Stability for the spectral gap in positive curvature.

Max Fathi
Joint work with Ivan Gentil and Jordan Serres

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Classical topic in geometry : optimizing a geometric quantity subject to a constraint.

Eg. : Isoperimetric problem. among all shapes with fixed volume the sphere minimizes the perimeter.

Today : optimization for manifolds satisfying a curvature constraint.

A first framework

We consider a smooth N-dimensional Riemannian manifold (M, g) whose Ricci curvature tensor satisfies

$$\mathrm{Ric} \geq (N-1)g.$$

The constant is chosen so that the sphere with unit radius satisfies this bound.

One way of defining the Ricci curvature tensor is via the Bochner formula :

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle = ||\nabla^2 f||_{HS}^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

From the arithmetic-geometric inquality, this implies

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle \geq \frac{1}{N}(\Delta f)^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

Lichnerowicz ('58) : the first positive eigenvalue of $-\Delta$ is larger than that of the sphere of radius one, which is N.

Bonnet-Myers Theorem : the diameter is maximized by the sphere.

Obata ('62) : both bounds are *rigid* : among all smooth N-manifolds with $\mathrm{Ric} \geq N-1$, the sphere is the only equality case.

Once a rigidity result holds, the next question is stability : if a space almost optimizes the desired bound, is it close to a true optimizer ?

Cheng ('75), Croke ('82) : the diameter is almost maximal iff the spectral gap is almost minimal.

Gromov ('81) : the class of spaces satisfying the curvature bound is precompact in the class of metric spaces endowed with measures, with respect to the measured Gromov-Hausdorff convergence.

A weak form of stability would be to show that a sequence of spaces that asymptotically saturates the bound converges to a sphere.

The answer is no (Anderson '90) : such a sequence may converge to a spherical suspension. To get full convergence in dimension n, we need convergence of λ_N to N (Petersen '99, Aubry '05).

However, if λ_1 is close to n, the manifold almost contains a piece that is close to a circle. More generaly, if λ_k is close to N for $k \leq N$, then the manifold contains a piece close to a k-sphere (Bertrand '05).

Defining curvature on metric-measure spaces, following Ambrosio, Gigli and Savaré, Erbar, Kuwada and Sturm

Given a metric-measure space (M, d, μ), we can

- Define a slope $|\nabla f|(x) = \limsup \frac{|f(x)-f(y)|}{d(x,y)}$ for locally lipschitz functions.
- Define the Cheeger energy $\int |\nabla f|^2 d\mu$ for general functions by approximation. Infinitessimally Hilbertian space : assume the Cheeger energy is quadratic.
- Define $\int \langle \nabla f, \nabla g \rangle d\mu$ by polarization.
- Define a Laplacian by integration by parts.

Once the Laplacian is defined, we can define a curvature-dimension condition $\mathrm{RCD}(K, N)$ via the Bochner inequality

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2.$$

Not well-defined pointwise if the space is non-smooth, so defined in a weak sense, by integration against nice test functions.

In this framework, spherical suspensions also satisfy the curvature-dimension condition $\mathrm{RCD}(N-1, N)$, and saturate the spectral gap/diameter bound. Cheeger & Colding ('96), Ketterer ('15) : they are the only ones.
One common feature : the eigenfunction is the cosine of the distance to some point, and the pushforward of the volume measure by the eigenfunction always follows a symmetrized (scaled) $\mathrm{Beta}(N/2, N/2)$ distribution, that is a measure with density proportional to $(1-x^2)^{N/2-1}$.
Cavaletti, Mondino & Semola ('19) :

$$||f - \cos(d(x, x_0))||_2 \leq C(N)(\lambda_1 - N)^{1/(8N+4)}.$$

Moreover,

$$\pi - \mathrm{Diam}(M) \leq C(N)(\lambda_1 - N)^{1/N}.$$

Theorem

Consider an $RCD(N-1,N)$ *space* (M,d,μ)*, and assume that* $\lambda_1 \leq N+\epsilon$*. Then if* f *is an eigenfunction with eigenvalue* λ_1 *with* $||\nabla f||_2 = \sqrt{N/(N+1)}$*, we have*

$$W_1(\mu \circ f^{-1}, \mathrm{Beta}(N/2)) \leq C(N)\epsilon.$$

Here W_1 stands for the L^1 Wasserstein distance from optimal transport :

$$W_1(\rho_1,\rho_2) = \sup_{g\ 1-lip} \int g d\rho_1 - \int g d\rho_2.$$

$C(N)$ can be made explicit, and the order of magnitude in ϵ is sharp.

