Volume product, polytopes and finite dimensional Lipschitz-free spaces.

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Mainly based on
Matthew Alexander, M. F. and Artem Zvavitch. Polytopes of Maximal Volume Product. Discrete and Computational Geometry 62 (3) (2019)
Matthew Alexander, M. F., Luis C. García-Lirola and Artem Zvavitch. Geometry and volume product of finite dimensional Lipschitz-free spaces, submitted.

Online Asymptotic Geometric Analysis seminar
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Blaschke-Santaló inequality

The polar and the volume product of a symmetric convex body $K \subset \mathbb{R}^n$ are $K^\circ = \{y \in \mathbb{R}^n; \langle y, x\rangle \leq 1, \forall\, x \in K\}$ and $\mathcal{P}(K) = |K||K^\circ|$. The volume product is invariant with respect to inversible linear transform.

Blaschke–Santaló inequality

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with equality if and only if K is an ellipsoid.

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- Blaschke (1923) for $n \leq 3$, Santaló (1949) for $n > 3$.
- Saint-Raymond (1981), Petty (1985) for the equality case.
- Proofs using Steiner symmetrization: Meyer–Pajor (1990).
- Functional forms: Ball (1986), Artstein-Avidan–Klartag–Milman (2004), Fradelizi–Meyer (2007), Lehec (2009).
- Stability Results: Böröczky (2010), Barthe–Böröczky–Fradelizi (2014).
- Harmonic Analysis based proof: Bianchi–Kelly (2015).
- Random polytope forms and more general measures than volume: Cordero-Erausquin–Fradelizi–Paouris–Pivovarov (2015).
- Maximum for polytopes with fixed number of vertices: Meyer–Reisner (2011) for $n = 2$ and Alexander–Fradelizi–Zvavitch (2019) for $n \geq 3$.

Mahler's conjecture and Hanner polytopes

Mahler's conjecture, symmetric case

$$\mathcal{P}(K) \geq \mathcal{P}([-1,1]^n) = \frac{4^n}{n!},$$

with equality if and only if K is a Hanner polytope.

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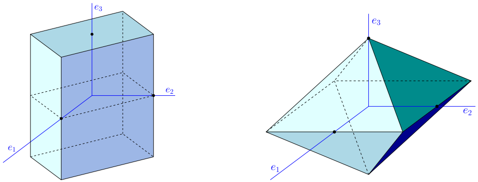
For $K \subset \mathbb{R}^{n_1}$, $L \subset \mathbb{R}^{n_2}$ with $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, we construct in $\mathbb{R}^n$:

- $K \oplus_\infty L = K + L$ the ℓ_∞-sum: $\|(x_1, x_2)\|_{K \oplus_\infty L} = \max\{\|x_1\|_K, \|x_2\|_L\}$
- $K \oplus_1 L =$ conv$(K \cup L)$ their ℓ_1-sum: $\|(x_1, x_2)\|_{K \oplus_1 L} = \|x_1\|_K + \|x_2\|_L$



$(I \oplus_1 I) \oplus_\infty I$ $\qquad$ $(I \oplus_\infty I) \oplus_1 I$

A **Hanner polytope** is the iterated ℓ_1 or ℓ_∞ sum of segments.
If $K \subset \mathbb{R}^n$ is a Hanner polytope, then $\mathcal{P}(K) = \frac{4^n}{n!}$.

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- Unconditional bodies (Saint-Raymond 1981), equality case (Meyer 1986), (Reisner 1987).
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- Polytopes with few vertices: (Lopez–Reisner 1998), (Meyer–Reisner 2006).
- Functional forms: (Klartag–Milman 2005), (F.–Meyer 2008-10), (Lehec 2009), (Gordon–F.–Meyer–Reisner 2010).
- A body with a point of positive curvature is not a minimizer: (Stancu, 2009), (Reisner–Schütt–Werner 2010), (Gordon–Meyer 2011).
- Close to Hanner polytopes/Unconditional bodies: (Nazarov–Petrov–Ryabogin–Zvavitch 2010), (Kim 2013), (Kim–Zvavitch 2013).
- Convex bodies with 'many' symmetries: (Barthe–Fradelizi 2010).
- It follows from Viterbo's conjecture in symplectic geometry, (Artstein-Avidan–Karasev–Ostrover 2014).
- Hyperplane sections of ℓ_p-balls and Hanner polytopes, (Karasev 2019).

