Symmetry problems for variational functionals: from continuous to discrete

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OUTLINE

I. *Discrete isoperimetric-type inequalities*

[symmetry obtained via concavity]

II. *Discrete Faber-Krahn-type inequalities*

[symmetry obtained via first variation]

III. *Overdetermined boundary value problems on polygons*

[symmetry obtained via reflection]

Admissible shapes

- $\mathscr{K}^n := \Big\{$ convex bodies in $\mathbb{R}^n \Big\}$
- $\mathscr{P}_n := \Big\{$ convex n-gons in $\mathbb{R}^2 \Big\}$

Shape functionals

- First Dirichlet eigenvalue $\begin{cases} -\Delta u_K = \lambda_1(K) u_K & \text{in } K \\ u_K = 0 & \text{on } \partial K \end{cases}$
- Torsional rigidity $\begin{cases} -\Delta u_K = 1 & \text{in } K \\ u_K = 0 & \text{on } \partial K \end{cases}$
- Capacity ($\rightsquigarrow$ Log-Capacity $n = 2$) $\begin{cases} -\Delta u_K = 0 & \text{in } \mathbb{R}^n \setminus K \\ u_K = 1 & \text{on } \partial K \\ u_K \to 0 & \text{as } |x| \to +\infty \end{cases}$

Definition: The *isoperimetric quotient for the volume functional* is the scale-invariant functional given by

$$\mathscr{I}_V(K) := \frac{d}{dt}|K+tB|^{\frac{1}{n}}\Big|_{t=0^+} = \frac{1}{n}|K|^{\frac{1}{n}-1}|\partial K| \qquad \forall K \in \mathscr{K}^n$$

(B = unit ball)

Theorem [ISOPERIMETRIC INEQUALITY]

(i) $\mathscr{I}_V(K) \geq \mathscr{I}_V(B) \qquad \forall K \in \mathscr{K}^n$

(ii) $\mathscr{I}_V(P) \geq \mathscr{I}_V(P_n^*) \qquad \forall P \in \mathscr{P}_n$

(P_n^* = regular n-gon)

[Burago-Zalgaller '88, De Giorgi '58, Fusco '04]

Q: Is there a variational analogue of the isoperimetric inequality?

Let $F = \lambda_1$ (first Dirichet Laplacian eigenvalue), τ (torsion), Cap (capacity)

Note that F is homogeneous of degree α under domain dilation: if $\alpha = -2$ for λ_1, $\alpha = n+2$ for τ, $\alpha = n-2$ for Cap, it holds

$$F(tK) = t^{\alpha} F(K) \qquad \forall K \in \mathscr{K}^n, \ \forall t \in \mathbb{R}_+$$

Definition: The *isoperimetric quotient for F* is the scale-invariant functional given by

$$\mathscr{I}_F(K) := \frac{d}{dt} F(K+tB)^{\frac{1}{\alpha}} \Big|_{t=0^+} = \frac{1}{\alpha} F(K)^{\frac{1}{\alpha}-1} \underbrace{\frac{d}{dt} F(K+tB)\Big|_{t=0^+}}_{\text{How does it read?}}$$

(B = unit ball)

Hadamard's formula for variational energies

$$\frac{d}{dt} F(K+tL)\Big|_{t=0^+} = \mathrm{sign}(\alpha) \int_{S^{n-1}} h_L \, d\mu_K \qquad \forall K, L \in \mathscr{K}^n$$

- h_L= support function of L $\qquad h_L(\xi) := \sup_{x\in L}(x \cdot \xi)$
- μ_K= first variation measure of K $\qquad \mu_K = (\nu_K)_\sharp\big(|\nabla u_K|^2 \mathscr{H}^{n-1} \llcorner \partial K\big)$

$$\int_{S^{n-1}} \varphi \, d\mu_K := \int_{\partial K} \varphi \circ \nu_K |\nabla u_K|^2 \, d\mathscr{H}^{n-1}$$

$$\int_{S^{n-1}} 1 \, d\mu_K = \int_{\partial K} |\nabla u_K|^2 \, d\mathscr{H}^{n-1} =: |\mu_K|$$

Ex. $K = P \;\Rightarrow\; \mu_P = \sum_i \int_{S_i} |\nabla u_P|^2 \delta_{\nu_i}, \quad |\mu_P| = \sum_i \int_{S_i} |\nabla u_P|^2$

