Rank of Erdos-Renyi Graphs

Margalit Glasgow

Joint work with Patrick DeMichele, Alex Moreira

Discrete Random Matrices

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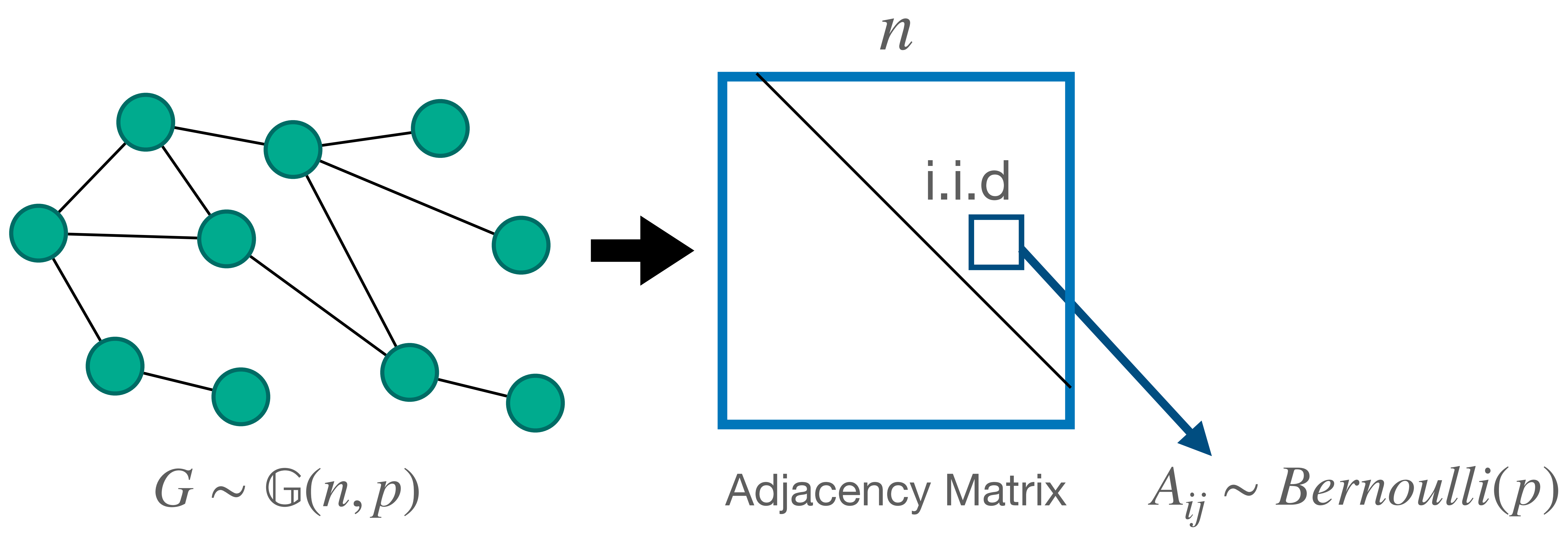

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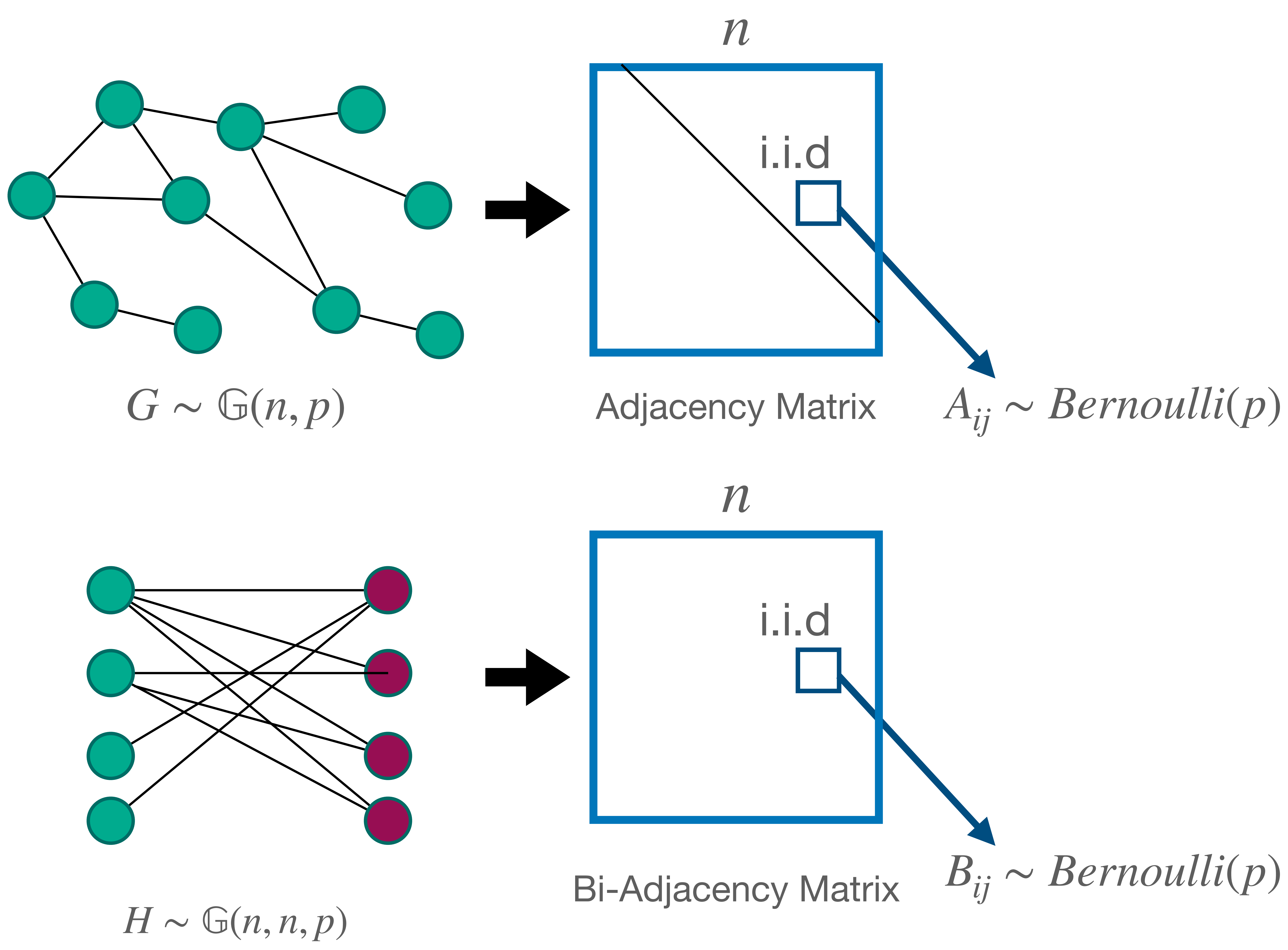

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For $d \geq 3$, random d-regular graphs are invertible with high probability.
[Huang ’18]

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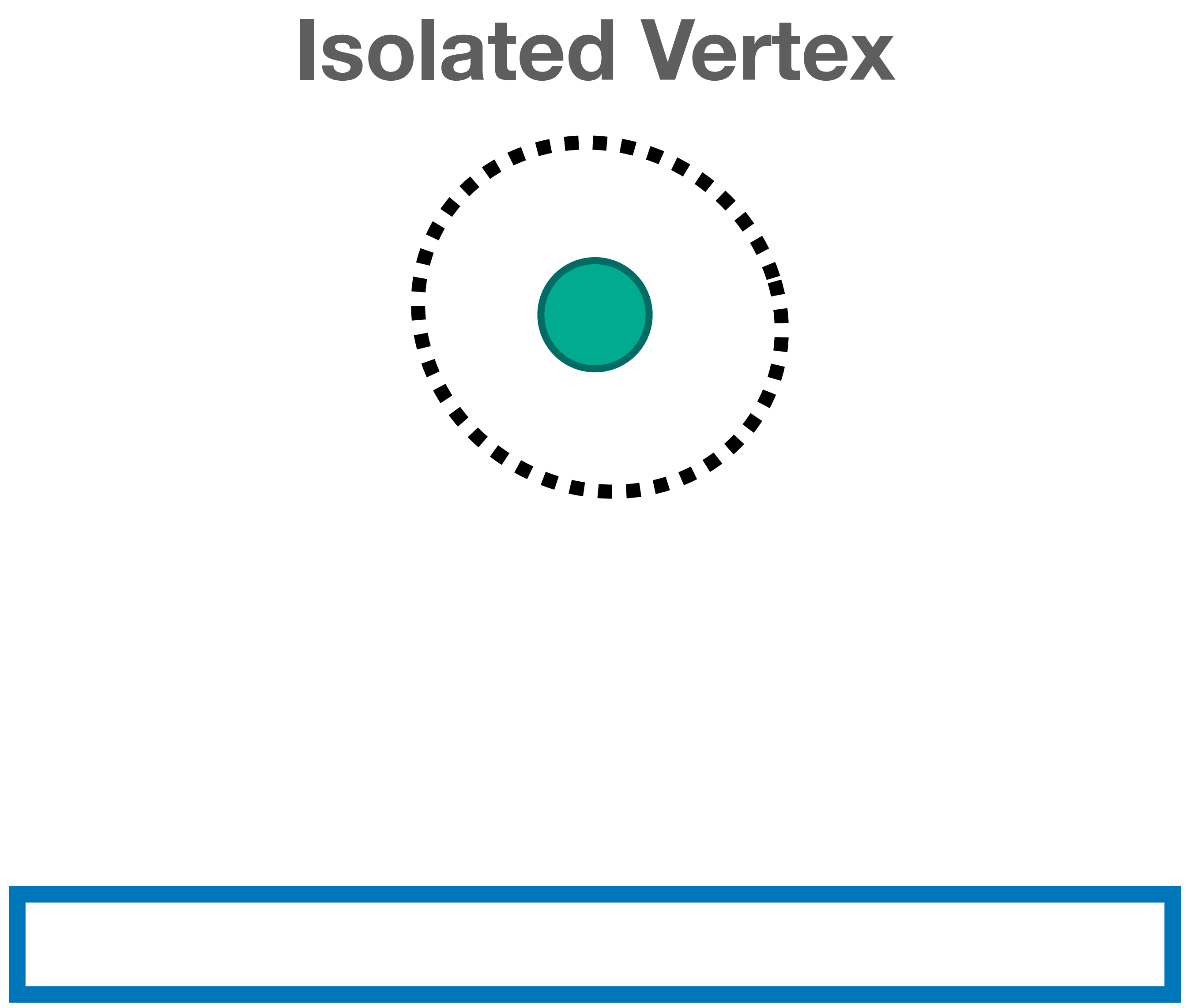

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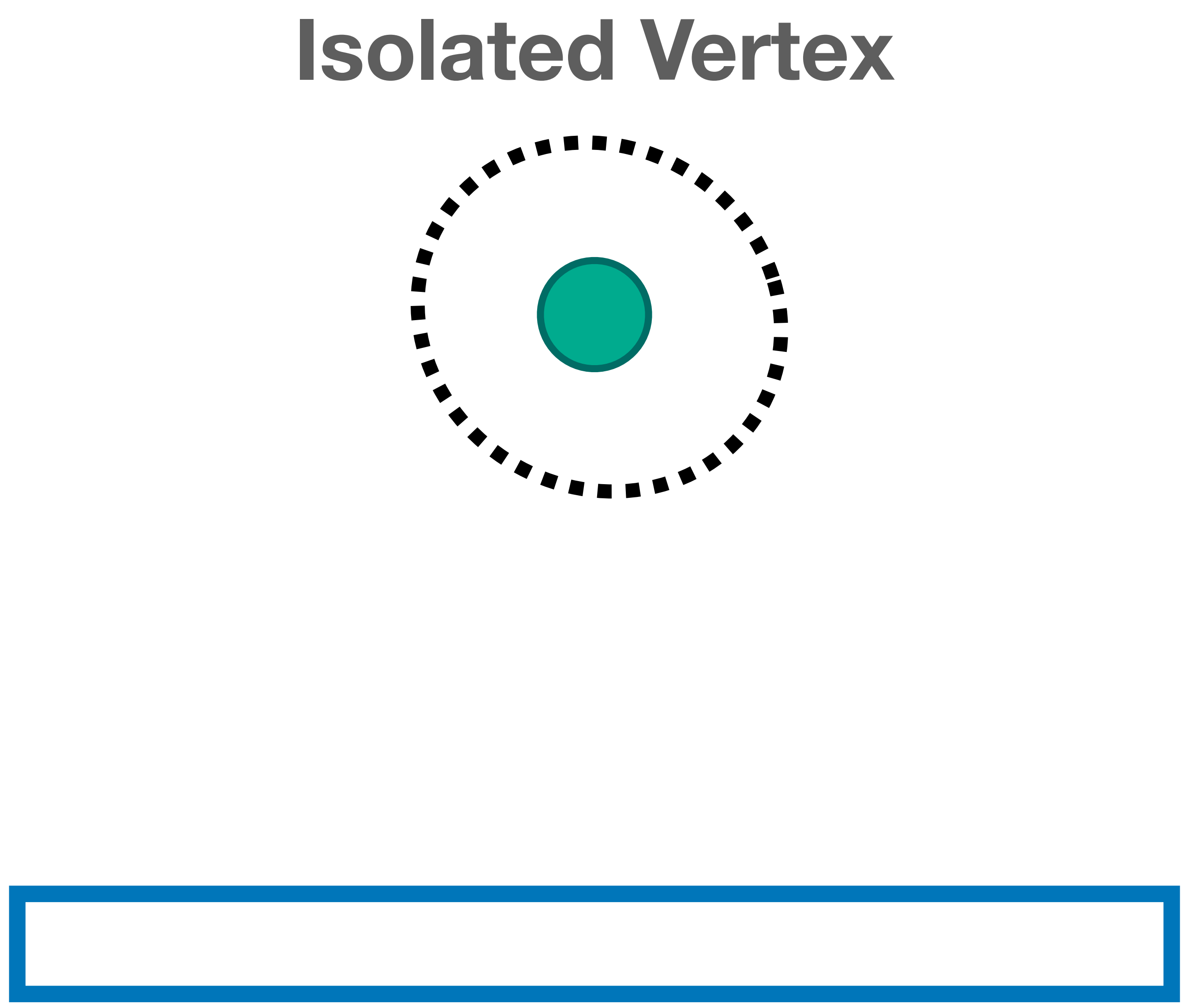

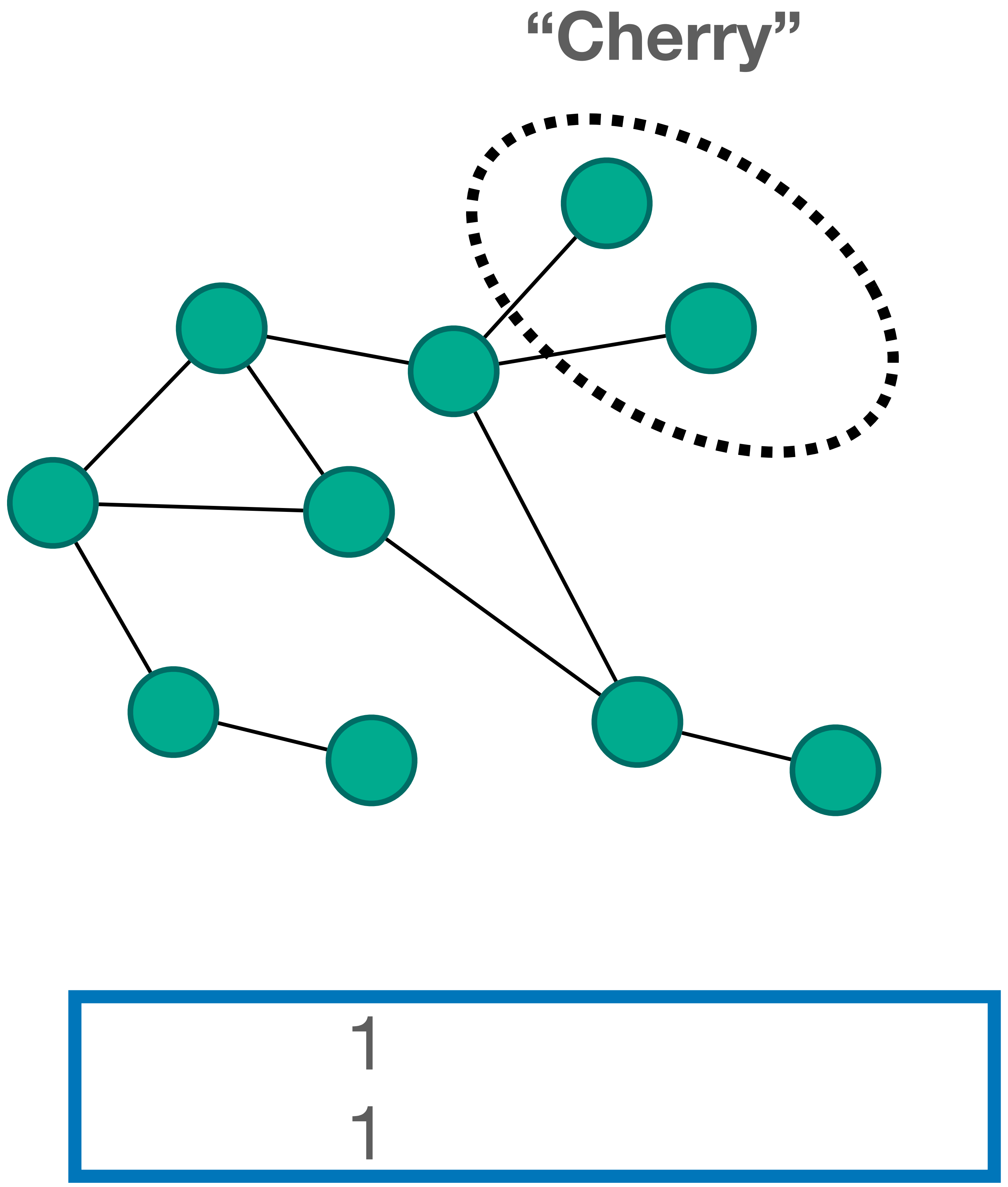

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Can we understand the rank of G by removing these structures?