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Shadow systems

- Definition (Shephard '64): $(K_t)_t$ is a shadow system if $K_t = P_t(C)$ is the projection on $\mathbb{R}^n$ parallel to $e_{n+1} - t\theta$ of a closed convex set C in $\mathbb{R}^{n+1}$, where $\theta \in S^{n-1}$ is fixed.

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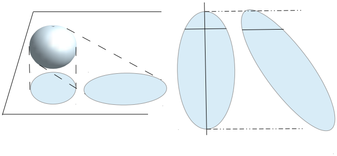

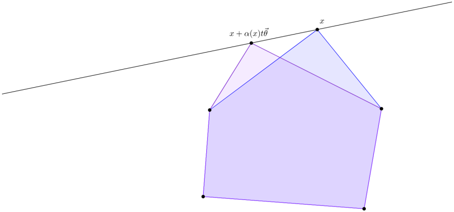

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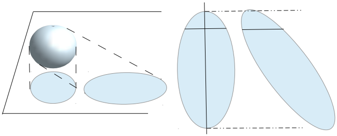

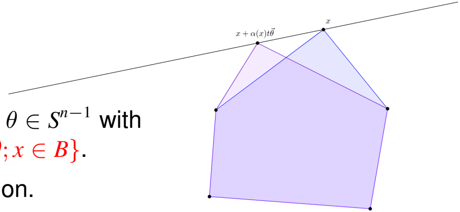

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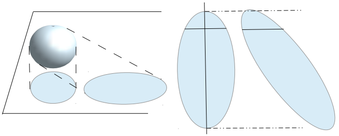

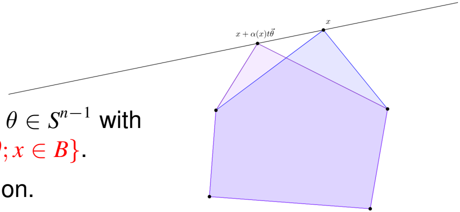

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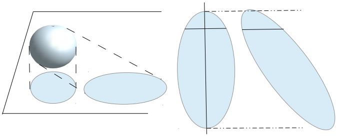

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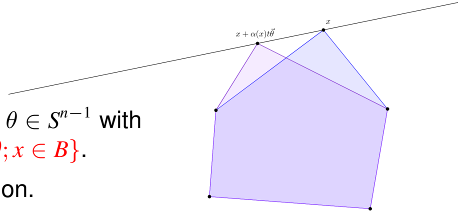

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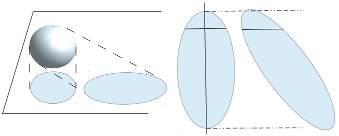

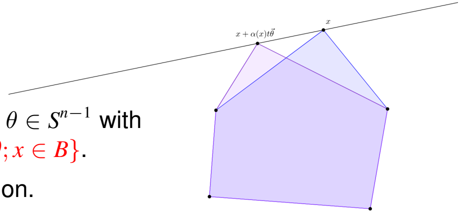

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- Campi-Gronchi '06: If K_t are symmetric, then $t \mapsto |(K_t)^\circ|^{-1}$ is convex.
- Meyer-Reisner '06: In general $t \mapsto \min_x |(K_t - x)^\circ|^{-1}$ is convex. Moreover if $t \mapsto |K_t|$ is affine then $t \mapsto \mathcal{P}(K_t)$ is quasi-concave and if it is constant then $K_t = A_t K_0$ where A_t affine.
- Cordero-Erausquin-F.-Paouris-Pivovarov '15: Generalization of Campi-Gronchi to more measures.

Shadow systems of polytopes

Let $n \geq 2$ and $m \geq n+1$. Let K be a polytope with at most m vertices. Let $0 \leq k \leq n-1$ and F be a k-face of K. Let x in the relative interior of F and u in the normal cone. Let $K_t = \mathrm{conv}(K, x+tu)$, $t > 0$.

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3) $1 \leq k \leq n-1$: Alexander-F.-García-Lirola-Zvavitch (submitted): generalization of the case $k = n-1$ and application to the maximizer of the volume product among Lipschitz-free balls. See below.

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Finite metric spaces and graphs

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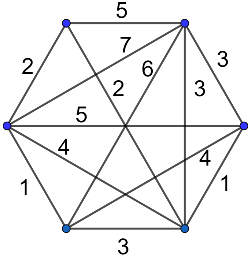

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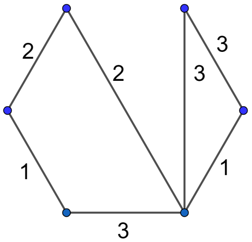

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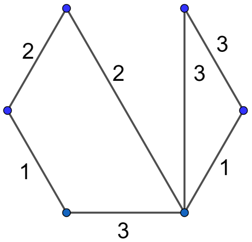

- (a_i, a_j) is an edge of the graph if and only if $d(a_i, a_j) < d(a_i, z) + d(z, a_j)$ for all $z \in M \setminus \{a_i, a_j\}$.