Remark: Perfect analogy with the case of volume (μ_K= surface area measure)

[Jerison '96, Colesanti-Fimiani '10, Xiao '18]

In particular:

- Taking $L = K$:

$$\frac{d}{dt}F((1+t)K)\Big|_{t=0^+} = \operatorname{sign}(\alpha)\int_{S^{n-1}} h_K\, d\mu_K = \frac{d}{dt}(1+t)^\alpha F(K)\Big|_{t=0^+} = \alpha F(K)$$

$$\Rightarrow F(K) = \frac{1}{|\alpha|}\int h_K\, d\mu_K \quad \textit{representation formula for } F$$

(for circumscribed polygons, reduces to $F(K) = \frac{1}{|\alpha|}\rho_K|\mu_K|$)

- Taking $L = B$:

$$\frac{d}{dt}F(K+tB)\Big|_{t=0^+} = \operatorname{sign}(\alpha)\int_{S^{n-1}} 1\, d\mu_K = \operatorname{sign}(\alpha)\int_{\partial K} |\nabla u_K|^2\, d\mathscr{H}^{n-1}$$

$$\Rightarrow \mathscr{I}_F(K) = \frac{1}{\alpha}F(K)^{\frac{1}{\alpha}-1}\frac{d}{dt}F(K+tB)\Big|_{t=0^+} = \frac{1}{|\alpha|}F(K)^{\frac{1}{\alpha}-1}\int_{\partial K}|\nabla u_K|^2\, d\mathscr{H}^{n-1}$$

expression of the isoperimetric quotient of F

Theorem [VARIATIONAL ISOPERIMETRIC INEQUALITY]

Let $F = \lambda_1, \tau,$ or Cap.

(i) $\mathscr{I}_F(K) \geq \mathscr{I}_F(B) \qquad \forall K \in \mathscr{K}^n$ [Bucur-F.-Lamboley '12]

(ii) $\mathscr{I}_F(P) \geq \mathscr{I}_F(P_n^*) \qquad \forall P \in \mathscr{P}_n$ [Bucur-F., in progress]

Among convex bodies (resp. convex n-gons) with prescribed energy, the ball (resp. the regular n-gon) is the MOST STABLE.

Sketch of proof.

Brunn-Minkowski inequality

$$(BM) \qquad F^{\frac{1}{\alpha}}\big((1-t)K+tL\big) \geq (1-t)F^{\frac{1}{\alpha}}(K)+tF^{\frac{1}{\alpha}}(L) \qquad \forall K,L \in \mathscr{K}^n,\ \forall t \in [0,1]$$

with equality iff K,L are homothetic

[Brascamp-Lieb '75, Borell '83-'85, Caffarelli-Jerison-Lieb '96, Colesanti' 05]

Minkowski's first inequality (infinitesimal form of BM)

$$(M1) \qquad \int_{S^{n-1}} h_L \, d\mu_K \geq |\alpha| F^{1-\frac{1}{\alpha}}(K) F^{\frac{1}{\alpha}}(L) \qquad \forall K,L \in \mathscr{K}^n$$

(i) Continuous case: Take $K \in \mathscr{K}^n$ arbitrary and $L = B$ in (M1).

$$\underbrace{\int_{S^{n-1}} h_B \, d\mu_K}_{=|\mu_K|} \geq |\alpha| F^{1-\frac{1}{\alpha}}(K) F^{\frac{1}{\alpha}}(B)$$

$$\Downarrow$$

$$\mathscr{I}_F(K) = \frac{1}{|\alpha|} F^{\frac{1}{\alpha}-1}(K) |\mu_K| \geq F^{\frac{1}{\alpha}}(B) = \mathscr{I}_F(B)$$

(ii) Discrete case:

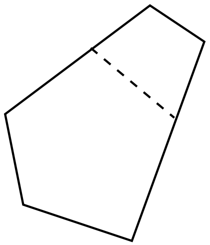
STEP 1: Take $K = P \in \mathscr{P}_n$ arbitrary and $L = P_0$ in (M1)

