# Strengthened inequalities for the mean width and the $\ell$-norm

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# Disclaimer

**This is talk is based on joint work with Károly J. Böröczky (Budapest, Hungary) and Daniel Hug (Karlsruhe, Germany).**

- In many geometric inequalities and extremal problems, Euclidean balls and simplices are often the extremizers.
- One classical example is the isoperimetric (and Ball's reverse isoperimetric) inequality.
- Another example is the Urysohn inequality that Euclidean balls minimize the mean width of convex bodies of given volume.

- A prominent geometric extremal problem with simplices as extremizers is the following:
- For each convex body $K$, there exists a unique ellipsoid of maximal volume in $K$ (John ellipsoid), and a unique ellipsoid of minimal volume containing $K$ (Löwner ellipsoid).
- Ball (1991): simplices maximize the volume of $K$ given the volume of the John ellipsoid of $K$, and thus simplices determine the extremal "inner" volume ratio.
- Barthe (1998): simplices minimize the volume of $K$ given the volume of the Löwner ellipsoid of $K$, hence simplices determine the extremal "outer" volume ratio (see also Lutwak, Yang, Zhang).
- In all these cases, equality was characterized by Barthe (1998).

# The mean width

- For a convex body $K$ in $\mathbb{R}^n$, we write $h_K(u) = \max_{x \in K} \langle x, u \rangle$ for its support function on $S^{n-1}$.
- The mean width of $K$ is given by

$$W(K) = \frac{1}{n\kappa_n} \int_{S^{n-1}} (h_K(u) + h_K(-u))\, du,$$

where the integration over the unit sphere $S^{n-1}$ is with respect to the spherical Lebesgue measure.

# The $\ell$-norm

- For a convex body $K \subset \mathbb{R}^n$, $o \in \mathrm{int}\, K$, let gauge function of $K$ is

$$\|x\|_K = \min\{t \geq 0 : x \in tK\}, \qquad x \in \mathbb{R}^n.$$

- The $\ell$-norm of $K$ is

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(dx) = \mathbb{E}\|X\|_K,$$

where $\gamma_n$ is the standard Gaussian measure in $\mathbb{R}^n$ with density function $x \mapsto \sqrt{2\pi}^{-n} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^n$, and $X$ is a standard Gaussian random vector with distribution $\gamma_n$.

- The $\ell$-norm of $K$ can also be expressed as (Barthe (1998))

$$\ell(K) = \int_{\mathbb{R}^n} \mathbb{P}(\|X\|_K > t)\, dt = \int_0^\infty (1 - \gamma_n(tK))\, dt.$$

# The connection between the mean width and the $\ell$-norm

- The polar body of $K$ is
$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall y \in K\}.$$
- Then
$$\ell(K) = \tfrac{\ell(B^n)}{2} W(K^\circ)$$
with
$$\lim_{n \to \infty} \frac{\ell(B^n)}{\sqrt{n}} = 1.$$

- $\Delta_n$ regular simplex inscribed into $B^n$
- $\Delta_n^\circ$ regular simplex circumscribed around $B^n$

**Theorem (Barthe (1998), Schmuckenschläger (1999))**

*Let $K$ be a convex body in $\mathbb{R}^n$.*

(i) *If $B^n \supset K$ is the Löwner ellipsoid of $K$, then $\ell(K) \leq \ell(\Delta_n)$, and if $B^n \subset K$ is the John ellipsoid of $K$, then $W(K) \leq W(\Delta_n^\circ)$. Equality holds in either case if and only if $K$ is a regular simplex.*

(ii) *If $B^n \subset K$ is the John ellipsoid of $K$, then $\ell(K) \geq \ell(\Delta_n^\circ)$, and if $B^n \supset K$ is the Löwner ellipsoid of $K$, then $W(K) \geq W(\Delta_n)$. Equality holds in either case if and only if $K$ is a regular simplex.*

The two statements in (i) are equivalent to each other, and the same is true for (ii).