From the Bochner formula applied to the eigenfunction f, we have

$$\frac{1}{2}\Delta(|\nabla f|^2 + (1+\epsilon)f^2) \geq -C(N)\epsilon f^2.$$

Setting $h = |\nabla f|^2 + (1+\epsilon)f^2$, we have

$$||\Delta h||_1 = 2||(\Delta h)_-||_1 \leq C(N)\epsilon.$$

Since the kernel of Δ is the set of constant functions, we expect h to be almost constant.
For a spherical suspension, the eigenfunction would be $\cos(d(x, x_0))$ for a well-chosen point x_0, and h is indeed constant.

Theorem

If the space is $RCD(N-1,N)$*, then for any centered function* g*, we have*

$$||g||_1 \leq C(N)||\Delta g||_1.$$

L^1 version of a Gaussian inequality of Meyer in L^p for $p > 1$. Discrete versions studied by Pisier, Naor & Schechtman, Eskenazis & Ivanisvili. Fails for the Gauss space.

Semigroup proof, uses ultracontractivity.

From this lemma we deduce that $|\nabla f|^2 + f^2$ is L^1-close to its average, up to an error of order ϵ.

Moreover, by integration by parts, for any smooth test function $g : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\int fg(f)d\mu = -(\lambda_1)^{-1} \int (\Delta f)g(f)d\mu = \lambda_1^{-1} \int g'(f)|\nabla f|^2 d\mu.$$

Hence we have the approximate integration by parts formula

$$\int fg(f)d\mu \approx N^{-1} \int g'(f)(1 - f^2)d\mu.$$

For $\nu = \mu \circ f^{-1}$ we have

$$\int xg(x)d\nu \approx N^{-1} \int g'(x)(1 - x^2)d\nu.$$

If a smooth density ρ exactly satisfies the previous integration by parts formula, we would have

$$Nx\rho = 2x\rho - (1 - x^2)\rho'$$

that is $\rho(x)$ is proportional to $(1 - x^2)^{N/2-1}$. This is exactly a (symmetrized) Beta distribution.

Stein's method

So we are brought to the problem of comparing two probability measures that almost satisfy the same integration-by-parts formula. This is the typical situation Stein's method was designed to address. The gaussianity of eigenfunctions on manifolds via Stein's method was studied by E. Meckes ('08).

Broad principle : aim to bound the L^1 Wasserstein distance

$$W_1(\rho, \nu) := \sup_{g\ 1-lip} \int g d\rho - \int g d\nu.$$

Assume we have a linear operator L such that

$$\int (Lh) d\nu = 0 \quad \forall h$$

Typically, L is the generator of a Markov process with invariant measure μ.

If we rewrite an arbitrary 1-lipschitz function g as

$$g - \int g d\nu = Lh$$

and identify a family of functions $\mathcal{H}$ that contains at least one solution for any f, then trivially

$$W_1(\mu, \nu) \leq \sup_{h \in \mathcal{H}} \int Lh d\mu.$$

The whole problem is then reduced to deriving non-trivial information on solutions to reduce the size of $\mathcal{H}$.

Here $Lf = a(x)f' + b(x)f$, so we can analyze solutions by solving ODEs. In the general setting, must typically analyze PDE or finite-difference operators.

Theorem (Goldstein & Reinert '13, Döbler '15)

If ρ is supported in $[-1, 1]$ then

$$W_1(\rho, \mathrm{Beta}(N/2)) \leq \sup_{||h'||_\infty \leq 4N} \int (1 - x^2)h' - Nxhd\rho.$$

Need an extra cutoff procedure on f to apply this result, to remove values larger than 1 or smaller than -1. Extra error of order ϵ.

The argument can be abstractified independently of the application considered here, giving a variant of a result of E. Meckes :

Theorem

Let f be an eigenfunction of a reversible diffusion generator on a manifold with invariant probability measure μ, with eigenfunction λ, and normalized so that $\int |\nabla f|^2 d\mu = N/(N+1)$. Then

$$W_1(f, \mathrm{Beta}(N/2)) \leq C(N)(|||\nabla f|^2 + f^2 - 1||_1 + |\lambda - N|).$$

- In the $\mathrm{RCD}(1, \infty)$ setting, same strategy (with a problematic lemma) leads to a bound of order $\epsilon \log(1/\epsilon)$ to the Gaussian measure. Improves earlier results of De Philippis & Figalli, Courtade & F., Bertrand & F.
- Also makes sense for negative values of N. Model space is a generalized Cauchy distribution, and can get $O(\epsilon \log(1/\epsilon))$ bound under extra regularity and integrability assumptions on the eigenfunction when $N < -2$.

Thanks!