$P_0 :=$ the circumscribed polygon with the same incircle and same normals as P

$$\underbrace{\int_{S^{n-1}} h_{P_0}\, d\mu_P}_{\rho(P_0)|\mu_P|} \geq |\alpha| F^{1-\frac{1}{\alpha}}(P) F^{\frac{1}{\alpha}}(P_0)$$

$$\Downarrow$$

$$\mathscr{I}_F(P) = \frac{1}{|\alpha|} F^{\frac{1}{\alpha}-1}(P)|\mu_P| \geq \frac{F^{\frac{1}{\alpha}}(P_0)}{\rho(P_0)} = \mathscr{I}_F(P_0)$$



STEP 2: Show that, if $P_0 \in \mathscr{P}_n$ is circumscribed, it holds

$$\mathscr{I}_F(P_0) = \frac{F^{\frac{1}{\alpha}}(P_0)}{\rho(P_0)} \geq \mathscr{I}_F(P_n^*)$$

$\rightsquigarrow$ shape optimization problem under INRADIUS constraint

▷ For the volume functional: $\mathscr{I}_V(P_0) = \sqrt{\sum_i \cot(\theta_i/2)}$

▷ For variational functionals: via construction of trial functions
$\rho(P_0) = const. \ \Rightarrow \ \lambda_1(P_0) \leq \lambda_1(P_n^*)$ [Solynin '92] □

For $F = \lambda_1, \tau, \mathrm{Cap}$, **balls** are optimal domains under **volume** constraint,
A discrete version of this symmetry result is a challenging open problem!

$$\mathscr{E}(K) := \lambda_1(K)^{1/2}|K|^{1/n}$$

(i) $\mathscr{E}(K) \geq \mathscr{E}(B) \qquad \forall K \in \mathscr{K}^n$ *FABER-KRAHN INEQUALITY '23*

(ii) $\mathscr{E}(P) \geq \mathscr{E}(P_n^*) \qquad \forall P \in \mathscr{P}_n$ *PÓLYA CONJECTURE '51*

proved for $n = 3, 4$, open for $n \geq 5$.

1) The approach by symmetrization

2) The approach by concavity

3) A lucky case

1) The approach by symmetrization

(i) *Continuous case*: $u \in H^1_0(\Omega) \rightsquigarrow u^* \in H^1_0(\Omega^*)$ Schwarz symmetrization

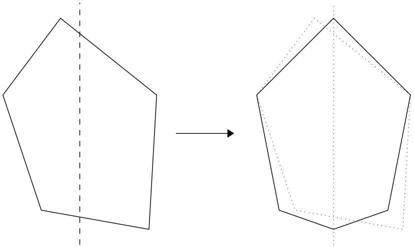
$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_\Omega|^2}{\int_\Omega |u_\Omega|^2} \geq \frac{\int_\Omega |\nabla u^*_\Omega|^2}{\int_\Omega |u^*_\Omega|^2} \geq \lambda_1(\Omega^*)$$

(ii) *Discrete case*: $u \in H^1_0(\Omega) \rightsquigarrow u^\sharp \in H^1_0(\Omega^\sharp)$ Steiner symmetrization

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_\Omega|^2}{\int_\Omega |u_\Omega|^2} \geq \frac{\int_\Omega |\nabla u^\sharp_\Omega|^2}{\int_\Omega |u^\sharp_\Omega|^2} \geq \lambda_1(\Omega^\sharp)$$

Which directions to choose?

- $n = 3$: Given any triangle, a sequence of Steiner symmetrizations w.r.t. the mediators of each side converges to the equilateral triangle
- $n = 4$: Given any quadrilateral, a sequence of 3 Steiner symmetrizations transforms it into a rectangle
- $n \geq 5$: *No way to reach P_n^* by preserving the number of sides*



2) The approach by concavity

– in the continuous setting, using (M1) with the two bodies K and B exchanged allows to prove that balls minimize λ_1 under *perimeter constraint*.