- A reverse form of the Urysohn inequality is still not known in general.
- Giannopoulos, Milman, Rudelson (2000) proved a reverse Urysohn inequality for zonoids.
- Hug, Schneider (2011) established reverse inequalities of other intrinsic and mixed volumes for zonoids.
- A related classical open problem is that among all simplices contained in the Euclidean unit ball, the inscribed regular simplex has the maximal mean width (comprehensive survey: Litvak (2018).

# The range of $W(K)$ and $\ell(K)$ in the above Theorem

If $K$ is a convex body in $\mathbb{R}^n$ whose Löwner ellipsoid is $B^n$, then the monotonicity of the mean width yields

$$W(\Delta_n) \leq W(K) \leq W(B^n) = 2,$$

where, according to Böröczky (1994)

$$W(\Delta_n) \sim 4\sqrt{\frac{2 \ln n}{n}} \quad \text{as } n \to \infty.$$

If $K$ is a convex body in $\mathbb{R}^n$ whose John ellipsoid is $B^n$, then

$$2 = W(B^n) \leq W(K) \leq W(\Delta_n^\circ)$$

with $W(\Delta_n^\circ) \sim 4\sqrt{2n \ln n}$.

# Isotropic measures

An important concept in the proof is the notion of an isotropic measure on the unit sphere. We call a Borel measure $\mu$ on the unit sphere $S^{n-1}$ isotropic if

$$\int_{S^{n-1}} u \otimes u \, \mu(du) = \mathrm{I}_n. \tag{1}$$

This condition is equivalent to

$$\langle x, x \rangle = \int_{S^{n-1}} \langle u, x \rangle^2 \, \mu(du) \quad \text{for } x \in \mathbb{R}^n.$$

In this case, $\mu(S^{n-1}) = n$.
We say that the isotropic measure $\mu$ on $S^{n-1}$ is centered if

$$\int_{S^{n-1}} u \, \mu(du) = o.$$

- If $\mu$ is a centered isotropic measure on $S^{n-1}$, then $|\operatorname{supp}\mu| \geq n+1$, with equality if and only if $\mu$ is concentrated on the vertices of a regular simplex and each vertex has measure $n/(n+1)$.
- Isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by Kannan, Lovász and Simonovits (1995) as well as in Bourgain's hyperplane conjecture (slicing problem); see, for instance, Barthe and Cordero-Erausquin (1993), Guedon and Milman (2011), Klartag (2009), Artstein-Avidan, Giannopoulos, Milman (2015) and Alonso-Gutiérrez, Bastero (2015).

- The emergence of isotropic measures on $S^{n-1}$ originates from Ball's (1989, 1991) crucial insight that John's characteristic condition for a convex body to have the unit ball as its John or Löwner ellipsoid can be used to give the Brascamp-Lieb inequality a convenient form which is ideally suited for many geometric applications.
- John's condition (equivalence by Ball (1992)) states that $B^n$ is the John ellipsoid of a convex body $K$ containing $B^n$ if and only if there exist distinct unit vectors $u_1, \ldots, u_k \in \partial K \cap S^{n-1}$ and positive weights $c_1, \ldots, c_k > 0$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n,$$

$$\sum_{i=1}^{k} c_i u_i = o.$$

- In particular, the measure $\mu$ on $S^{n-1}$ with support $\{u_1, \ldots, u_k\}$ and $\mu(\{u_i\}) = c_i$ for $i = 1, \ldots, k$ is isotropic and centered.
- In addition, $B^n$ is the Löwner ellipsoid of a convex body $K \subset B^n$ if and only if there exist $u_1, \ldots, u_k \in \partial K \cap S^{n-1}$ and $c_1, \ldots, c_k > 0$ satisfying the same two conditions.
- According to John (1948) (also Gruber, Schuster (2005), one may assume that $k \leq n(n+3)/2$.
- It follows from John's characterization that $B^n$ is the Löwner ellipsoid of a convex body $K \subset B^n$ if and only if $B^n$ is the John ellipsoid of $K^\circ$.

