures

So far, the passage to infinite dimensions in the setting of Nilpotent Lie groups required the condition $|\nabla N| \geq c$ outside the unit ball which is not true for homogeneous norms introduced.

For Kaplan norm in Heisenberg group: $N = (|x|^2 + 16|z|^2)^{\frac{1}{4}}$, and
$$|\nabla N| = \frac{|x|}{N}.$$
Choose $|x| = 0$ and $|z|$ large.

Then, $|\nabla N|$ does not satisfy $|\nabla N| \geq c$ outside the unit ball $\{N(x,z) < 1\}$.

Gibbs measures: Setup



We have a Carnot group $\mathbb{G}$, and we give it a d-dimensional integer lattice structure: $\mathbb{G}^{\mathbb{Z}^d}$.For any compact $\Lambda \subset \mathbb{Z}$, denote the potential U_Λ^w by

$$U_\Lambda^w(x_\Lambda) := \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda, i\sim j} \beta V(x_i, x_j) + \sum_{i\in\Lambda, j\notin\Lambda, i\sim j} \beta V(x_i, w_j),$$

where $\phi \in C^1(\mathbb{G}, \mathbb{R})$ is the phase and $V \in C^1(\mathbb{G}\times\mathbb{G}, \mathbb{R})$ is the interaction with strength $\beta \geq 0$. Let $\mathbb{E}_\Lambda^w := \frac{1}{Z_\Lambda^w} e^{-U_\Lambda^w} dx_\Lambda$ be the local Gibbs measure and ν be the associated global measure satisfying $\nu\mathbb{E}_\Lambda^w = \nu$ for all compact $\Lambda \subset \mathbb{Z}$. Denote $|\nabla_\Lambda f|^2 = \sum_{i\in\Lambda} |\nabla_i f|^2$ and $|\nabla f|^2 = |\nabla_{\mathbb{Z}} f|^2$.

Gibbs measures: Hypothesis

Consider the following two hypotheses:

(H1) For any $i \in \mathbb{Z}$, the (U-bound $\rightarrow$) weak U-Bound

$$\sum_{j:j\sim i} \nu(f^q |\nabla_j V(x_i, x_j)|^q) \leq A \left(\nu|\nabla_i f|^q + \nu|f|^q + \sum_{m=0}^{\infty} C_\beta^m \nu|\nabla_{\{i-1-m,i+1+m\}} f|^q \right)$$

holds for some constants $A > 0$ and $C_\beta \in [0,1)$ such that $A\beta$ and C_β vanish as $\beta \rightarrow 0$.

(H2) For any $i \in \mathbb{Z}$, the weak q-Poincaré inequality

$$\nu \mathbb{E}_i^w |f - \mathbb{E}_i^w f|^q \leq B_{SG} \left(\nu|\nabla_i f|^q + \sum_{m=0}^{\infty} C_\beta^m \nu|\nabla_{\{i-1-m,i+1+m\}} f|^q \right)$$

holds for some constants $B_{SG} > 0$ and the same $C_\beta \in [0,1)$ such that $A\beta$ and $B_{SG}\beta \rightarrow 0$ as $\beta \rightarrow 0$.

Gibbs measures: Theorem

Theorem (EBD, Y. Qiu, B. Zegarliński, and M. Zhang, 2022)

*Suppose **(H1)** and **(H2)** are satisfied, then there exists $\tilde{\beta} > 0$ such that for all $\beta \in [0, \tilde{\beta})$ the global Poincaré inequality*

$$\nu|f - \nu f|^q \leq c_{SG} \nu |\nabla f|^q$$

holds for some constant $c_{SG} > 0$.

[redacted]

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

No LSI

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Force LSI by introducing a singularity through an additive or multiplicative term.

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Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

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Proof idea of U-Bound Inequality

Theorem (EBD and B. Zegarliński, 2021)

Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ *with* $a \in (0, \infty)$ *be as above and* $g : [0, \infty) \to [0, \infty)$ *be a differentiable increasing function such that* $g''(N) \leq g'(N)^3 N^3$ *on* $\{N \geq 1\}$*. Let* $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$ *be a probability measure, and* Z *the normalization constant. Then, for all locally Lipschitz functions* f,

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \tag{7}$$

holds outside the unit ball $\{N < 1\}$ *with* C *and* D *positive constants and* $q \geq 2$.

Here, $\mathcal{U} = \frac{g'(N)}{N^2}$. First Question: How to choose $\mathcal{U}$?