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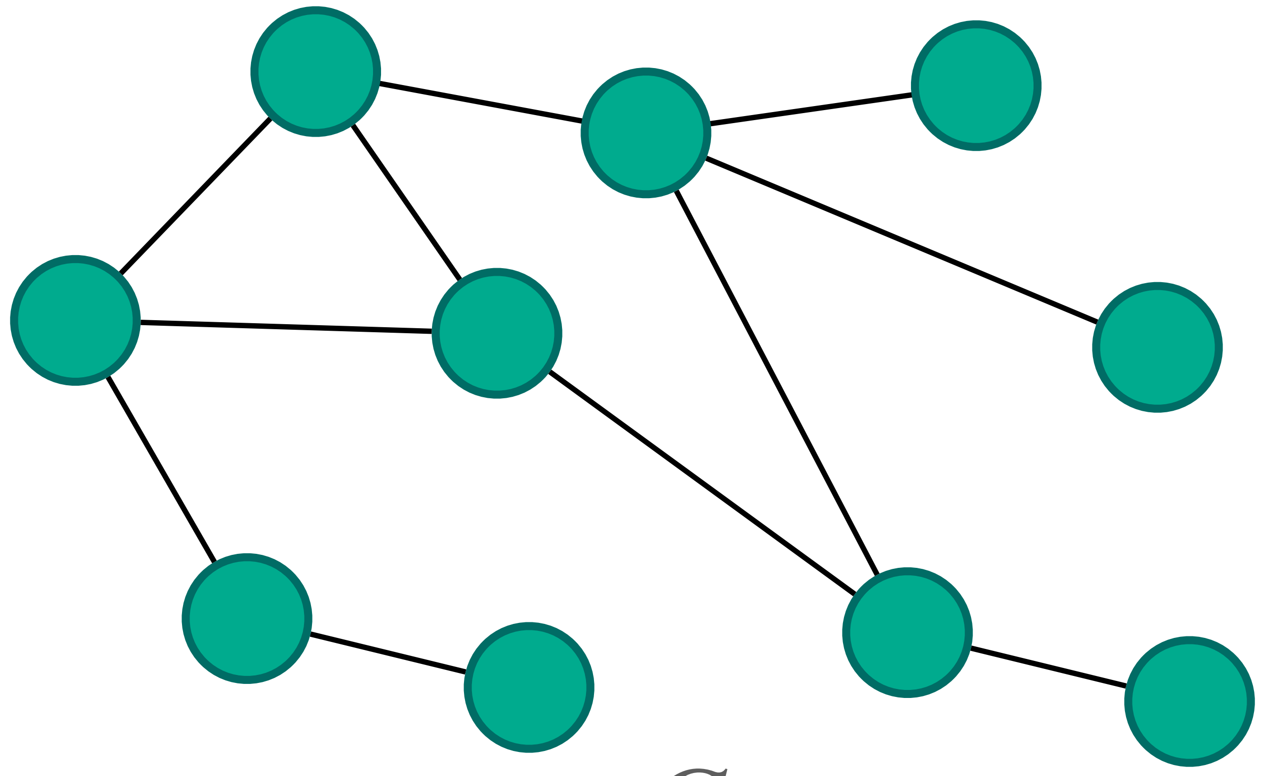

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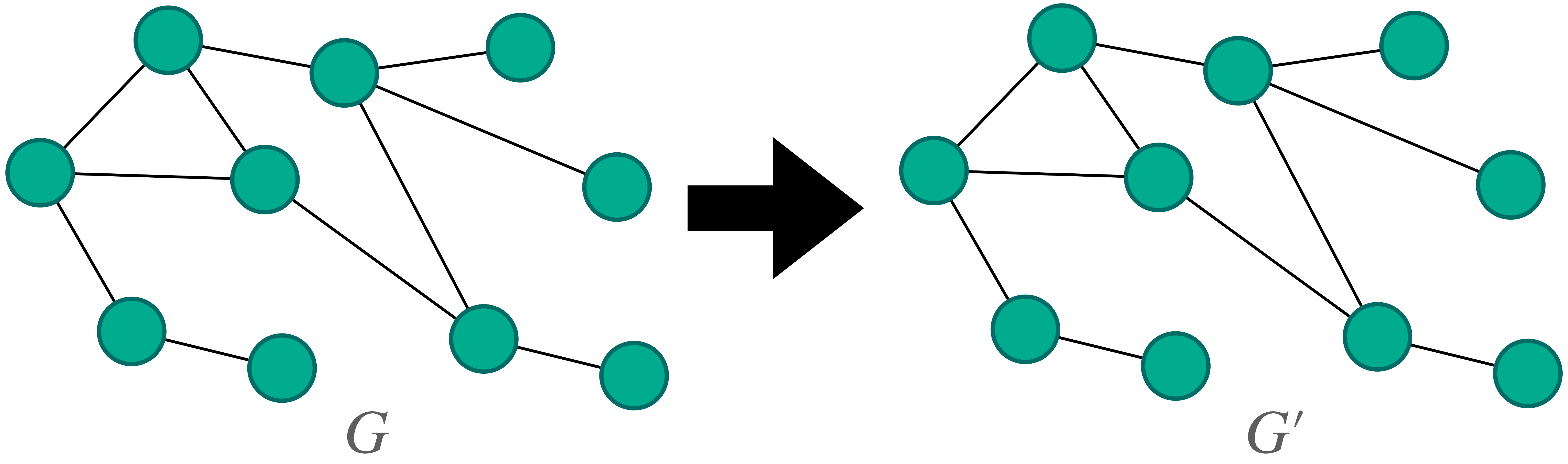

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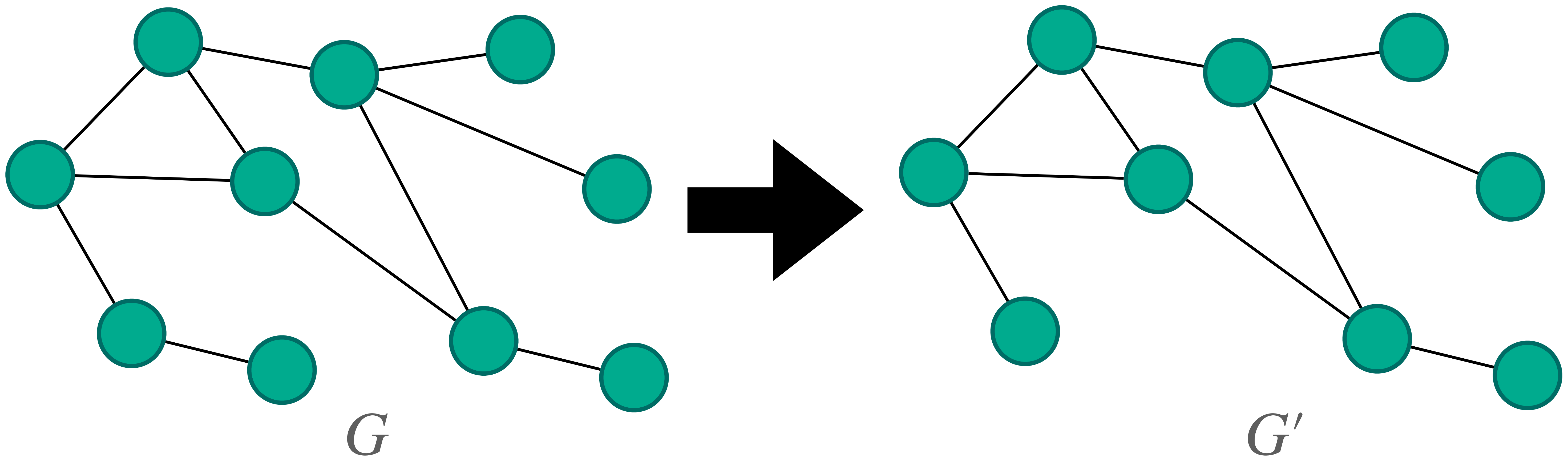

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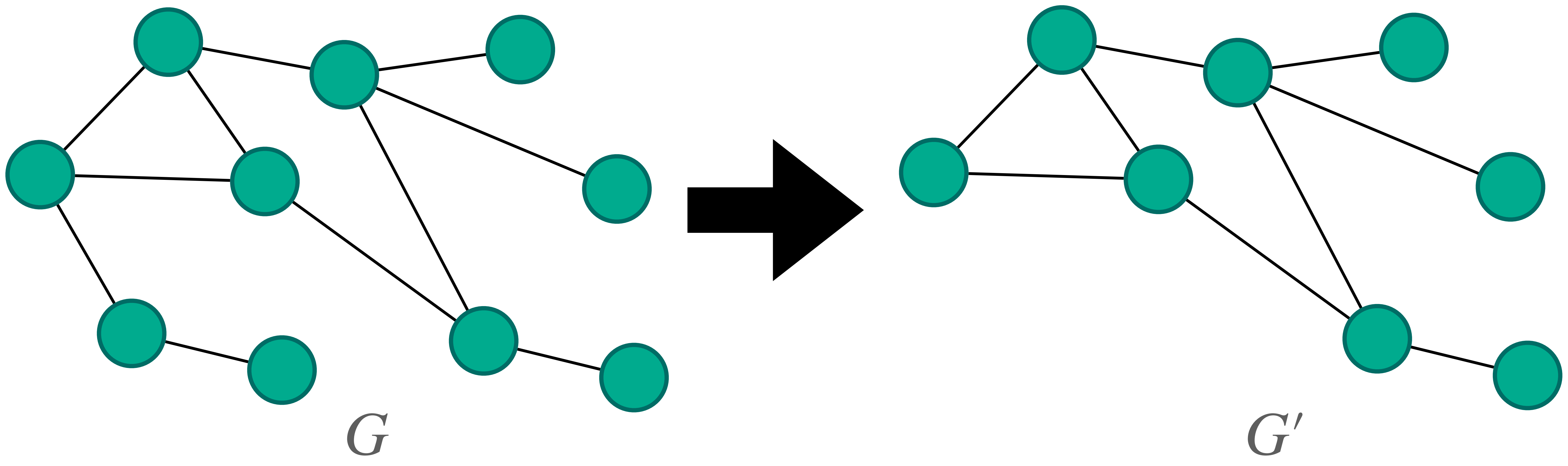

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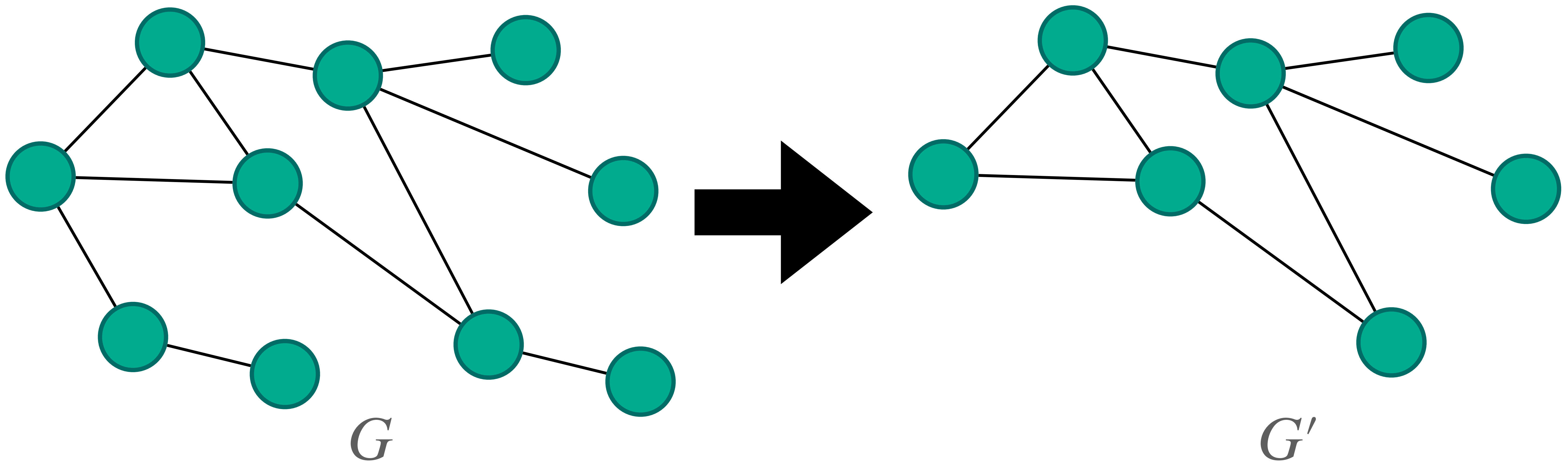

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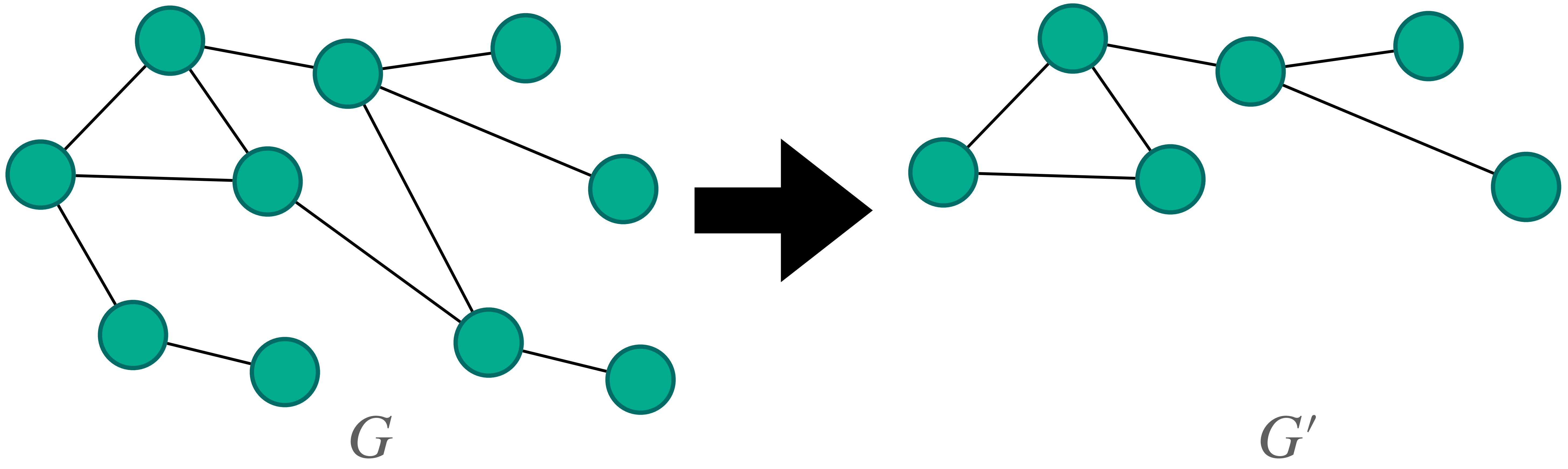

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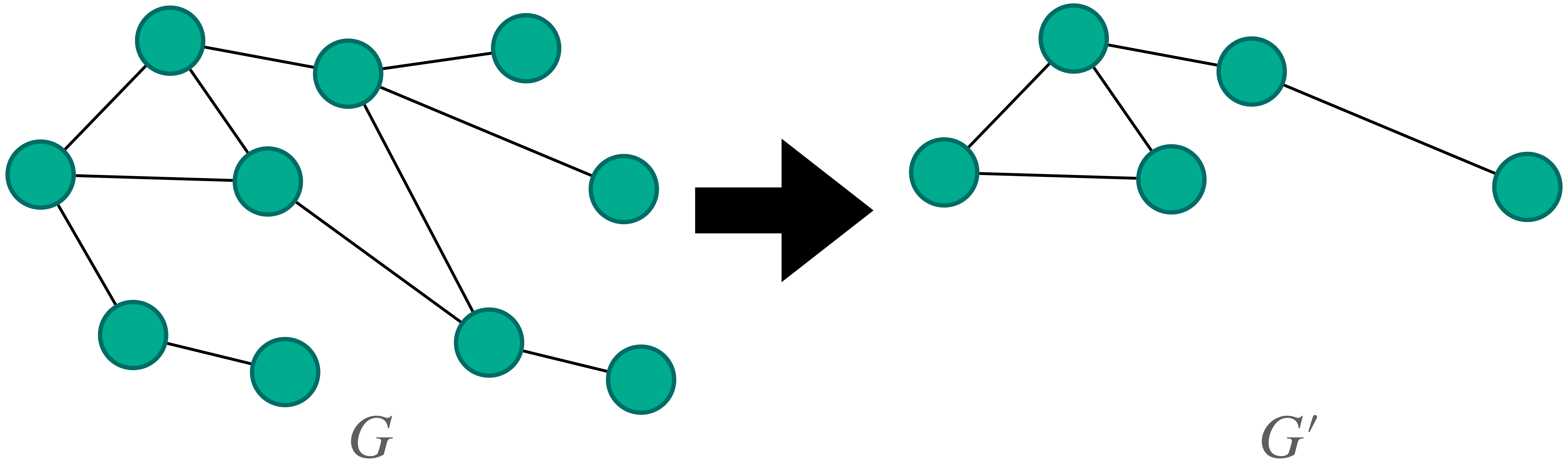