- If $\mu$ is a centered isotropic measure on $S^{n-1}$, then $o \in \operatorname{int} Z_\infty(\mu)$, where

$$Z_\infty(\mu) = \operatorname{conv} \operatorname{supp} \mu.$$

- For our purposes, the study of $Z_\infty(\mu)$ can be reduced to discrete measures using the following result of Böröczky, Hug (2017):

### Lemma (K.J. Böröczky, D. Hug (2017))

*For any centered isotropic measure $\mu$ on $S^{n-1}$, there exists a discrete centered isotropic measure $\mu_0$ on $S^{n-1}$ such that*

$$\operatorname{supp} \mu_0 \subset \operatorname{supp} \mu \quad \textit{and} \quad |\operatorname{supp} \mu_0| \leq 2n^2.$$

- It follows that the previous theorem is equivalent to the following statements proved by Li, Leng (2012).

## Theorem (Li, Leng (2012))

*If $\mu$ is a centered isotropic measure on $S^{n-1}$, then* $\ell(Z_\infty(\mu)) \leq \ell(\Delta_n)$, $W(Z_\infty(\mu)^\circ) \leq W(\Delta_n^\circ)$, $\ell(Z_\infty(\mu)^\circ) \geq \ell(\Delta_n^\circ)$ *and* $W(Z_\infty(\mu)) \geq W(\Delta_n)$, *with equality in either case if and only if* $|\operatorname{supp}\mu| = n+1$.

- Results similar to the above theorem are proved by Ma (2017) in the $L_p$ setting.

- Our main goal is to provide stronger stability versions of the Theorem of Barthe and Schmuckenshcläger, and that of Li and Leng.

We will use the following metrics:
The Hausdorff distance between compact subsets $X$ and $Y$ of $\mathbb{R}^n$:

$$\delta_H(X, Y) = \max\{\max_{y \in Y} d(y, X), \max_{x \in X} d(x, Y)\},$$

where $d(x, Y) = \min_{y \in Y} \|x - y\|$.
The Symmetric difference distance for convex bodies $K$ and $C$:

$$\delta_{\mathrm{vol}}(K, C) = V(K \backslash C) + V(C \backslash K).$$

- Let $O(n)$ be the orthogonal group of $\mathbb{R}^n$.

We first phrase our main theorems in terms of the $\ell$-norm:

**Theorem (K.J. Böröczky, F.F., D. Hug (2020))**

*Let $B^n$ be the Löwner ellipsoid of a convex body $K \subset B^n$ in $\mathbb{R}^n$, let $c = n^{26n}$ and let $\varepsilon \in (0,1)$. If $\ell(K) \geq (1-\varepsilon)\ell(\Delta_n)$, then there exists a $T \in O(n)$ such that*

(i) $\delta_{\mathrm{vol}}(K, T\Delta_n) \leq c\,\sqrt[4]{\varepsilon}$,

(ii*) $\delta_H(K, T\Delta_n) \leq c\,\sqrt[4]{\varepsilon}$.

**Theorem (K.J. Böröczky, F.F., D. Hug (2020))**

*Let $B^n$ be the John ellipsoid of a convex body $K \supset B^n$ in $\mathbb{R}^n$ and let $\varepsilon > 0$. If $\ell(K) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$, then there exists a $T \in O(n)$ such that*

(i) $\delta_{\mathrm{vol}}(K, T\Delta_n^\circ) \leq c\,\sqrt[4]{\varepsilon}$ *for* $c = n^{27n}$,

(ii) $\delta_H(K, T\Delta_n^\circ) \leq c\,\sqrt[4n]{\varepsilon}$ *for* $c = n^{27}$.