– in the discrete setting, using a similar strategy allows to prove that the regular n-gon minimizes λ_1:

- under perimeter constraint, in the restricted class of *equiangular* n-gons;
- under *symmetric content* constraint:

$$\sigma(P) := \inf \left\{ \frac{1}{n} \sum_{i=1}^{n} h_P(v_i) \ : \ (v_1, \ldots, v_n) \in \mathscr{E}_n(S^1) \right\}$$

where $\mathscr{E}_n(S^1)$:=the class of *equidistributed* n-tuples of vectors in S^1

3) A lucky case: the Cheeger constant

Given $\Omega \subset \mathbb{R}^2$ with finite measure

$$h(\Omega) := \inf\left\{ \frac{\mathrm{Per}(A,\mathbb{R}^2)}{|A|} \ : \ A \text{ measurable}, \ A \subseteq \Omega \right\}.$$

A solution is called a Cheeger set of Ω

[Alter, Buttazzo, Carlier, Caselles, Chambolle, Comte, Figalli, Fridman, Fusco, Kawohl, Krejčiřík, Lachand-Robert, Leonardi, Maggi, Novaga, Pratelli]

- It is the limit as $p \to 1^+$ of the first Dirichlet eigenvalue of Δ_p.
- It satisfies the Faber-Krahn inequality $h(\Omega) \geq h(\Omega^*)$.

Theorem [DISCRETE F.K. FOR THE CHEEGER CONSTANT] [Bucur-F. '16]
Among all simple polygons with a given area and at most n sides, the regular *n*-gon minimizes the Cheeger constant.

$$h(P) \geq h(P_n^*) \qquad \forall P \in \mathscr{P}_n\,.$$

- No convexity is needed on the admissible polygons.
- The same result holds true for log-capacity [Solynin-Zalgaller '04] .

Sketch of proof – the case of convex polygons (easy!)

If P minimizes the Cheeger constant among polygons in $\mathscr{P}_n$ with the same area, it is *Cheeger regular*, and consequently [Kawohl-Lachand Robert '06]

$$h(P) = \frac{|\partial P| + \sqrt{|\partial P|^2 - 4|P|(\Lambda(P) - \pi)}}{2|P|} \qquad \text{with } \Lambda(P) = \sum_i \cot\left(\frac{\theta_i}{2}\right).$$

By the discrete isop. inequality (Step 1), $\mathscr{I}_V(P) = \frac{1}{2}\frac{|\partial P|}{|P|^{1/2}} \geq \mathscr{I}_V(P_0) = \Lambda(P)^{1/2}$

$$h(P) \geq \frac{|\partial P| + \sqrt{4\pi|P|}}{2|P|}.$$

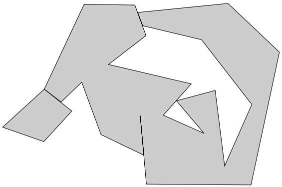
By the discrete isop. inequality (Step 2), $\mathscr{I}_V(P) \geq \mathscr{I}_V(P_n^*)$

$$h(P) \geq \frac{|\partial P_n^*| + \sqrt{4\pi|P_n^*|}}{2|\Omega_n^*|} = h(P_n^*).$$

□

Sketch of proof – the case of general polygons

- Since $\mathscr{P}_n$ is not closed in the Hausdorff complementary topology, we enlarge the class of competitors to $\overline{\mathscr{P}_n}$ (thus allowing self-intersections).

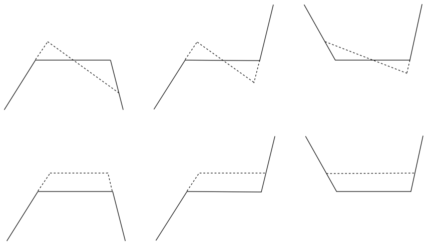


- For such "generalized polygons", we introduce a notion of *relaxed Cheeger constant*. In this framework, we obtain an *existence result*, and a *representation formula* for an optimal generalized polygon.

▷ We introduce a Lagrange multiplier μ and we use first order shape derivatives to get some *stationarity conditions*:

$$\frac{d}{d\varepsilon}\Big(h(P_\varepsilon)+\mu|P_\varepsilon|\Big)\Big|_{\varepsilon=0}=0\ .$$

The deformations we use are *rotations* and *parallel movements* of one side.



▷ Via the stationarity conditions, we show that the boundary of an optimal generalized polygons contains *no self-intersections* and *no reflex angles*.

We are thus back to the case of simple convex polygons and we are done.

□

Q: Is it possible to attack Pólya conjecture for λ_1 by the same approach?

- Stationarity conditions in the previous proof are purely geometric! E.g.,

$$\frac{\sin\left(\frac{\beta-\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)}\ell h(P)=\frac{\cos^2\left(\frac{\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)}$$

- For λ_1, stationarity conditions are no longer purely geometric!