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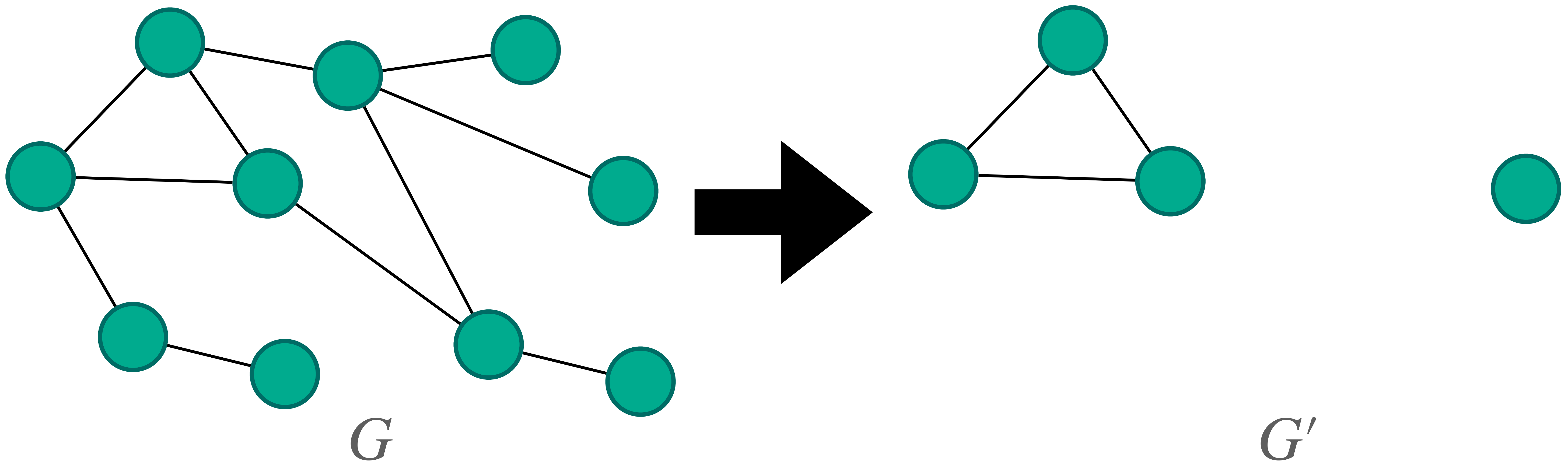

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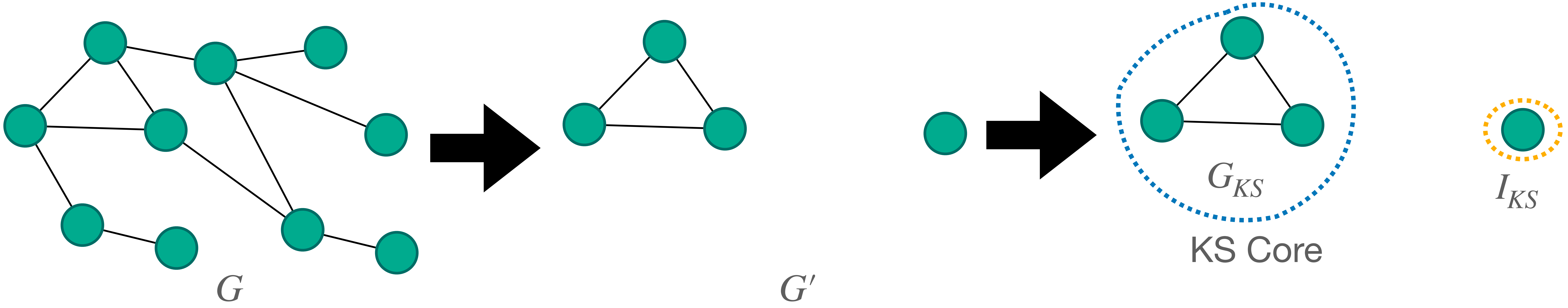

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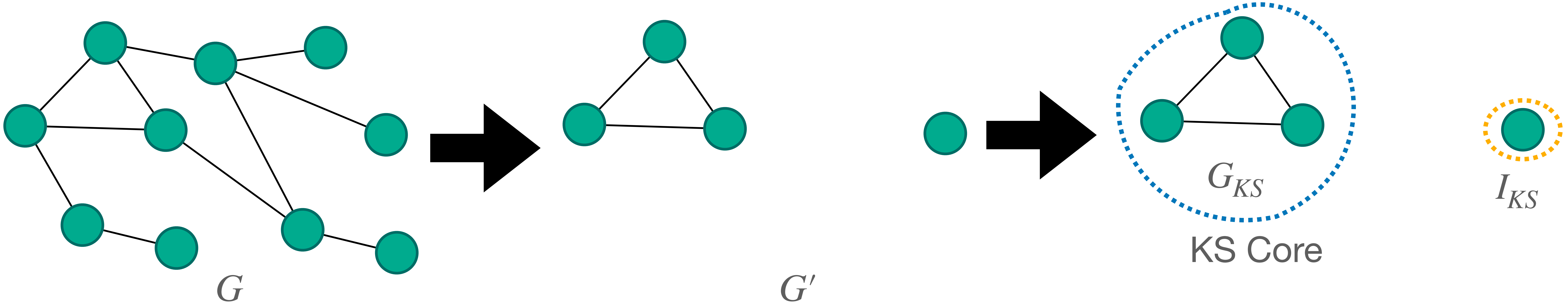

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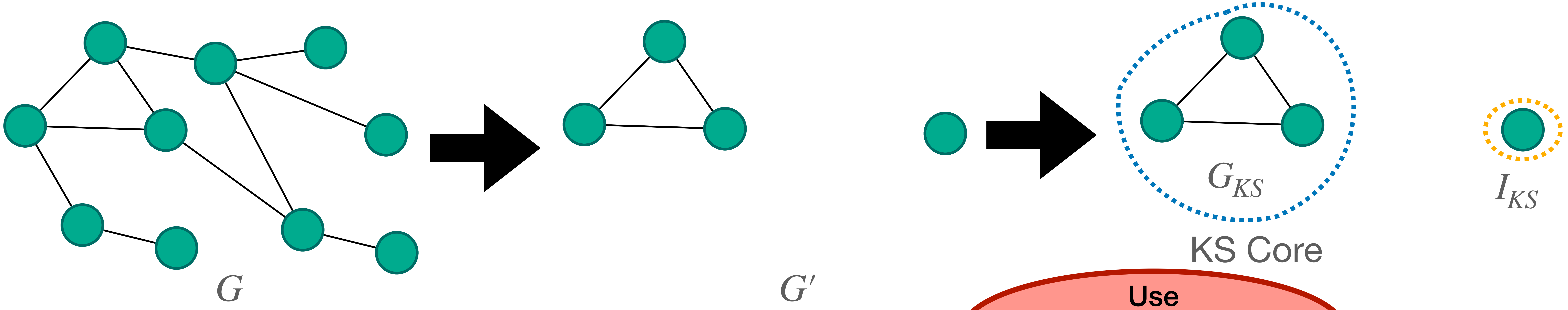

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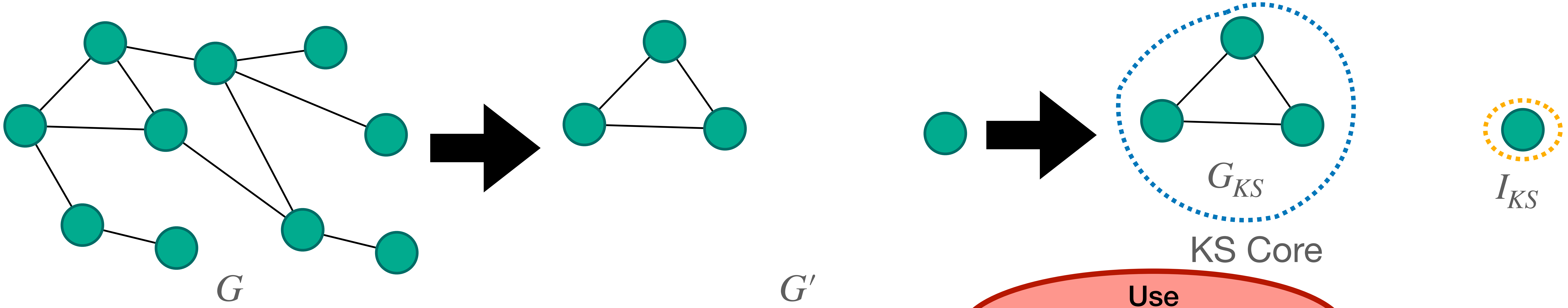

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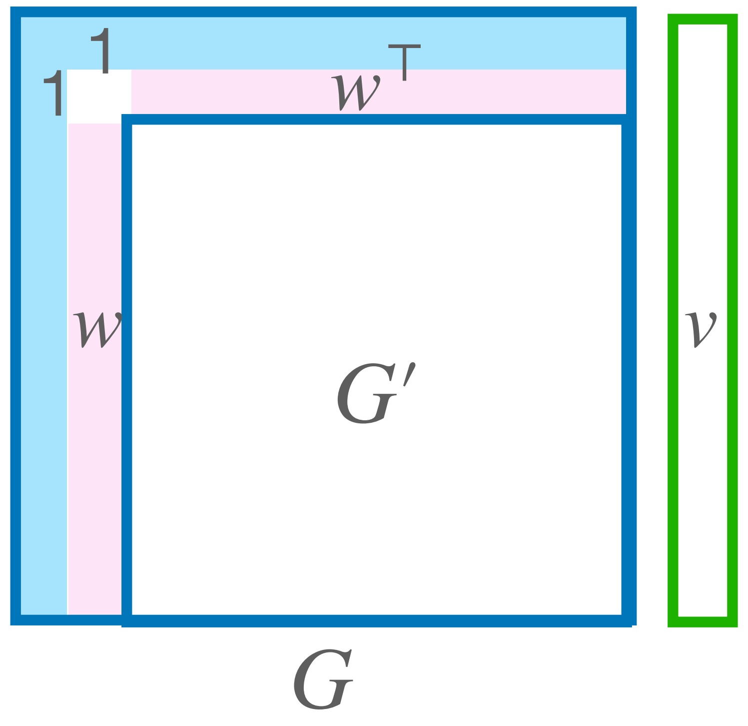

$$Gv = 0 \Leftrightarrow G'v' = 0$$

$$v = (-w^\top v', 0, v')$$

Prior Results

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Coja-Oghlan, Ergür, Gao, Hetterich, Rolvien '20: For $H \sim \mathbb{G}(n, n, p)$, with $p = \Theta(1/n)$, in probability,

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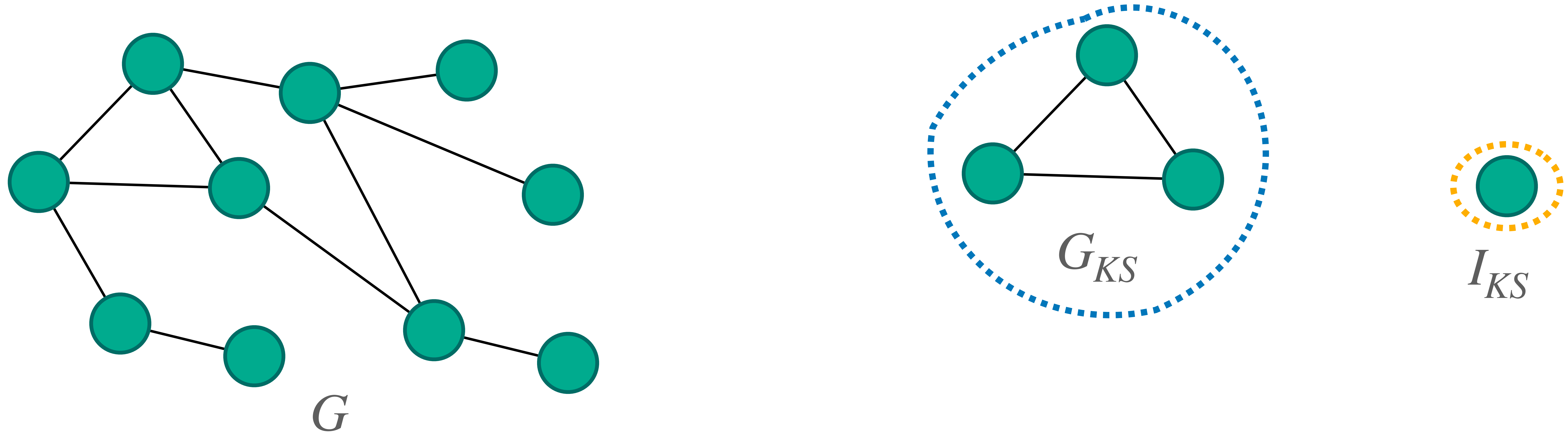

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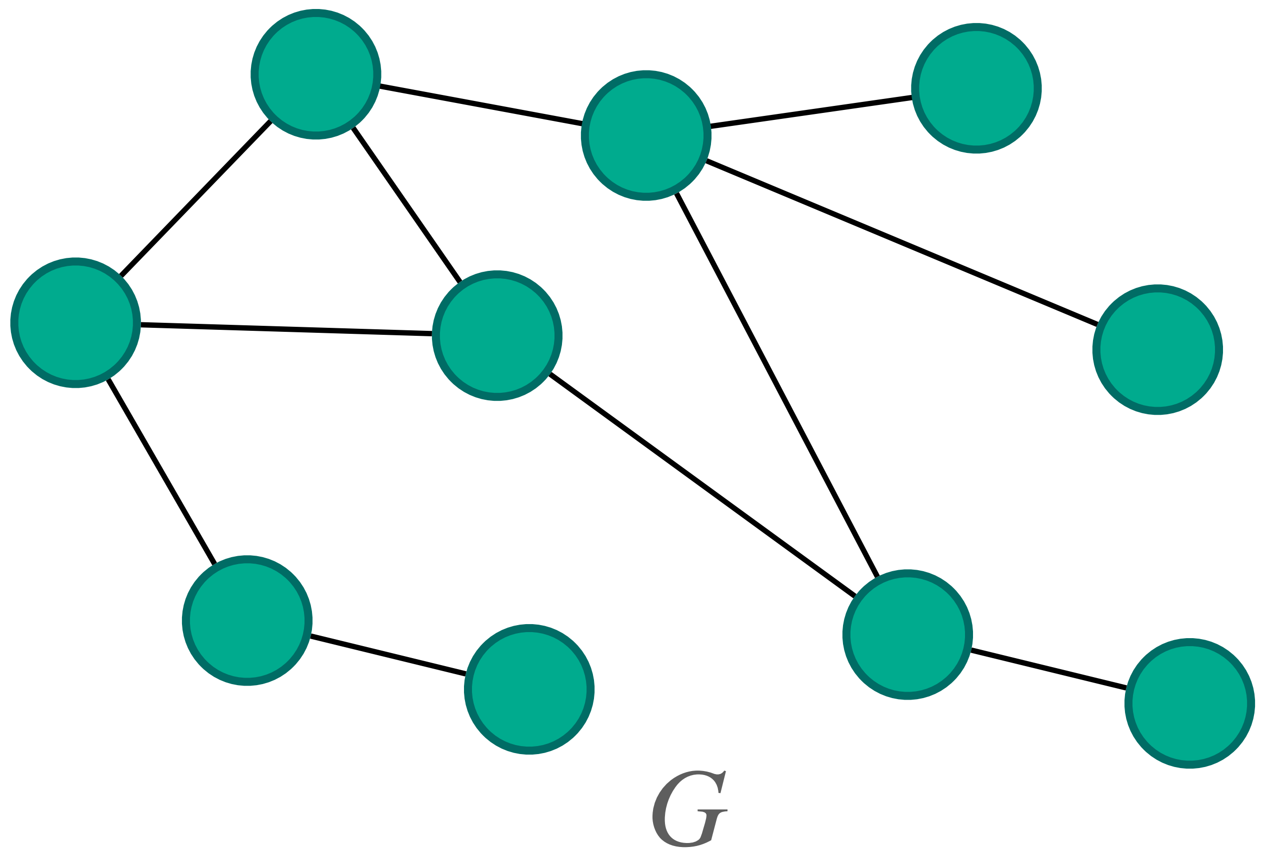