# The question of optimality

- For (i) and (ii) of the first theorem: add an $(n+2)$nd vertex $v_{n+2} \in S^{n-1}$ to the vertices of $\Delta_n$ with $\angle(v_{n+2}, v_1) = c_1\varepsilon$ for suitable $c_1 > 0$ depending on $n$ such that $v_1$ lies on the geodesic arc connecting $v_2$ and $v_{n+2}$. For $K = \mathrm{conv}\{v_1, \ldots, v_{n+2}\}$, $\ell(K) \geq (1-\varepsilon)\ell(\Delta_n)$ on the one hand, and $\delta_{\mathrm{vol}}(K, T\Delta_n) \geq c_2\varepsilon$ and $\delta_H(K, T\Delta_n) \geq c_2\varepsilon$ for suitable $c_2 > 0$ depending on $n$ and for any $T \in O(n)$ on the other hand.
- Using the polar of $K$ for (i) of the second theorem, possibly after decreasing $c_1$, we have $\ell(K^\circ) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$ while $\delta_{\mathrm{vol}}(K^\circ, T\Delta_n^\circ) \geq c_3\varepsilon$ for suitable $c_3 > 0$ depending on $n$ and for any $T \in O(n)$.
- For (ii) of the second theorem: Cut off $n+1$ regular simplices of edge length $c_4\sqrt[n]{\varepsilon}$ at the vertices of $\Delta_n^\circ$, for suitable $c_4 > 0$ depending on $n$, getting a polytope $\widetilde{K}$ with $\ell(\widetilde{K}) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$ and $\delta_H(\widetilde{K}, T\Delta_n^\circ) \geq c_5\sqrt[n]{\varepsilon}$ for suitable $c_5 > 0$ depending on $n$ and for any $T \in O(n)$.

In the case of the mean width, we have the following stability versions.

Corollary (K.J. Böröczky, F.F., D. Hug (2020))

*Let $K$ be convex body in $\mathbb{R}^n$.*

(i) *If $B^n$ is the John ellipsoid of $K \supset B^n$ and $W(K) \geq (1-\varepsilon)W(\Delta_n^\circ)$ for some $\varepsilon \in (0,1)$, then there exists a $T \in O(n)$ such that $\delta_H(K, T\Delta_n^\circ) \leq c\sqrt[4]{\varepsilon}$ for $c = n^{27n}$.*

(ii) *If $B^n$ is the Löwner ellipsoid of $K \subset B^n$ and $W(K) \leq (1+\varepsilon)W(\Delta_n)$ for some $\varepsilon > 0$, then there exists a $T \in O(n)$ such that $\delta_H(K, T\Delta_n) \leq c\sqrt[4n]{\varepsilon}$ for $c = n^{29}$.*

We also have a stronger form of the Theorem of Li and Leng:

**Theorem (K.J. Böröczky, F.F., D. Hug (2020))**

*Let $\mu$ be a centered isotropic measure on the unit sphere $S^{n-1}$, let $c = n^{28n}$, and let $\varepsilon \in (0,1)$. If one of the conditions*

(a*) $\ell(Z_\infty(\mu)) \geq (1-\varepsilon)\ell(\Delta_n)$ *or*

(b) $W(Z_\infty(\mu)^\circ) \geq (1-\varepsilon)W(\Delta_n^\circ)$ *or*

(c) $\ell(Z_\infty(\mu)^\circ) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$ *or*

(d) $W(Z_\infty(\mu)) \leq (1+\varepsilon)W(\Delta_n)$

*is satisfied, then there exists a regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ such that*

$$\delta_H(\operatorname{supp}\mu, \{w_1, \ldots, w_{n+1}\}) \leq c\,\varepsilon^{\frac{1}{4}}.$$

Let $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ form a John decomposition of the identity, and let and $f_1, \ldots, f_k$ be non-negative measurable functions on $\mathbb{R}$.

The rank one geometric Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i \rangle)^{c_i} \, dx \leq \prod_{i=1}^{k} \left( \int_{\mathbb{R}} f_i \right)^{c_i}.$$

The reverse rank one geometric Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^{k} c_i \theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i} \, dx \geq \prod_{i=1}^{k} \left( \int_{\mathbb{R}} f_i \right)^{c_i}.$$

In the reverse Brascamp–Lieb inequality, we always assume that $\theta_1, \ldots, \theta_k \in \mathbb{R}$.

- The rank one geometric BL inequality was identified by Ball (1989) as an essential case of the rank one BL inequality,
- The reverse form was proved by Barthe (1997,1998).
- According to Barthe (1998), if equality holds in either inequality and none of the functions $f_i$ is identically zero or a scaled version of a Gaussian, then there is an origin symmetric regular crosspolytope in $\mathbb{R}^n$ such that $u_1, \dots, u_k$ lie among its vertices.
- Conversely, equality holds in the Brascamp–Lieb inequality and the reverse Brascamp–Lieb inequality if either each $f_i$ is a scaled version of the same centered Gaussian, or if $k = n$ and $u_1, \dots, u_n$ form an orthonormal basis.