We need a technical lemma first:

Heisenberg $\mathbb{R}^{n+m}$

- $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$
- $|\nabla N|^2 = \frac{|x|^2}{N^2}$
- $|\Delta N| = (Q-1)|\nabla N|^2 = (n+2m-1)\frac{|x|^2}{N^2}$
- $\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}$

Step-2 Carnot $\mathbb{R}^{n+m}$

- $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$
- 1) $A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\,\frac{|x|^2}{N^2}$
- 2) $|\Delta N| \leq B\frac{|x|^2}{N^2}$
- 3) $\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}$
- $a, A, B, C \in (0, \infty)$

Anisotropic Heisenberg $\mathbb{R}^{2n+1}$

- $N = \frac{(B^2+t^2)^{\frac{1}{4n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2}-\frac{1}{4n}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{2}}}$

  where

  $$A = \frac{x_1^2}{2} + \frac{x_{n+1}^2}{2} + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$$

  and

  $$B = \frac{x_1^2}{4} + \frac{x_{n+1}^2}{4} + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$$

- $\frac{|x|^2}{2^{5+\frac{2}{n}}N^2} \leq |\nabla N|^2 \leq \frac{(2n+1)^2|x|^2}{8n^2N^2}$
- $|\Delta N| = (Q-1)|\nabla N|^2 = (2n+1)\frac{|x|^2}{N^2}$
- $\frac{x}{|x|} \cdot \nabla N \geq -\frac{|x|^2}{4nN}$. Problem: this term could be negative, and so we need the dimension $n > 5$.

How to choose $\mathcal{U}$?

For $q=2$,using integration by parts:

$$\int(\nabla N)\cdot(\nabla f)\,e^{-g(N)}d\lambda=-\int\nabla\left(\nabla N e^{-g(N)}\right)fd\lambda$$

$$=-\int\Delta N f e^{-g(N)}d\lambda+\int|\nabla N|^2 fg^{'}(N)\,e^{-g(N)}d\lambda.$$

Netx, using 1) and 2),

$$A\int\frac{|x|^2}{N^2}fg^{'}(N)e^{-g(N)}d\lambda-B\int\frac{|x|^2}{N^3}fe^{-g(N)}d\lambda\leq\int(\nabla N)\cdot(\nabla f)\,e^{-g(N)}d\lambda.$$

First candidate for $\mathcal{U}=\frac{|x|^2}{N^2}g^{'}(N)$ .We need $\mathcal{U}\to\infty$ "in all directions" to apply Hebisch-Zegarlinski Theorem 1 (2009). Recall that $N=(|x|^2+a|z|^2)^{\frac{1}{4}}$. For $|x|=0$, we can have $|z|^2\to\infty$, but $\mathcal{U}$ =0. So, the problem is around the $z-$axis.

Idea: Replace f by $\frac{f^2}{|x|^2}$:

Now we have the good candidate $\mathcal{U} = \frac{g'(N)}{N^2}$:

$$\int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda$$

$$= \int (\nabla N) \cdot \left[2f \frac{\nabla f}{|x|^2} - \frac{2f^2 \nabla |x|}{|x|^3} \right] e^{-g(N)} d\lambda$$

$$= \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int f^2 \frac{\nabla N \cdot x}{|x|^4} e^{-g(N)} d\lambda$$

See Technical Lemma

Heisenberg: $\nabla N \cdot x \geq 0$

$$\leq 2 \int \frac{f}{|x|^2} |\nabla N| |\nabla f| e^{-g(N)} d\lambda$$

$$\leq 2\sqrt{C} \int \frac{|f|}{N|x|} |\nabla f| e^{-g(N)} d\lambda.$$

Where the last two inequalities use the calculation of $\nabla N \cdot x$, from 3) and the upper bound on $|\nabla N|$ from 1).

Trial 1: Use Hardy's Inequality

Applying Cauchy's inequality with ϵ : $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ with $a = \frac{|f|}{N|x|} e^{-\frac{g(N)}{2}}$ and $b = \sqrt{C}|\nabla f| e^{-\frac{g(N)}{2}}$,

Bad term Good term

$$\int f^2 \left(A\frac{g'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda.$$

For $f \in C_0^\infty(\mathbb{R}^{n+m})$, we want to use Hardy's inequality:

$$\int \frac{f^2}{|x|^2} d\lambda \leq \frac{4}{(n-2)^2} \int |\nabla f|^2 d\lambda.$$

The bad term becomes:

$$\epsilon \int \frac{(\frac{fe^{\frac{-g(N)}{2}}}{N})^2}{|x|^2} d\lambda \leq \frac{4\epsilon}{(n-2)^2} \int |\nabla \frac{fe^{\frac{-g(N)}{2}}}{N}|^2 d\lambda$$

$$= \frac{\epsilon}{(n-2)^2} \int \frac{f^2 g'(N)^2}{N^2} |\nabla N|^2 d\mu + other \ \ terms.$$

This last term cannot be absorbed in the left-hand side of our $U-$Bound inequality, and Trial 1 fails.