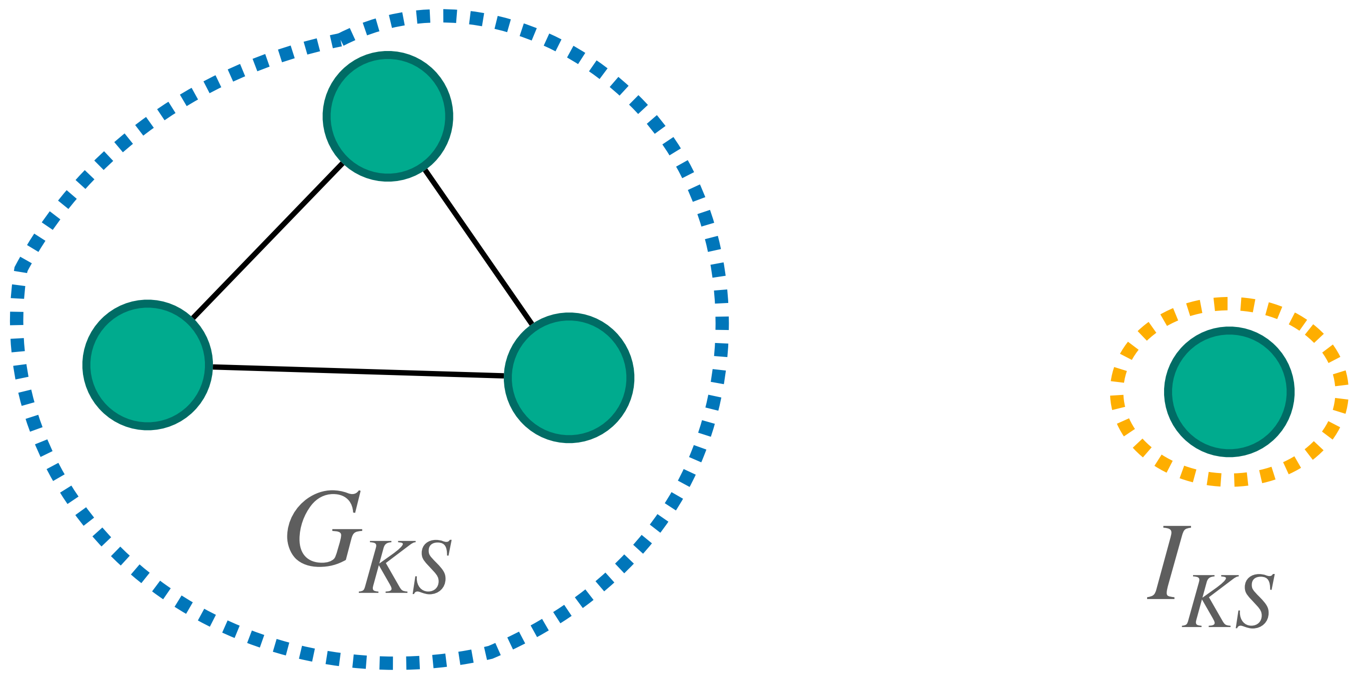

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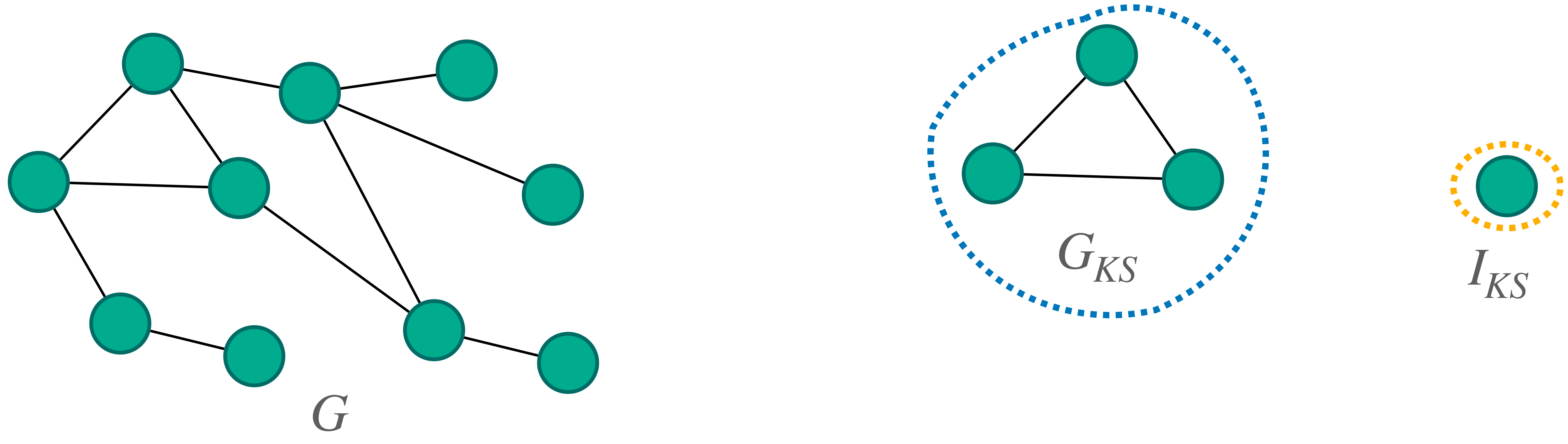

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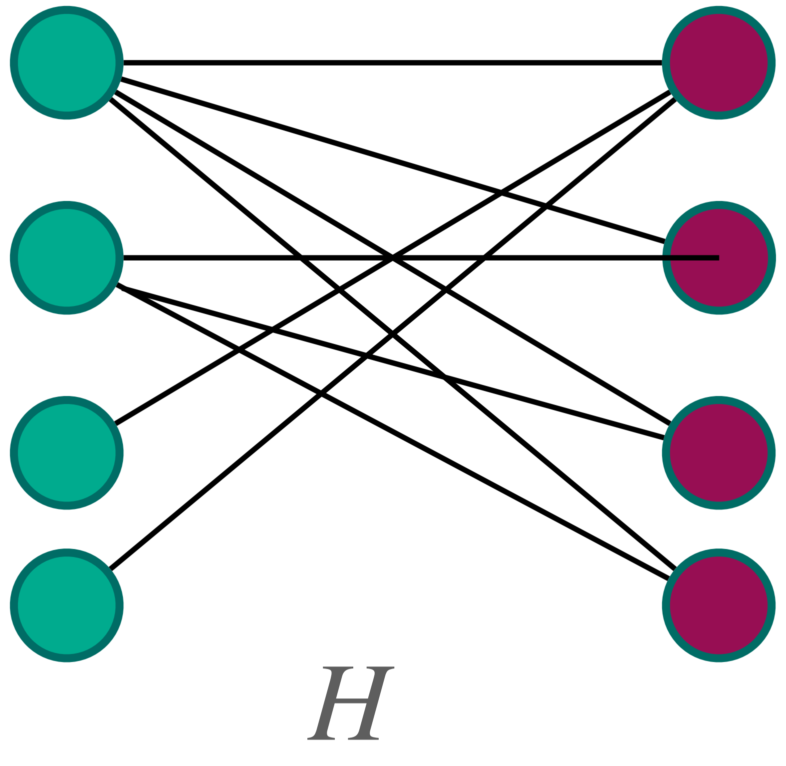

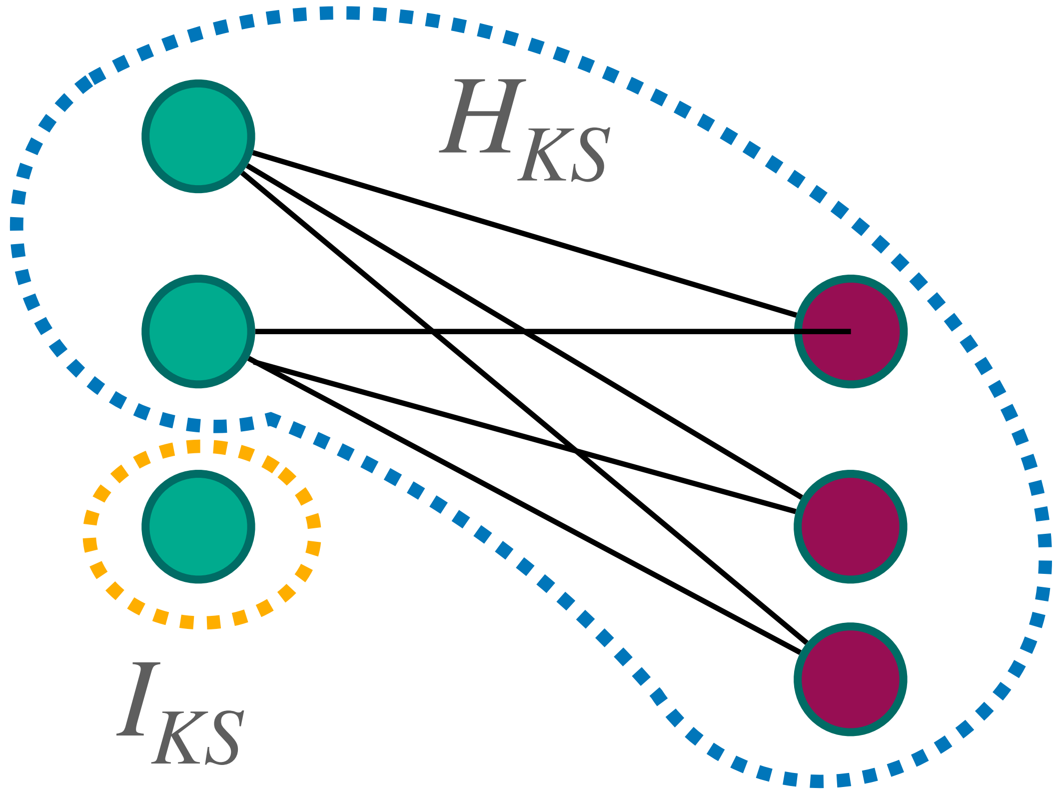

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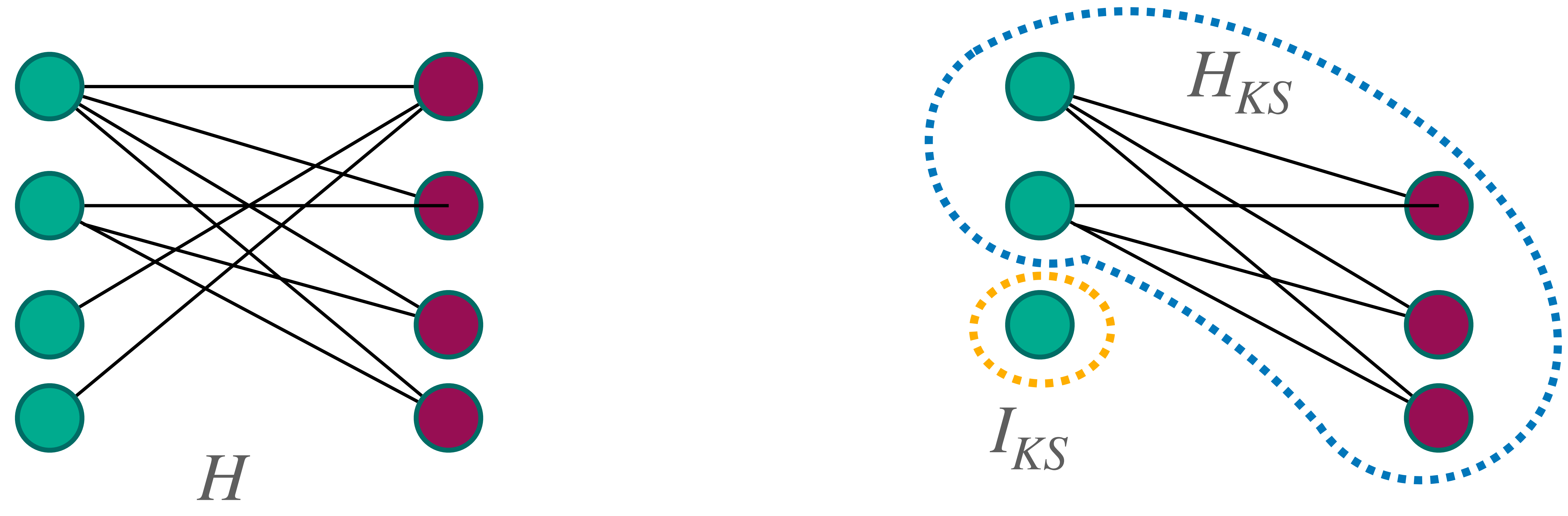

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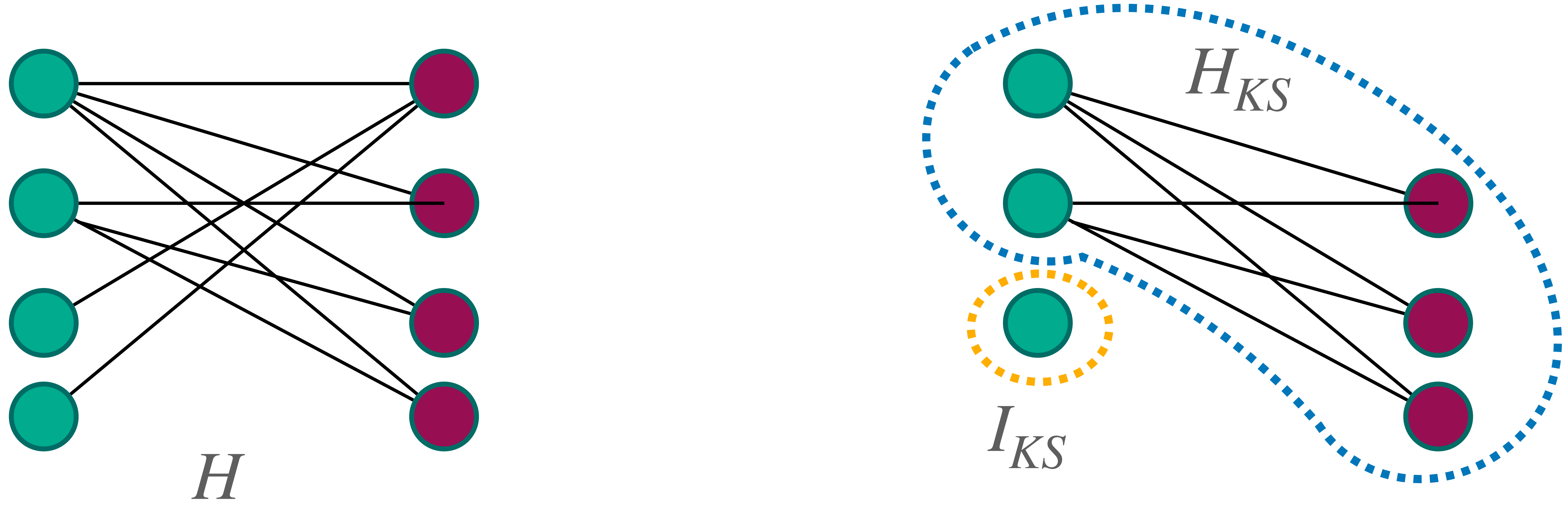

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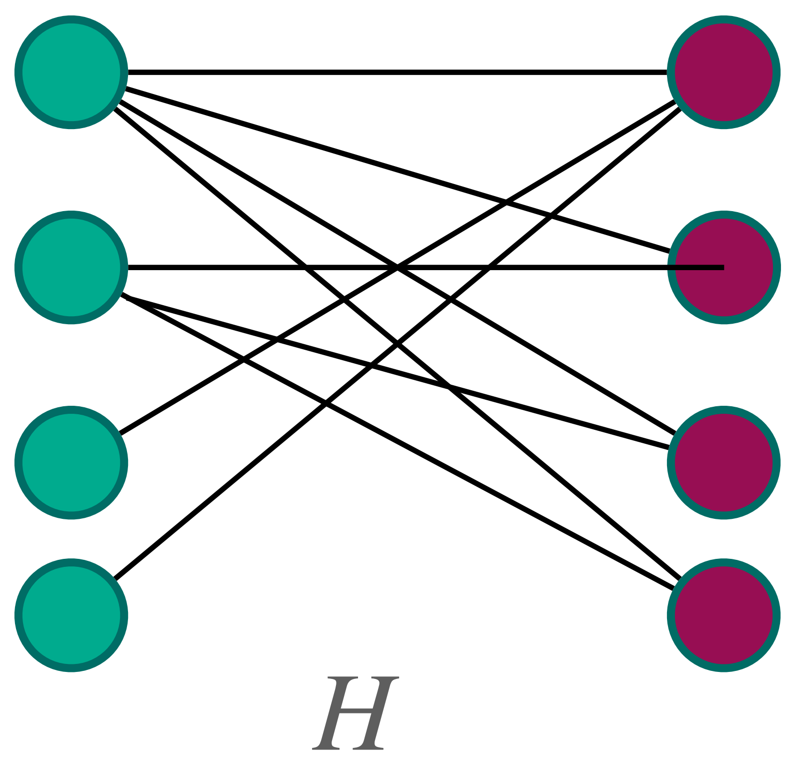