If $k \geq n$, $c_1, \dots, c_k > 0$ and $u_1, \dots, u_k \in S^{n-1}$ satisfy the John condition, then

(i) for any $t_1, \dots, t_k > 0$, we have

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i},$$

(ii) for $z = \sum_{i=1}^{k} c_i \theta_i u_i$ with $\theta_1, \dots, \theta_k \in \mathbb{R}$, we have

$$\|z\|^2 \leq \sum_{i=1}^{k} c_i \theta_i^2,$$

(iii) for $i = 1, \dots, k$, we have

$$c_i \leq 1,$$

(iv) and it holds that

$$c_1 + \cdots + c_k = n.$$

Inequality (i) is called the Ball-Barthe inequality by Lutwak, Yang, Zhang (2007) and Li, Leng (2012).

# Proof of the BL inequality: all $f_i = f$ log-concave

- $g(t) = \sqrt{2\pi}^{\,-1}\, e^{-t^2/2}$, $t \in \mathbb{R}$ standard Gaussian density
- $f$ log-concave probability density function on $\mathbb{R}$
- $T$ and $S$ transportation maps determined by

$$\int_{-\infty}^{x} f = \int_{-\infty}^{T(x)} g \quad \text{and} \quad \int_{-\infty}^{S(y)} f = \int_{-\infty}^{y} g.$$

- $f$ log-concave $\longrightarrow \exists$ open interval $I$ s.t. $f > 0$ on $I$ and zero on the complement of the closure of $I$, and $T : I \to \mathbb{R}$ and $S : \mathbb{R} \to I$ are inverses of each other.
- In addition, for $x \in I$ and $y \in \mathbb{R}$ we have

$$f(x) = g(T(x))\, T'(x), \qquad g(y) = f(S(y))\, S'(y).$$

For

$$\mathcal{C} = \{x \in \mathbb{R}^n : \langle u_i, x \rangle \in I \ \text{ for } i = 1, \ldots, k\},$$

we consider the transformation $\Theta : \mathcal{C} \to \mathbb{R}^n$ with

$$\Theta(x) = \sum_{i=1}^{k} c_i \, T(\langle u_i, x \rangle) \, u_i, \qquad x \in \mathcal{C},$$

which satisfies

$$d\Theta(x) = \sum_{i=1}^{k} c_i \, T'(\langle u_i, x \rangle) \, u_i \otimes u_i.$$

$d\Theta$ is positive definite and $\Theta : \mathcal{C} \to \mathbb{R}^n$ injective (Barthe (1997, 1998)).

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \prod_{i=1}^{k} f(\langle u_i, x\rangle)^{c_i}\, dx \\
&= \int_{\mathcal{C}} \prod_{i=1}^{k} f(\langle u_i, x\rangle)^{c_i}\, dx \\
&= \int_{\mathcal{C}} \left(\prod_{i=1}^{k} g(T(\langle u_i, x\rangle))^{c_i}\right) \left(\prod_{i=1}^{k} T'(\langle u_i, x\rangle)^{c_i}\right)\, dx \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{C}} \left(\prod_{i=1}^{k} e^{-c_i T(\langle u_i, x\rangle)^2/2}\right) \det\left(\sum_{i=1}^{k} c_i T'(\langle u_i, x\rangle)\, u_i \otimes u_i\right)\, dx \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathcal{C}} e^{-\|\Theta(x)\|^2/2} \det\left(d\Theta(x)\right)\, dx \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\|y\|^2/2}\, dy = 1.
\end{aligned}
$$

The BL inequality for non-negative log-concave functions follows by scaling

# A stability form of the Ball-Barthe inequality

**Theorem (K.J. Böröczky, D. Hug (2017))**

*If $k \geq n+1$, $t_1, \ldots, t_k > 0$, $c_1, \ldots, c_k > 0$ and $u_1, \ldots, u_k \in S^{n-1}$ satisfy the John condition, and there exist $\beta > 0$ and $n+1$ indices $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, k\}$ such that*

$$c_{i_1} \cdots c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \geq \beta,$$
$$c_{i_2} \cdots c_{i_{n+1}} \det[u_{i_2}, \ldots, u_{i_{n+1}}]^2 \geq \beta,$$