  This leads to study overdetermined boundary value problems on polygons.

  .... NEW TOPIC!

Let u_Ω be the torsion function of an open bounded domain $\Omega \subset \mathbb{R}^2$,
i.e. the unique solution to

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Serrin's symmetry result:

If u_Ω satisfies the following overdetermined boundary condition, Ω is a ball:

$$(*) \qquad\qquad |\nabla u_\Omega| = c \text{ on } \partial\Omega\,,$$

Remark: $(*)$ corresponds to the stationarity condition $\frac{d}{d\varepsilon}\left(\frac{\tau(\Omega_\varepsilon)}{|\Omega_\varepsilon|^2}\right)\Big|_{\varepsilon=0} = 0$

Conjecture (discrete version of Serrin's result)

Let u_P be the torsion function or first Dirichlet eigenfunction of a polygon P, with sides $S_i = [A_i, A_{i+1}]$ of length ℓ_i.

It u_P satisfies the overdetermined boundary conditions

$$\begin{cases} (Par) & \displaystyle\int_{S_i} |\nabla u_P|^2 = \kappa \ell_i \qquad \forall i \\ (Rot) & \displaystyle\int_{S_i} \Big(\frac{\ell_i}{2} - |x - A_i|\Big) |\nabla u_P|^2 = 0 \qquad \forall i\,, \end{cases}$$

then P is a regular polygon.

- (Par) is the stationarity condition under parallel movement of S_i
- (Rot) is the stationarity condition under rotation of S_i around its mid-point.

Theorem [SERRIN'S THEOREM FOR TRIANGLES] [F.-Velichkov '19]

Let u_T be the torsion function or first Dirichlet eigenfunction of a triangle T.

(i) *Triangular equidistribution*: For *any* triangle T, u_T satisfies

$$(Par) \qquad \int_{S_i} |\nabla u_T|^2 = \kappa \ell_i \qquad \forall i = 1,2,3.$$

(ii) *Triangular symmetry*: If u_T satisfies

$$(Rot) \qquad \int_{S_i} \Big(\frac{\ell_i}{2} - |x - A_i|\Big) |\nabla u_T|^2 = 0 \qquad \forall i = 1,2,3,$$

then T is a regular triangle.

Sketch of proof

(i) Triangular equidistribution.

Recall that (Par) corresponds to the the stationarity condition

$$\frac{d}{d\varepsilon}\Big(\lambda_1(T_\varepsilon)\,|T_\varepsilon|\Big)\Big|_{\varepsilon=0} = 0$$

when T_ε are obtained from T by parallel movement of one if its side.

But ALL triangles obtained this way are homothetic, and the shape functional

$$T \mapsto \lambda_1(T)|T|$$

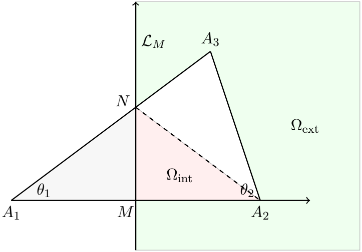
is invariant under dilations.

It follows that condition (Par) holds true for any triangle.

[Christianson '17]

(ii) Triangular symmetry.

We prove: $\theta_2 > \theta_1 \;\Rightarrow\; \int_{A_1}^{A_2} \Big(\frac{\ell_i}{2} - |x - A_i|\Big)|\nabla u_T|^2 < 0.$



Consider on the reflected triangle in red the difference $v = u_T - \widetilde{u}_T$.

It satisfies $-\Delta v = \lambda_1 v$ inside and is nonnegative on the boundary.

By the max principle, it is strictly positive inside.

Then, by Hopf boundary point principle $|\nabla u_T| > |\nabla \widetilde{u}_T|$ on $[M, A_2]$.

$$\int_M^{A_2} |x - M||\nabla u_T|^2 > \int_M^{A_2} |x - M||\nabla \widetilde{u}_T|^2 = \int_{A_1}^{M} |x - M||\nabla u_T|^2 .$$

OPEN PROBLEMS:

- **Pólya conjecture (for $n \geq 5$)**
- **discrete Serrin (for $n \geq 4$)**

THANK YOU FOR YOUR ATTENTION!