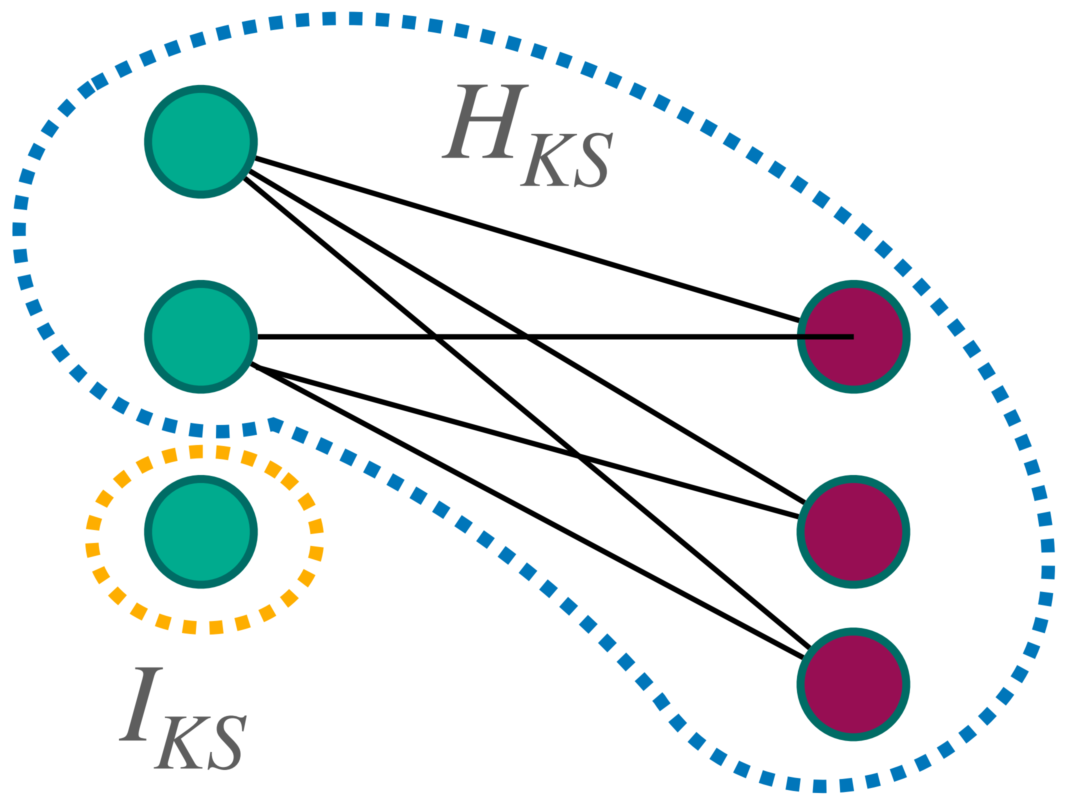

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Fact: Any k-minimal dependency must have $\geq 2k - 2$ non-zero entries

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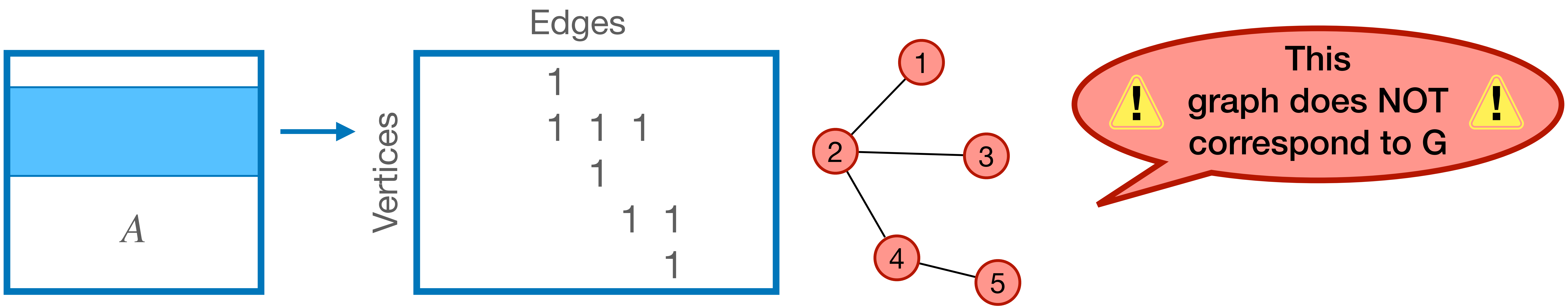

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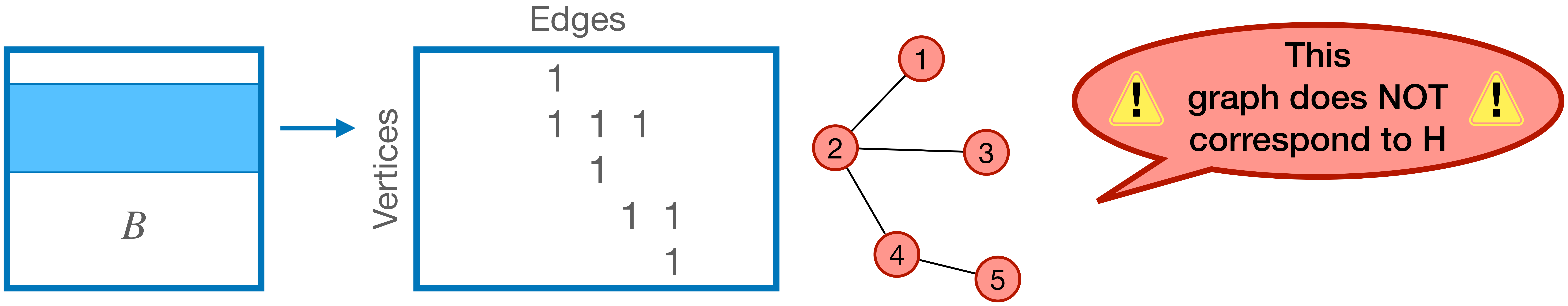

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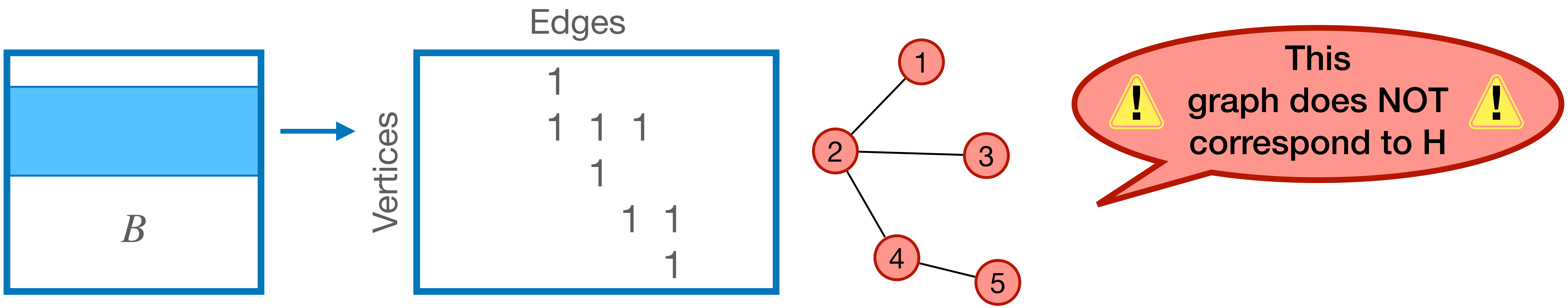

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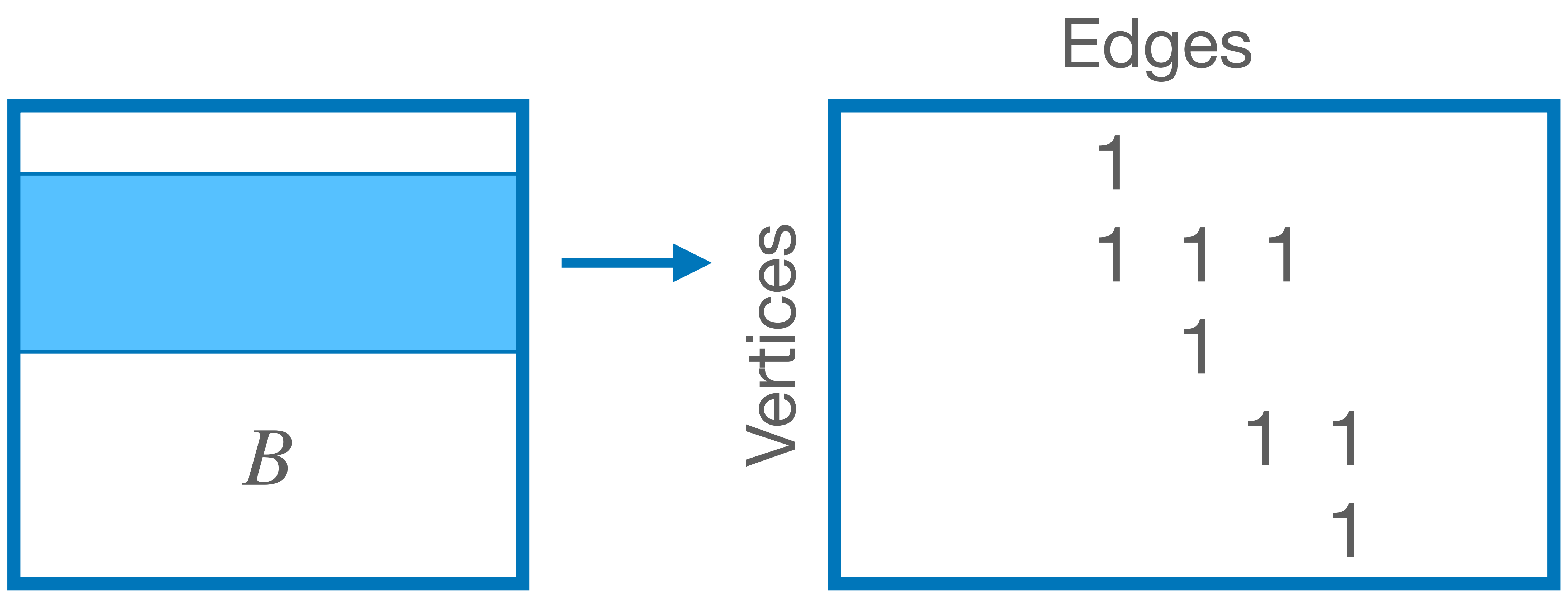

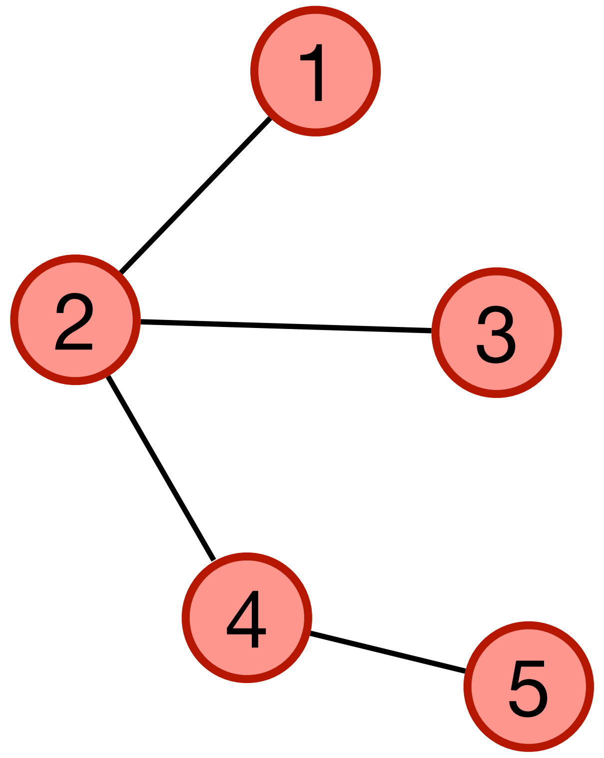

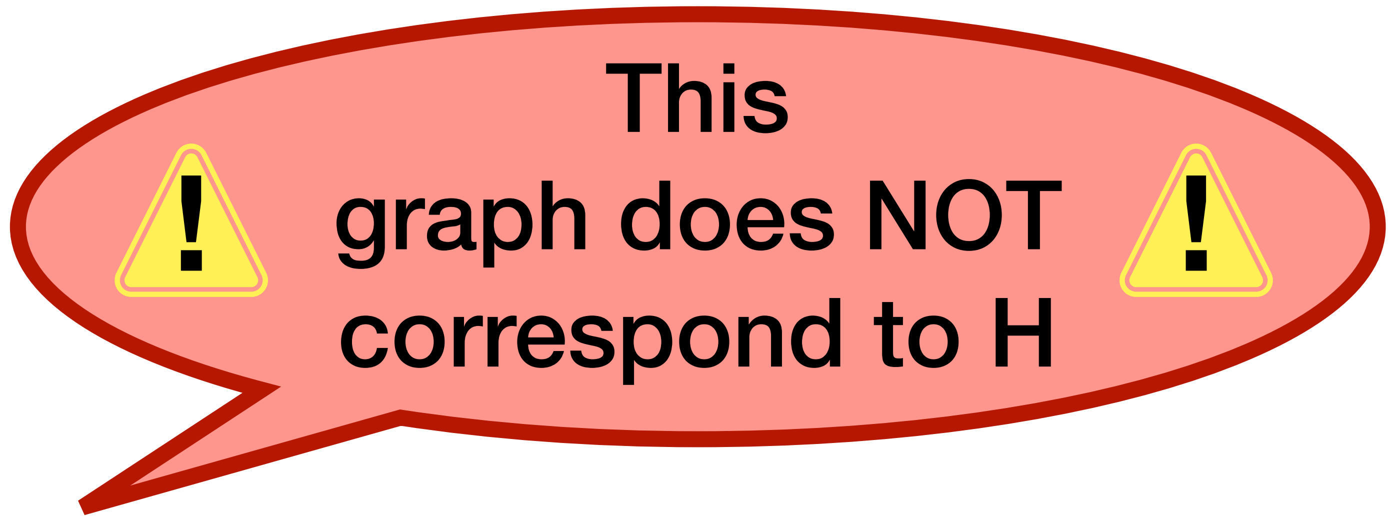

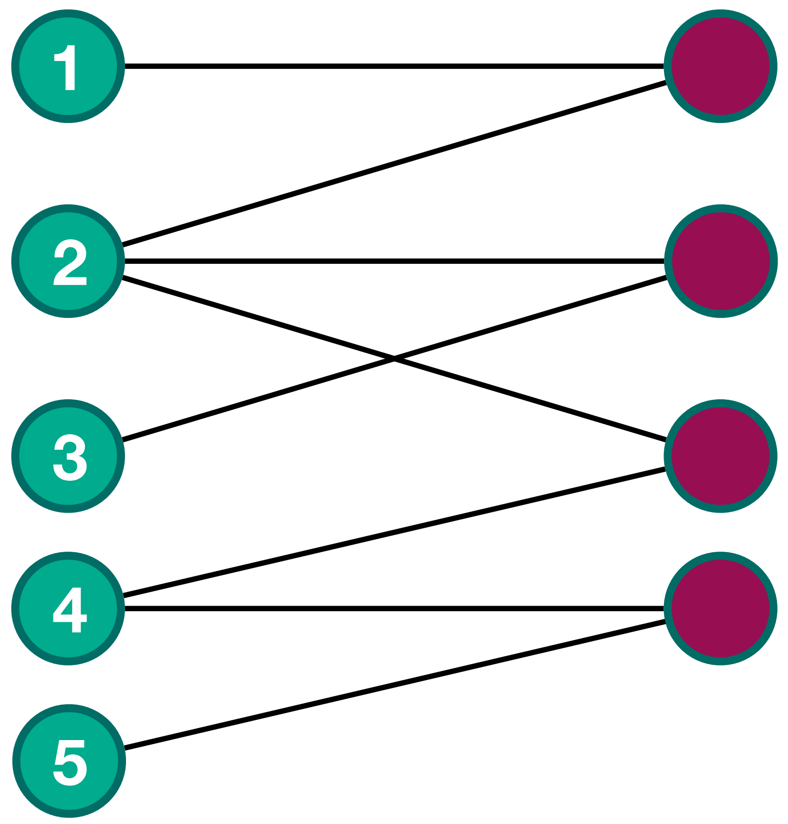

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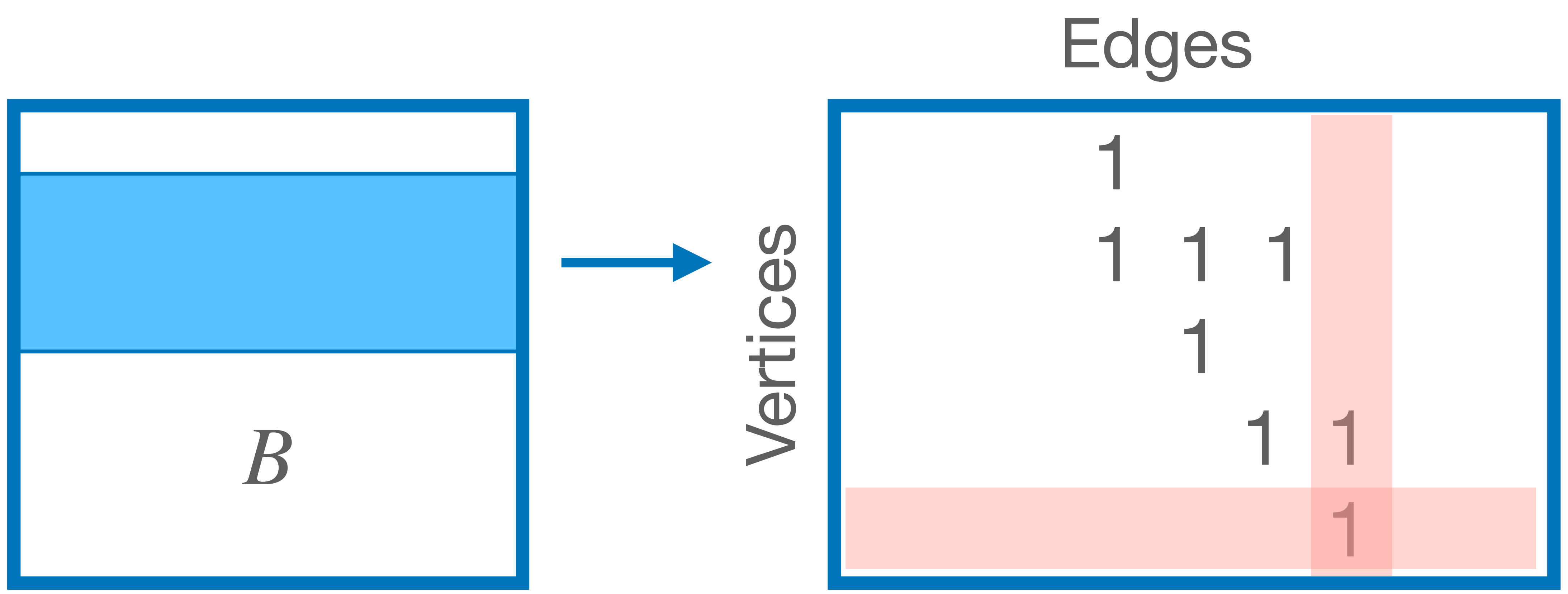