*then*

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \left(1 + \frac{\beta(t_{i_1} - t_{i_{n+1}})^2}{4(t_{i_1} + t_{i_{n+1}})^2}\right) \prod_{i=1}^{k} t_i^{c_i}.$$

Let $g$ be the standard Gaussian density $g(t) = \sqrt{2\pi}^{-1} e^{-t^2/2}$, $t \in \mathbb{R}$, and for $s \in \mathbb{R}$, let $g_s$ be the truncated Gaussian density

$$g_s(x) = \begin{cases} \left( \int_0^\infty g(t-s)\, dt \right)^{-1} g(x-s), & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

- Fix $e \in S^n \subset \mathbb{R}^{n+1}$, and identify $e^\perp \subset \mathbb{R}^{n+1}$ with $\mathbb{R}^n$.
- For $k \geq n+1$, let $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ be such that
$$\begin{aligned} \textstyle\sum_{i=1}^k c_i u_i \otimes u_i &= \mathrm{I}_n, \\ \textstyle\sum_{i=1}^k c_i u_i &= o. \end{aligned}$$
- For each $u_i$, we consider
$$\begin{aligned} \tilde{u}_i &= \tfrac{\sqrt{n}}{\sqrt{n+1}}\, u_i + \tfrac{1}{\sqrt{n+1}}\, e \in S^n, \\ \tilde{c}_i &= \tfrac{n+1}{n}\, c_i, \end{aligned}$$
- and hence
$$\sum_{i=1}^k \tilde{c}_i\, \tilde{u}_i \otimes \tilde{u}_i = \mathrm{I}_{n+1}.$$

# Stability versions of the BL and rBL inequalities for a special class of functions

Theorem (K.J. Böröczky, F.F., D. Hug (2020))

*With the above notation, let* $k \leq 2n^2$, *let* $s \in [0, 0.15]$ *and let* $\varepsilon \in (0, n^{-56n})$. *If*

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} g_s(\langle x, \tilde{u}_i \rangle)^{\tilde{c}_i} \, dx \geq 1 - \varepsilon, \text{ or}$$

$$\int_{\mathbb{R}^{n+1}}^{*} \sup_{x = \sum_{i=1}^{k} \tilde{c}_i \theta_i \tilde{u}_i} \prod_{i=1}^{k} g_s(\theta_i)^{\tilde{c}_i} \, dx \leq 1 + \varepsilon,$$

*then there exists a regular simplex with vertices* $w_1, \ldots, w_{n+1} \in S^{n-1}$ *and* $i_1 < \ldots < i_{n+1}$ *such that* $\angle(u_{i_j}, w_j) < n^{14n} \varepsilon^{1/4}$ *for* $j = 1, \ldots, n+1$.

We will actually use the Brascamp-Lieb inequality and its reverse for the function

$$\tilde{g}_s(t) = \mathbf{1}\{t \geq 0\} \exp\left(-\frac{(t-s)^2}{2}\right), \quad s \in \mathbb{R}.$$

**Theorem (K.J. Böröczky, F.F., D. Hug (2020))**

*Let $k \leq 2n^2$, let $s \in [0, 0.15]$ and let $\varrho \in (0,1)$. If for any regular simplex with vertices $w_1, \ldots, w_{n+1} \in S^{n-1}$ and any subset $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, n+1\}$ such that $\angle(u_{i_j}, w_j) \geq \varrho$, then*

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} \tilde{g}_s(\langle x, \tilde{u}_i \rangle)^{\tilde{c}_i} \, dx \leq (1 - n^{-56n} \varrho^4) \left( \int_{\mathbb{R}} \tilde{g}_s \right)^{n+1},$$

$$\int^{*}_{\mathbb{R}^{n+1}} \sup_{x = \sum_{i=1}^{k} \tilde{c}_i \theta_i \tilde{u}_i} \prod_{i=1}^{k} \tilde{g}_s(\theta_i)^{\tilde{c}_i} \, dx \geq (1 + n^{-56n} \varrho^4) \left( \int_{\mathbb{R}} \tilde{g}_s \right)^{n+1}.$$