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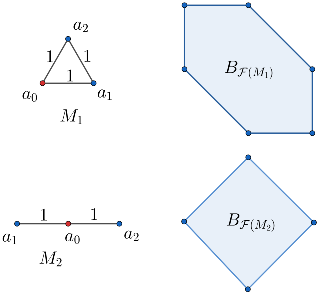
$$B_{\mathrm{Lip}_0(M)} := \{f; \|f\|_{\mathrm{Lip}_0} \leq 1\} = \left\{f; \langle f, \frac{e_i - e_j}{d(a_i, a_j)} \rangle \leq 1 \ \forall i \neq j\right\}$$

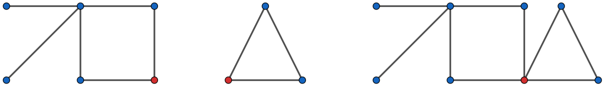
$$B_{\mathcal{F}(M)} := B^{\circ}_{\mathrm{Lip}_0(M)} = \mathrm{conv}\left\{\frac{e_i - e_j}{d(a_i, a_j)} : i \neq j\right\} \subset \mathbb{R}^n.$$

$B_{\mathcal{F}(M)}$ is a symmetric convex body of $\mathbb{R}^n$, a polytope having at most $n(n+1)$ vertices, called alcoved polyhedron, polytrope. $\mathcal{F}(M)$ is also called Arens-Eells, Wasserstein W_1, Kantorovich-Rubinstein, ...



When $\mathcal{F}(M)$ and $\mathcal{F}(M')$ are isometric?

- To (M,d) we associate a weighted graph $G = (V,E,d)$.
- The $\diamond$-sum of two metric spaces M and N is $M \diamond N$ obtained by identifying the distinguished points of M and N.



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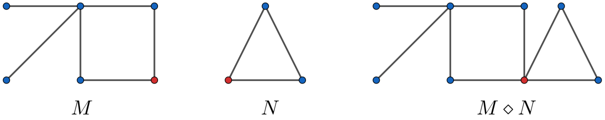

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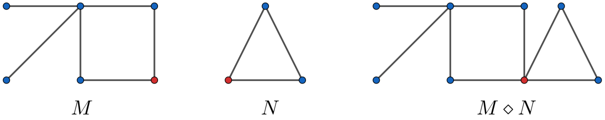

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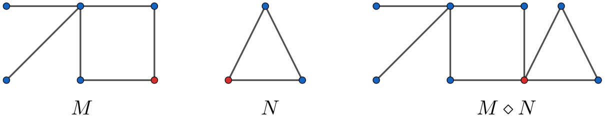

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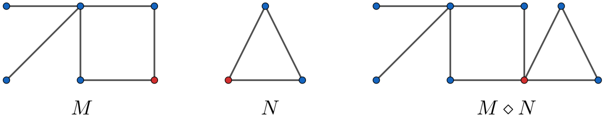

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Theorem (Alexander–F.–García-Lirola–Zvavitch, 2020+)

$\mathcal{F}(M)$ and $\mathcal{F}(M')$ are isometric if and only if $|M| = |M'|$ and there exists a cyclic bijection $\sigma : E \to E'$ such that $e \mapsto d(\sigma(e))/d(e)$ is constant on each 2-connected component of G.

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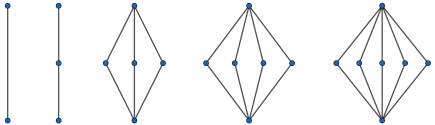
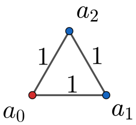


$$B_{K_{2,m}} = B_1^m \oplus_\infty [-1,1]$$

Maximizers of the volume product of a metric space

For $n = 2$, the maximizer is the complete graph K_3 with equal weights.



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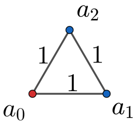

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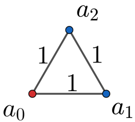

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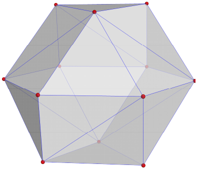
If $n \geq 3$ and M is K_{n+1} with equal weights, then $B_{\mathcal{F}(M)}$ is not simplicial!
Therefore it doesn't maximize the volume product.



Thank you!