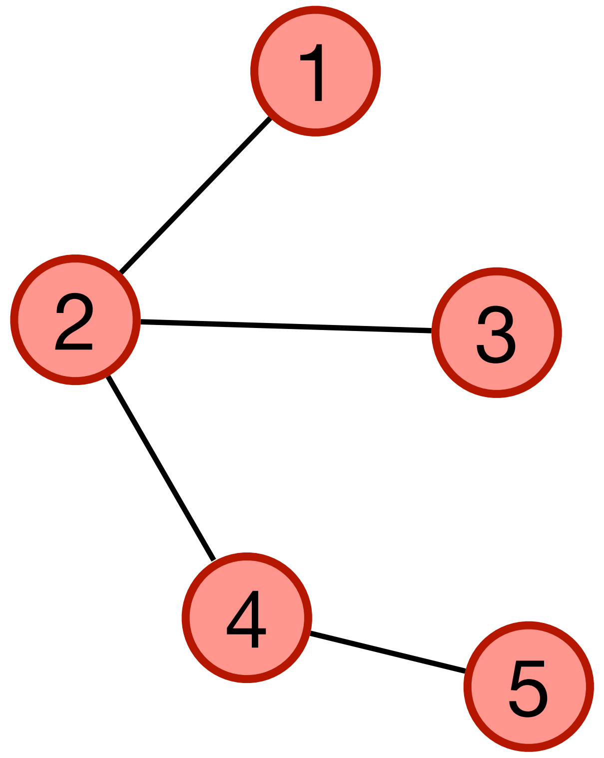

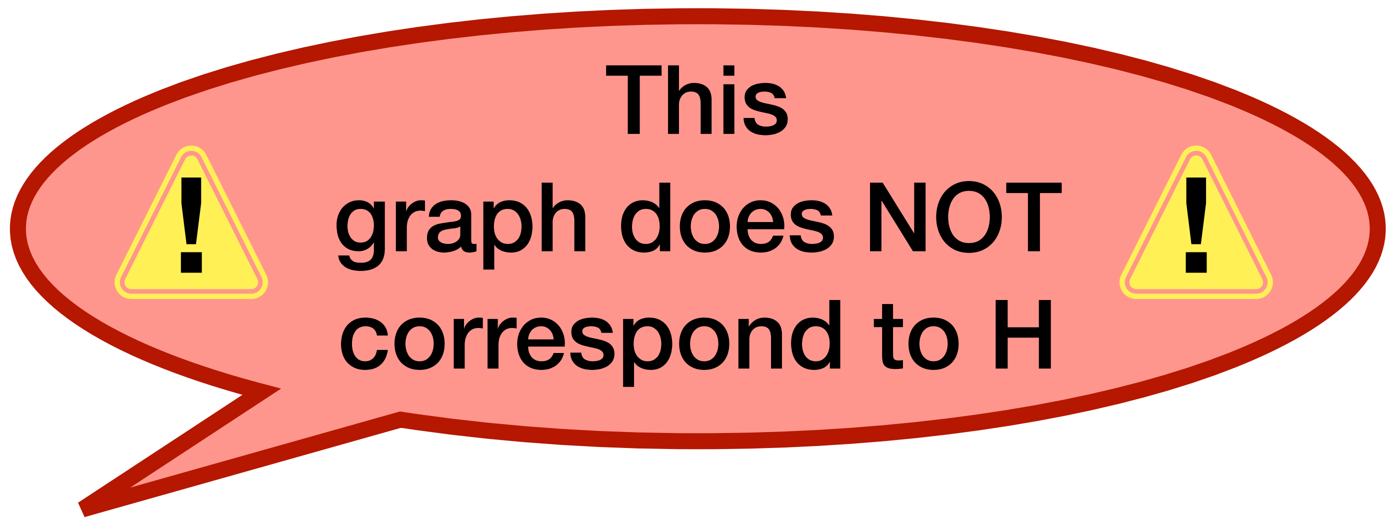

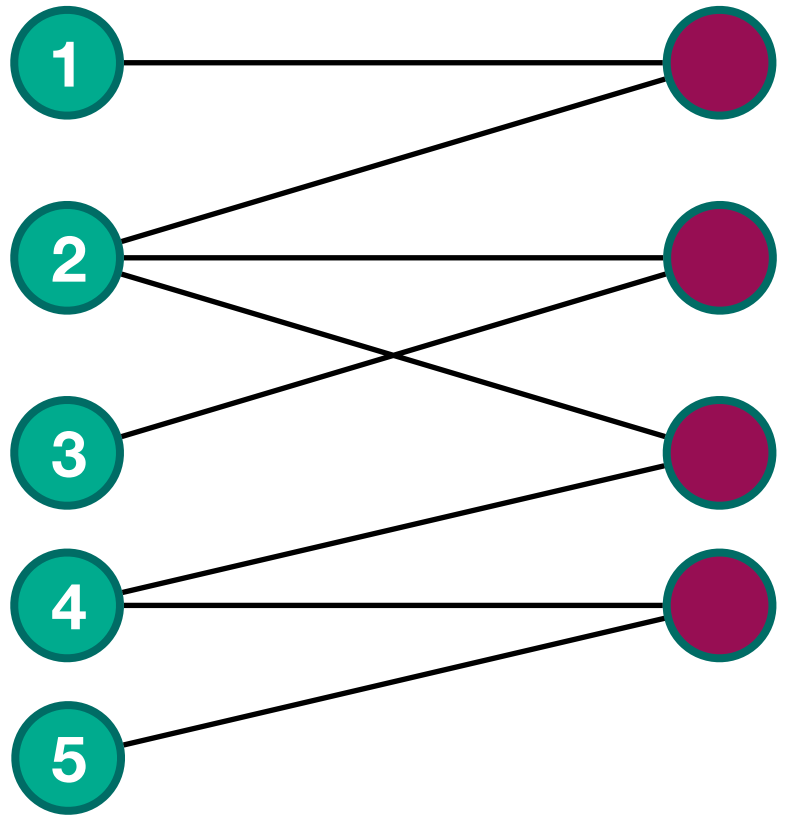

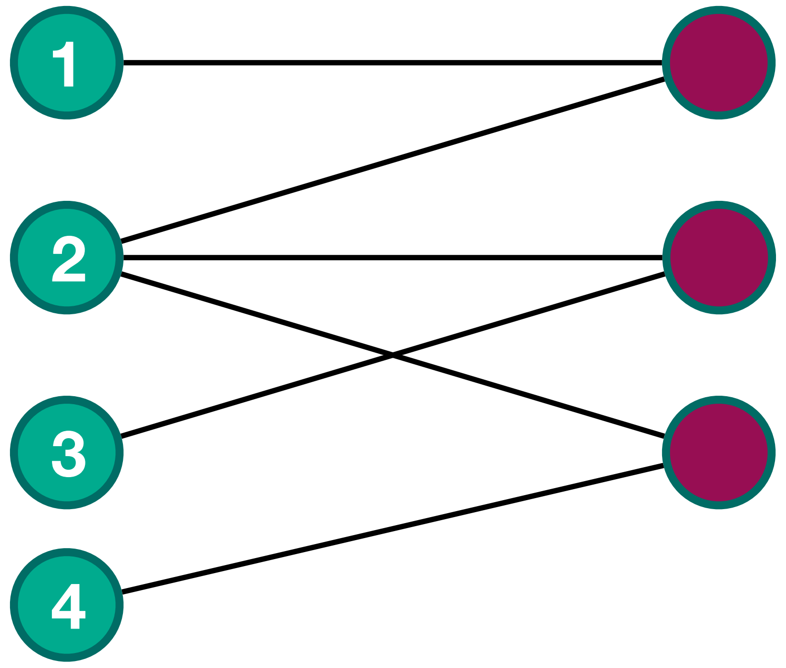

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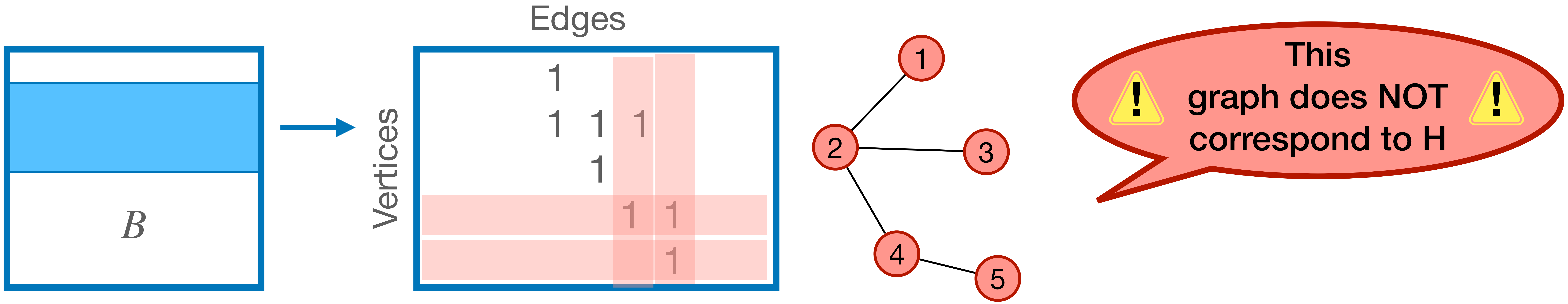

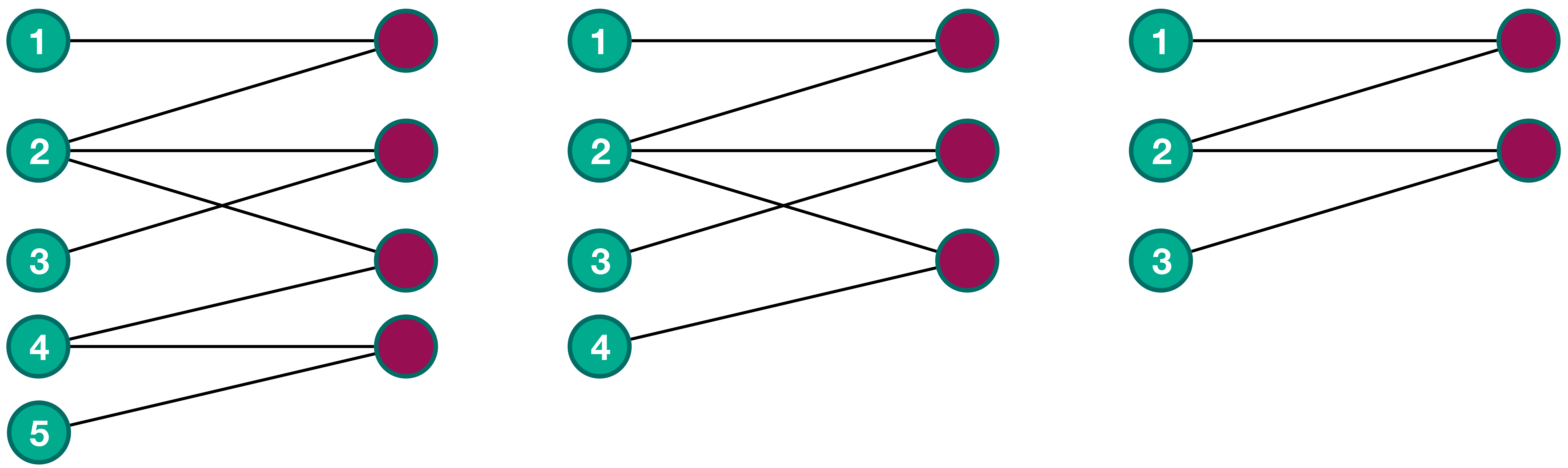

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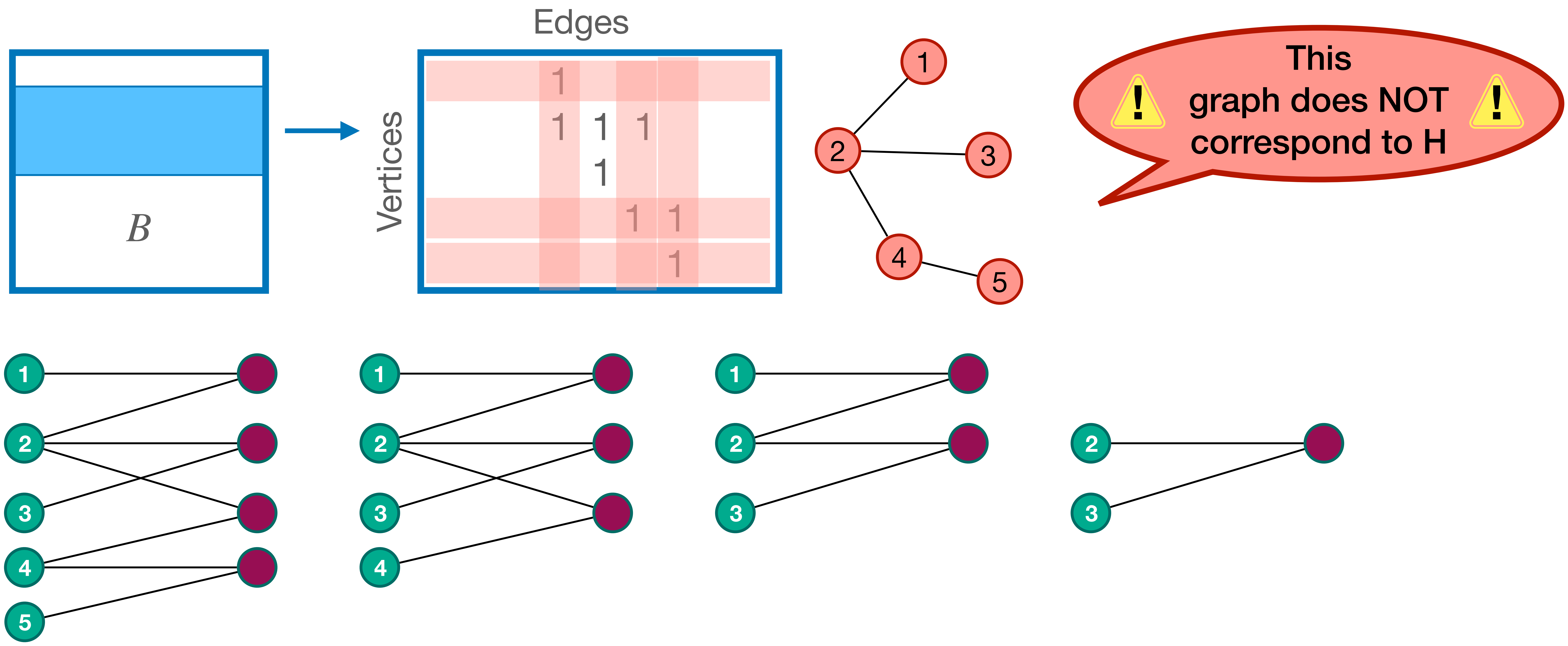