# An almost regular simplex

**Theorem (K.J. Böröczky, F.F., D. Hug (2020))**

*Let* $n+1 \leq k \leq 2n^2$, $u_1, \ldots, u_k \in S^{n-1}$ *and* $c_1, \ldots, c_k > 0$ *be such that*

$$\begin{aligned} \textstyle\sum_{i=1}^k c_i u_i \otimes u_i &= \mathrm{I}_n, \\ \textstyle\sum_{i=1}^k c_i u_i &= o, \end{aligned}$$

*and* $\ell(C) \geq (1-\varepsilon)\ell(\Delta_n)$ *holds for* $C = \mathrm{conv}\{u_1, \ldots, u_k\}$ *and* $\varepsilon \in (0, n^{-60n})$.

*Then for* $\eta = n^{15n}\varepsilon^{\frac{1}{4}} \in (0,1)$, *there exists a regular simplex with vertices* $w_1, \ldots, w_{n+1} \in S^{n-1}$ *and* $\{i_1, \ldots, i_{n+1}\} \subset \{1, \ldots, k\}$ *such that*

$$\angle(u_{i_j}, w_j) \leq \eta \quad \text{for } j = 1, \ldots, n+1.$$

## Sketch of the proof of (ii$^*$)

- Let $\ell(K) \geq (1-\varepsilon)\ell(\Delta_n)$, and assume that
$$0 < \varepsilon < n^{-100n}.$$
- $\mu_0$ is the centered discrete isotropic measure with $\operatorname{supp} \mu_0 = \{u_1, \ldots, u_k\}$ and $\mu_0(\{u_i\}) = c_i$ for $i = 1, \ldots, k$, and
$$C := Z_\infty(\mu_0) = \operatorname{conv}\{u_1, \ldots, u_k\}.$$
- Since $\ell(C) \geq \ell(K) \geq (1-\varepsilon)\ell(\Delta_n)$ and $0 < \varepsilon < n^{-60n}$, it follows that we may assume that the vertices $w_1, \ldots, w_{n+1}$ of $\Delta_n$ satisfy
$$\angle(u_i, w_i) \leq \eta \quad \text{for } \eta = n^{15n}\varepsilon^{\frac{1}{4}} \qquad \text{and} \qquad i = 1, \ldots, n+1.$$
- For the simplex
$$S_0 = \operatorname{conv}\{u_1, \ldots, u_{n+1}\} \subset K,$$
it follows from an earlier result of Böröczky and Hug (2017) that
$$\widetilde{\Delta}_n := (1 + 2n\eta)^{-1}\Delta_n \subset S_0 \subset K.$$

- Let $\xi > 0$ be minimal such that

$$K \subset (1+\xi)\widetilde{\Delta}_n = (1+\xi)(1+2n\eta)^{-1}\Delta_n.$$

- Then one can show (we omit the details here) that

$$(1-\varepsilon)\ell(\Delta_n) \le \ell(K) \le (1 - n^{-(4n+5)}\xi + 2n\eta)\ell(\Delta_n),$$

- and hence $\eta = n^{15n}\varepsilon^{\frac{1}{4}}$ implies

$$\xi \le n^{4n+5}(2n\eta + \varepsilon) < n^{23n}\varepsilon^{\frac{1}{4}}.$$

- It follows that

$$(1-2n\eta)\Delta_n \subset K \subset (1+\xi)\Delta_n.$$

- Since $\Delta_n \subset B^n$, $\eta = n^{15n}\varepsilon^{\frac{1}{4}} < n^{23n}\varepsilon^{\frac{1}{4}} =: \tilde{\xi}$ and $\xi < \tilde{\xi}$, we conclude for the Hausdorff distance that $\delta_H(K, \Delta_n) < n^{23n}\varepsilon^{\frac{1}{4}}$.
- If $\varepsilon \ge n^{-100n}$, the statement trivially holds as $\delta_H(M, \Delta_n) < 1$ for any convex body $M \subset B^n$ by the choice of $c = n^{26n}$.

$\square$

**Thank you for your attention.**