Trial 2: Use Hardy's Inequality with $(B_R \times B_1)$

Using Hardy's inequality on the bad term:

$$\epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda = \epsilon \int_{B_R \times B_1} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \underbrace{\epsilon \int_{(B_R \times B_1)^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda}_{:= I_2}$$

$$\leq \epsilon \int_{B_R \times B_1} \frac{|f|^2}{|x|^2} d\lambda + I_2$$

$$\leq \frac{4\epsilon}{(n-2)^2} \int_{B_R \times B_1} |\nabla f|^2 d\lambda + I_2$$

$$\leq \frac{4\epsilon C}{(n-2)^2} \int_{B_R \times B_1} |\nabla f|^2 d\mu + I_2.$$

Where the last line is true since we can bound $e^{-g(N)}$ from below on $B_R \times B_1$.

Regarding the complement:

$$(B_R \times B_1)^c = B_R^c \times B_1^c \cup B_R^c \times B_1 \cup B_R \times B_1^c.$$

On $B_R^c \times B_1^c$ and $B_R^c \times B_1$, we have $\dfrac{1}{|x|^2} \leq \dfrac{1}{R^2}$,so we avoid the singularity. However, on $B_R \times B_1^c$,we face the same problem as Trial 1.

Trial 3: Introduce $F = \{(x,z) \in \mathbb{R}^{n+m} : |x|\sqrt{g'(N)} < 1\}$

$$\int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda$$

$$\leq \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda$$

$$= \epsilon \int_F \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \epsilon \int_{F^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + GT$$

$$\leq \epsilon \int_F \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda + \epsilon \int \frac{g'(N)|f|^2}{N^2} e^{-g(N)} d\lambda + GT.$$

Final trial: Hardy's inequality on $F_r = \{(x,z) \in \mathbb{R}^{n+m} : |x|\sqrt{g'(N)} < r\}$, where $1 \le r \le 2$.

$$\epsilon \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda = \frac{\epsilon}{n-2} \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2} \nabla \cdot \left(\frac{x}{|x|^2}\right) d\lambda$$

$$= -\frac{2\epsilon}{n-2} \int_{F_r} \frac{fe^{\frac{-g(N)}{2}}}{N} \nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right) \cdot \frac{x}{|x|^2} d\lambda + boundary\ term$$

$$\le \frac{\epsilon}{2} \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda + \frac{2\epsilon}{(n-2)^2} \int_{F_r} \left|\nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + boundary\ term.$$

Where we have used Integration by parts in the first line, Cauchy's inequality in the last line, and $boundary\ term = \dfrac{\epsilon}{n-2} \displaystyle\int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} \frac{x_j < X_j I, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right) >}{\left|\nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\right|} dH^{n+m-1}$. So,

$$\epsilon \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda \le \frac{4\epsilon}{(n-2)^2} \int_{F_r} \left|\nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + boundary\ term.$$

Using the fact that $F \subset F_r \subset F_2$,

$$\epsilon \int_{F} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda \le \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left|\nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + boundary\ term.$$

Recover the full measure using the Coarea formula

$$\epsilon \int_1^2 \int_F \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda dr \leq \frac{4\epsilon}{(n-2)^2} \int_1^2 \int_{F_2} \left| \nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda dr$$

$$+ \frac{2\epsilon}{n-2} \int_1^2 \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} \frac{x_j < X_j I, \nabla_{euc} \left(|x| \sqrt{g'(N)} \right) >}{\left| \nabla_{euc} \left(|x| \sqrt{g'(N)} \right) \right|} dH^{n+m-1} dr$$

To recover the full measure in the boundary term, we use the Coarea formula:

$$\epsilon \int_F \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda \leq \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left| \nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda$$

$$+ \frac{2\epsilon}{n-2} \int_{\{1 < |x|\sqrt{g'(N)} < 2\}} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} x_j < X_j I, \nabla_{euc} \left(|x| \sqrt{g'(N)} \right) > d\lambda.$$

The remainder of the proof is to use the condition of the theorem, the technical lemma, the domain of integrations, and the given fields X_j,to find a suitable ϵ, which turns out to be satisfying $\left(\frac{10\epsilon}{n-2} + \epsilon \right) < A$. (Recall: 1) $A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\frac{|x|^2}{N^2}$.)

Thanks for your attention!