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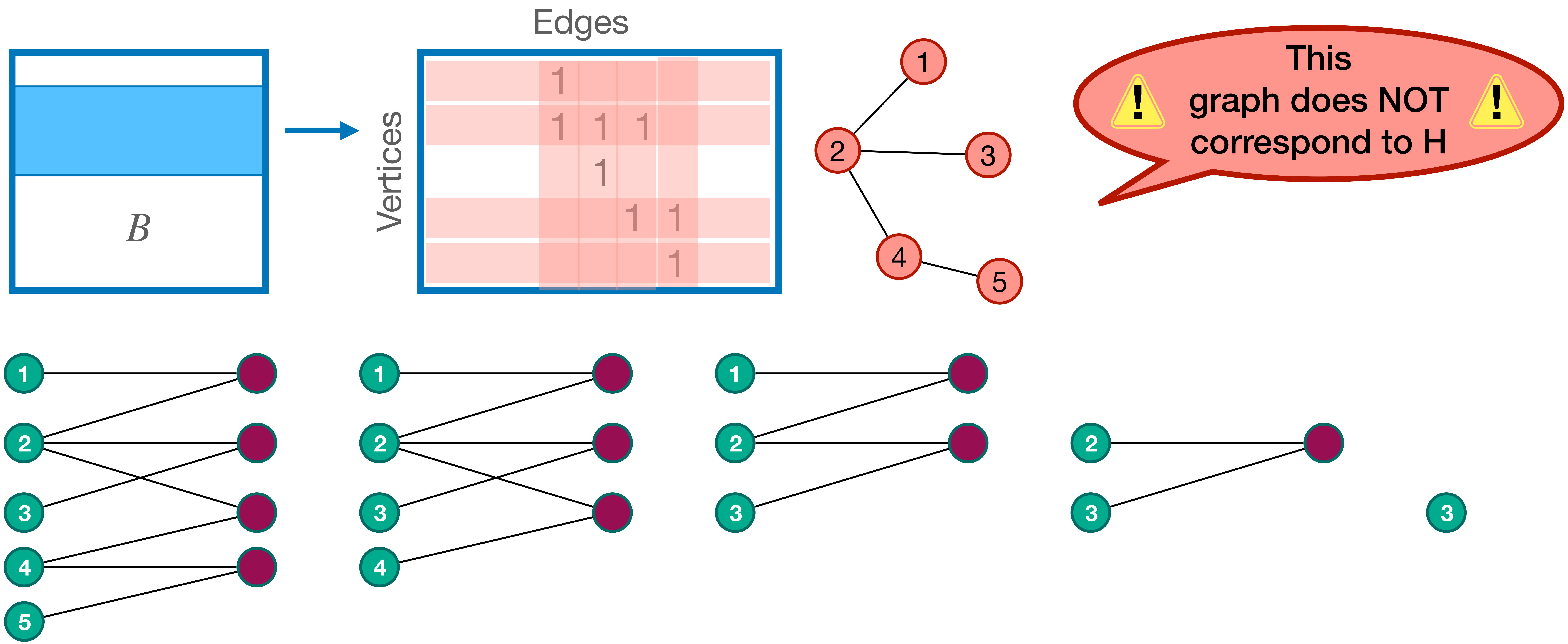

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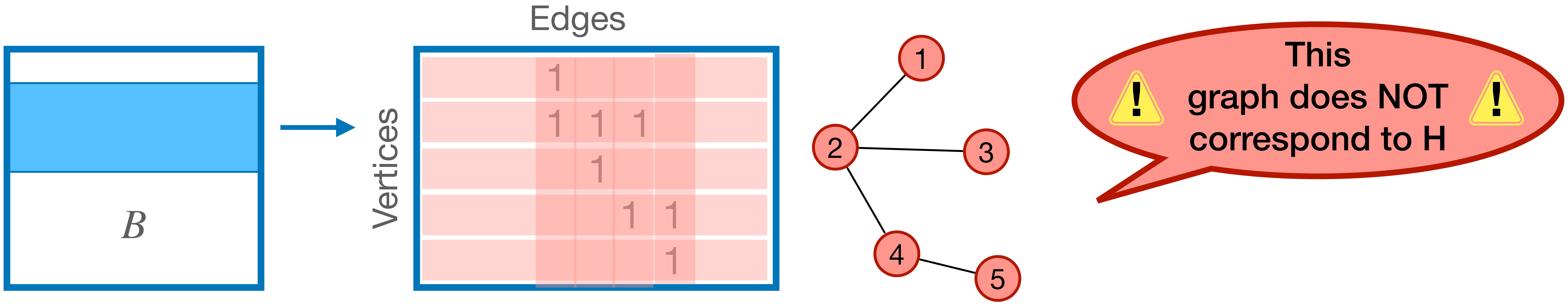

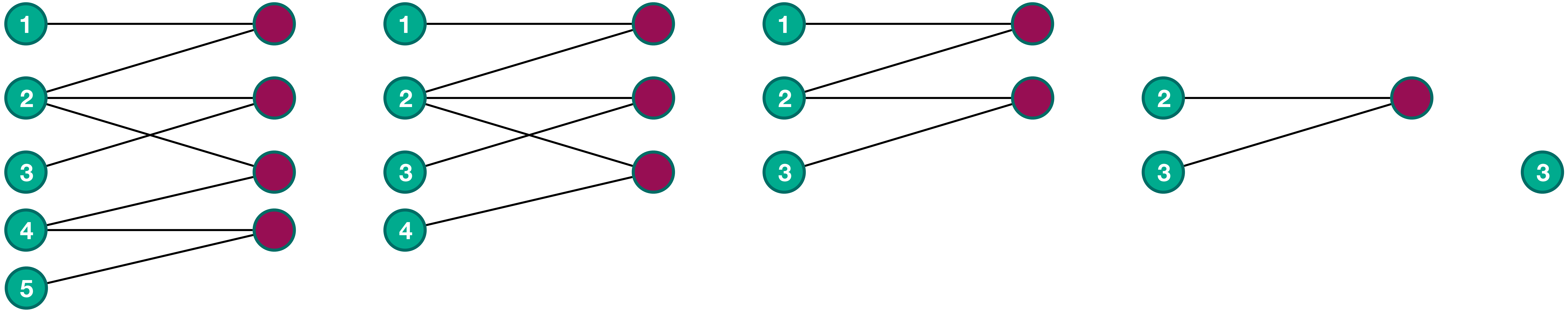

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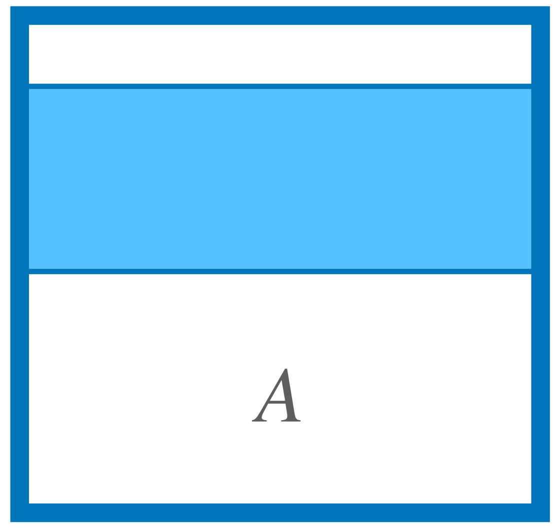

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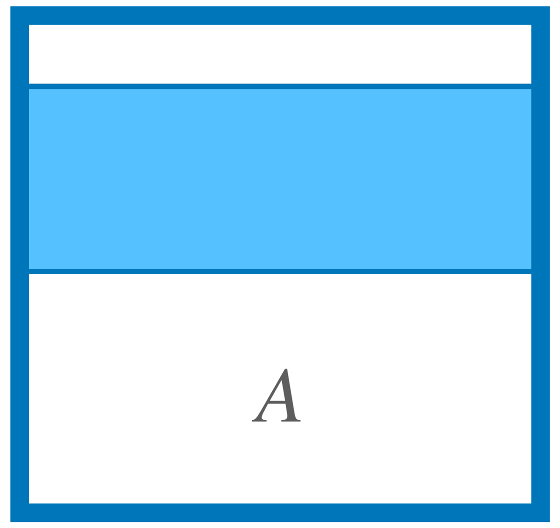

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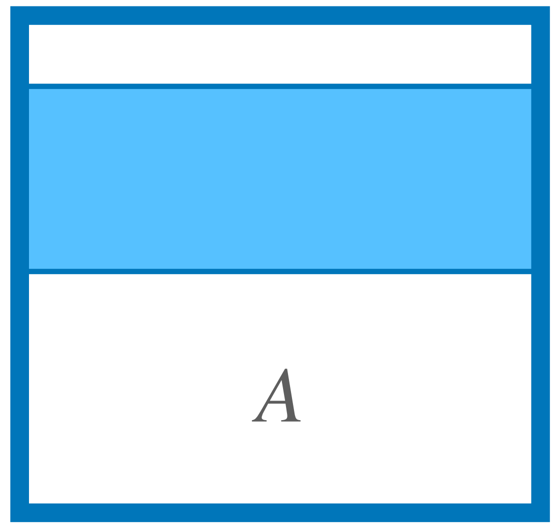

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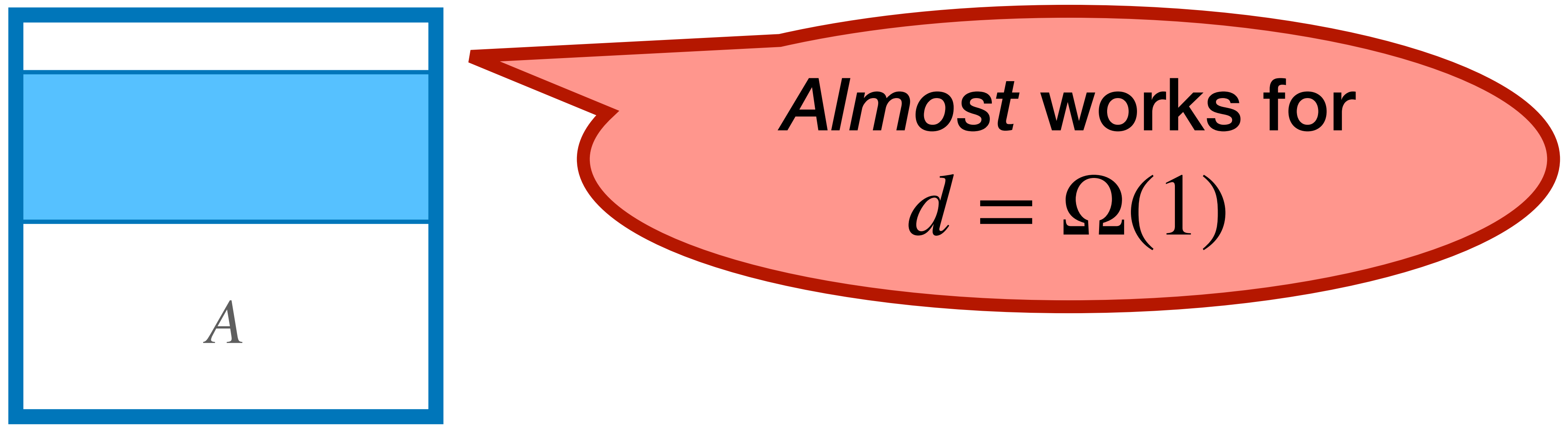

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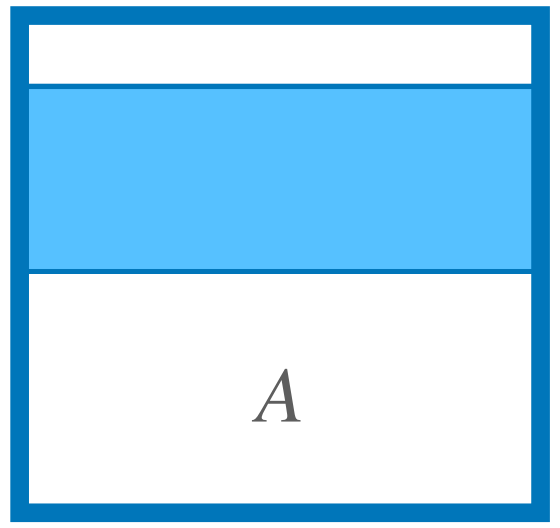

Almost works for $d = \Omega(1)$

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Requires $d = \omega(1)$

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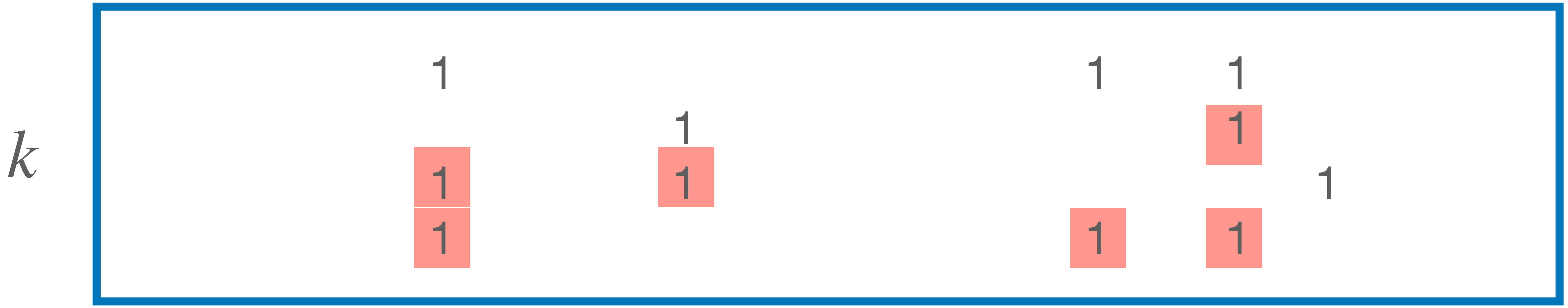

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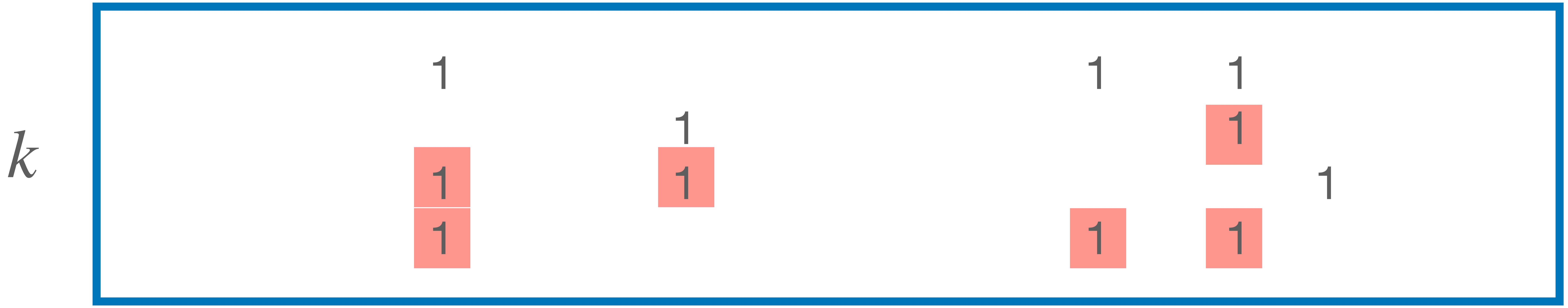

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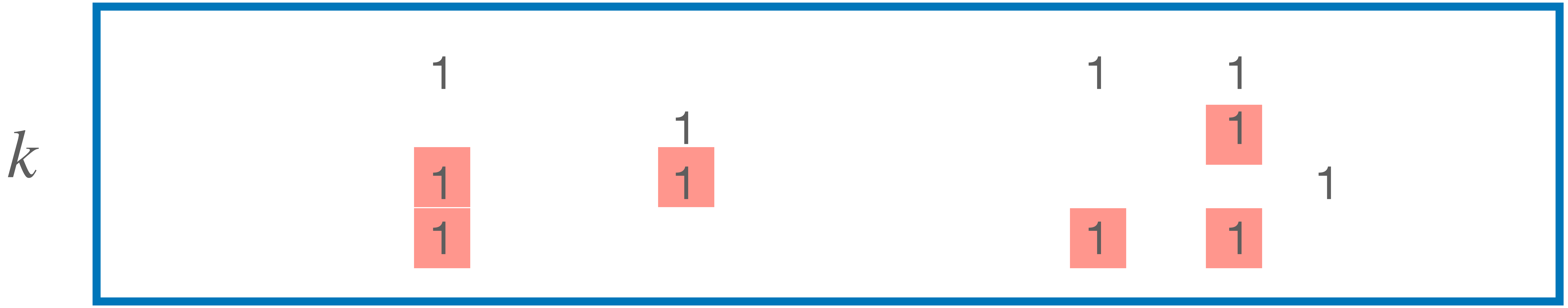

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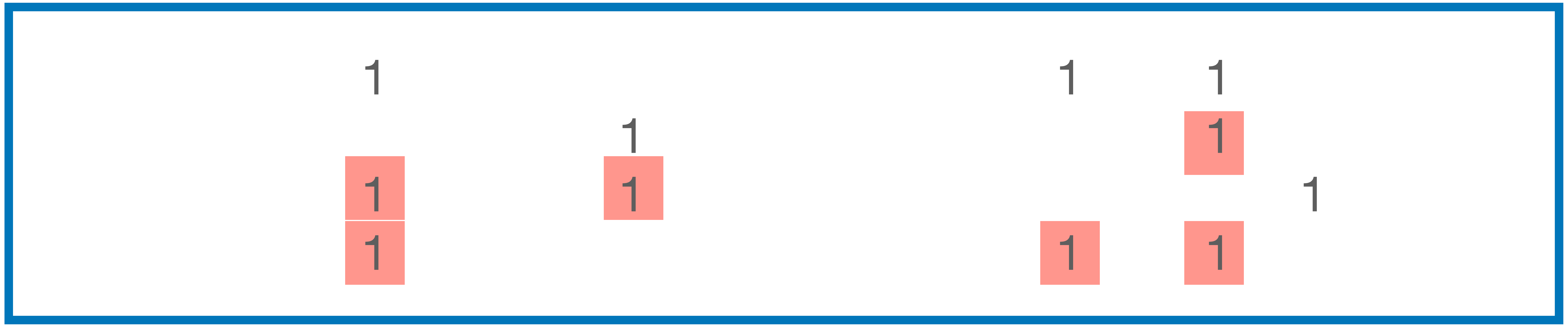
k



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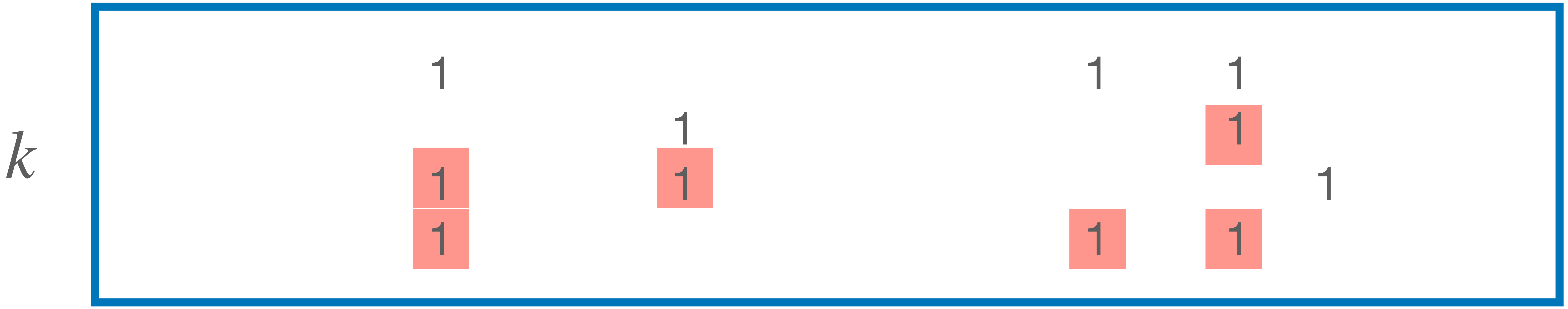

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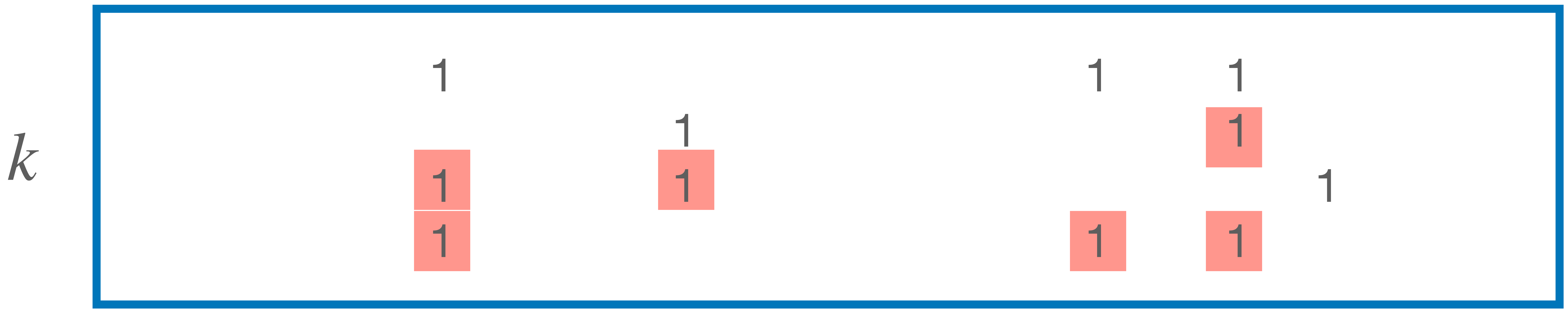

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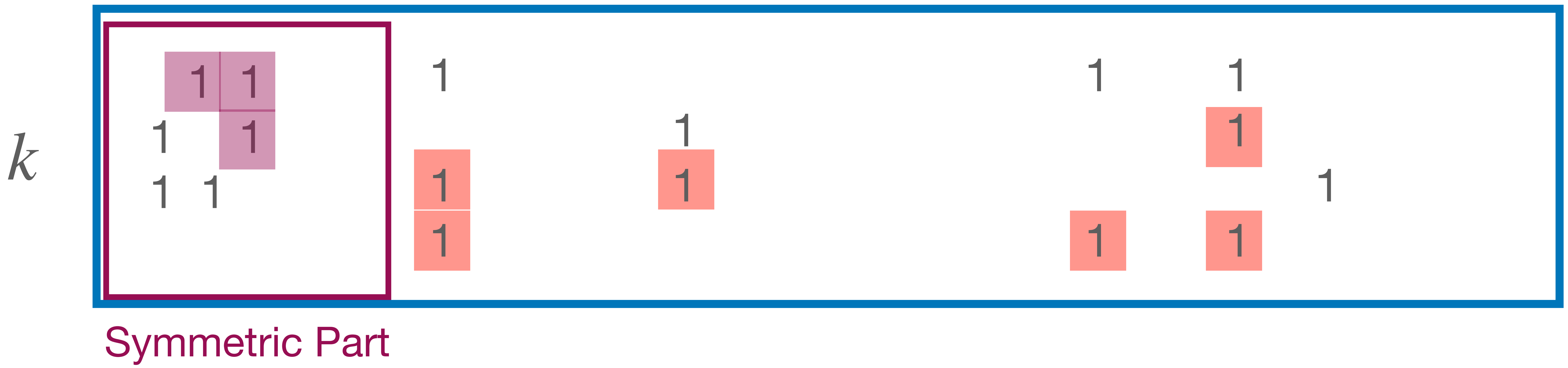

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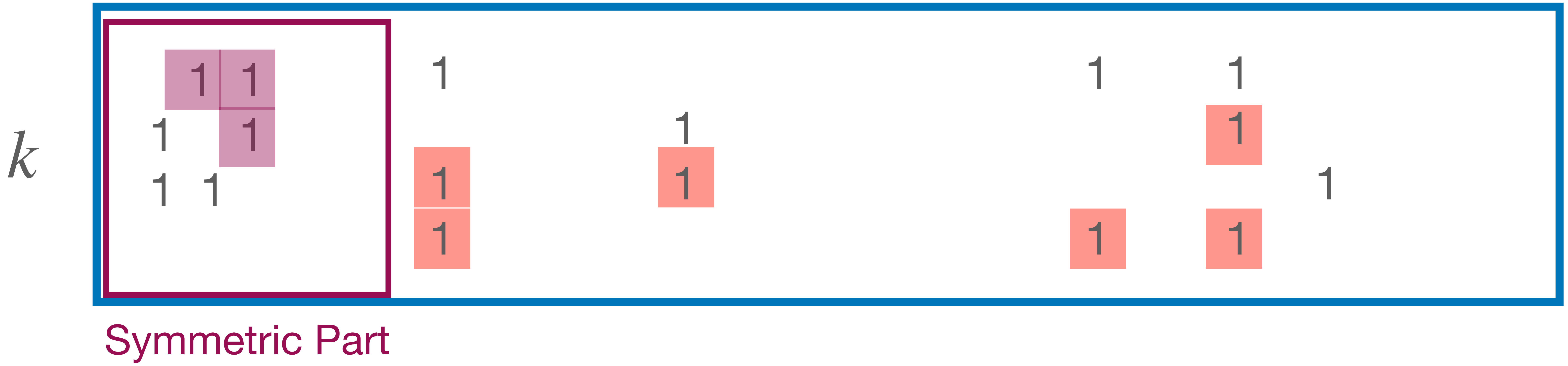

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Let $X_i \sim$ Bernoulli(p) for $i \in [n]$. Let $v \in \mathbb{R}^n$ have support of size at least m.
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Quadratic Littlewood Offord Theorem [Costello, Vu '06]:
Let $X_i \sim \text{Bernoulli}(p)$ for $i \in [n]$. Let $M \in \mathbb{R}^{n \times n}$ contain at least m columns with at least m non-zeros.
Then $\Pr[X^T M X = 0] \leq O\left(1/\sqrt[4]{pm}\right)$

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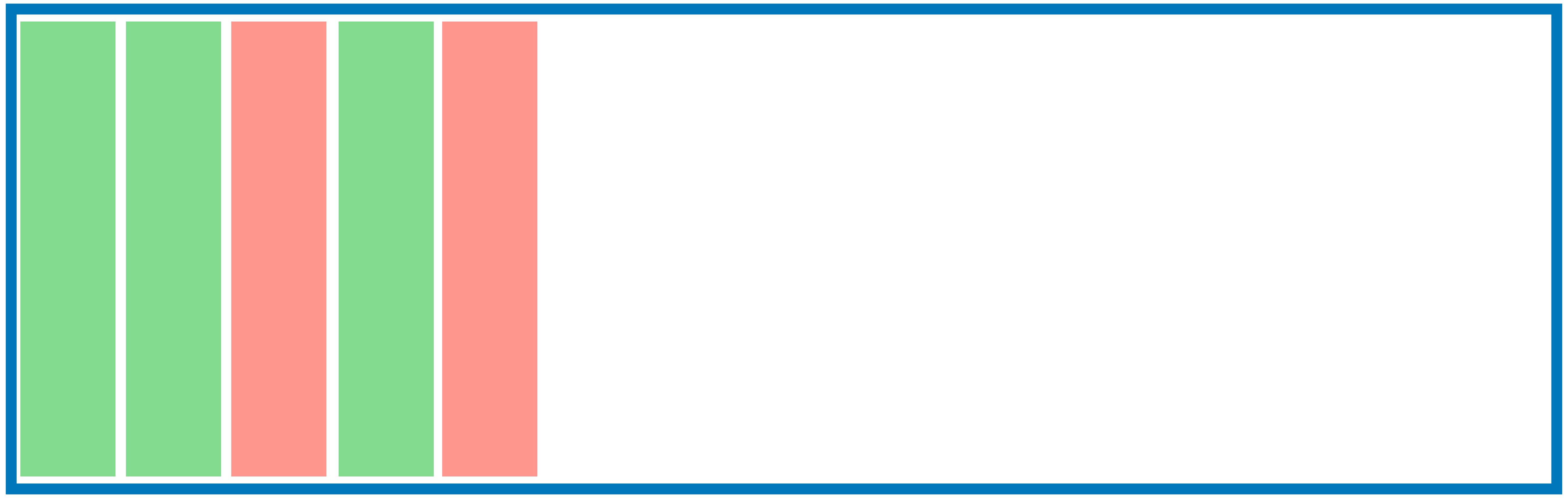

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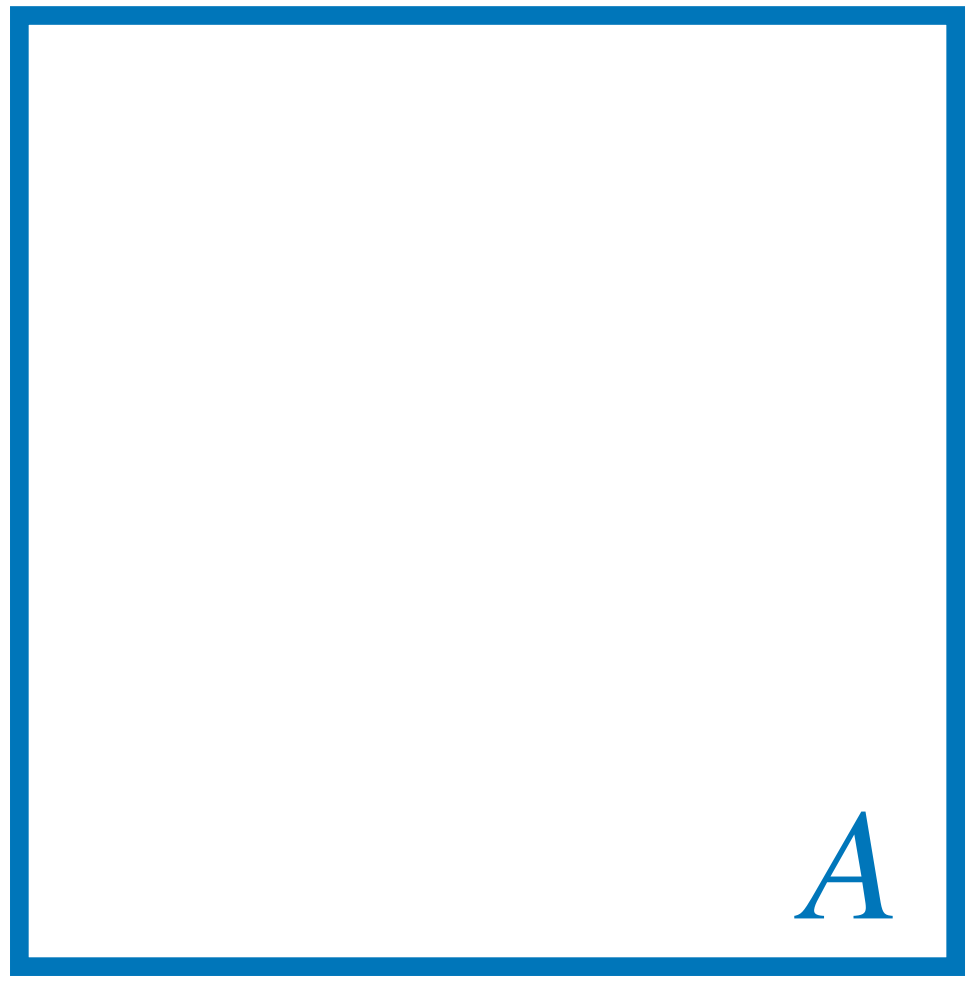

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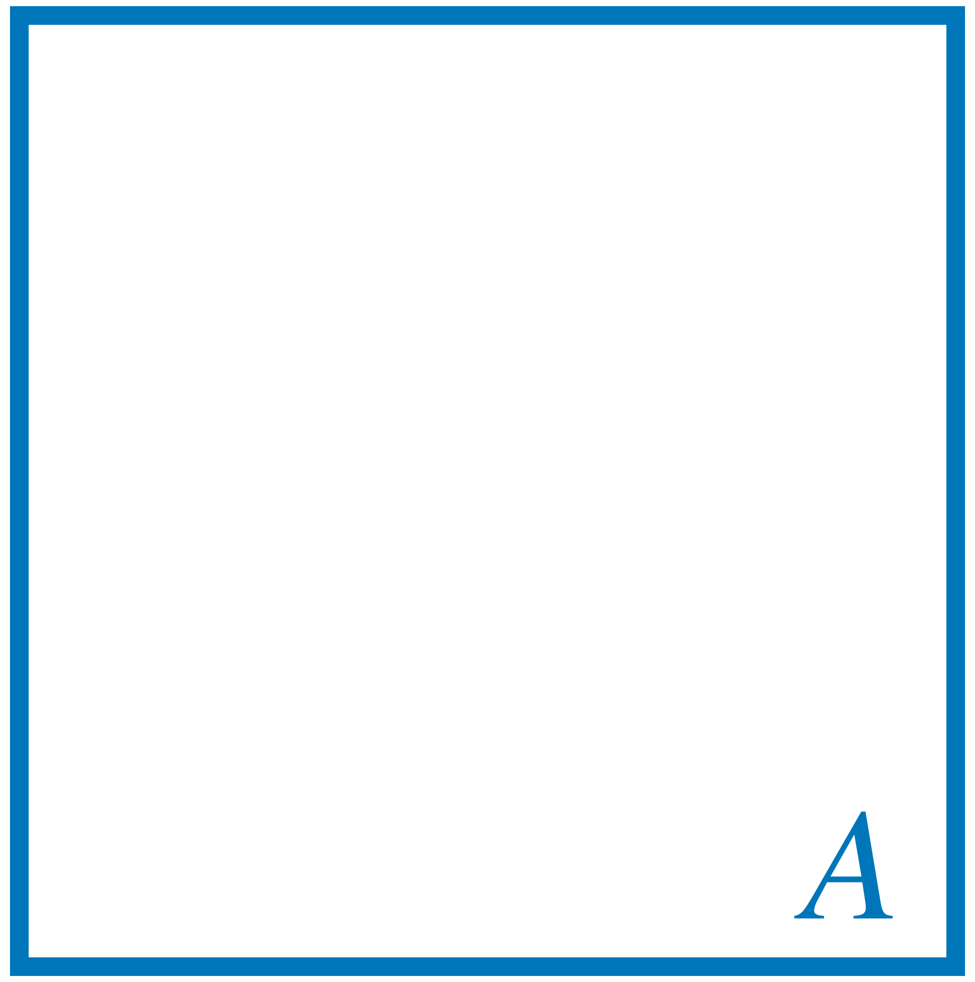

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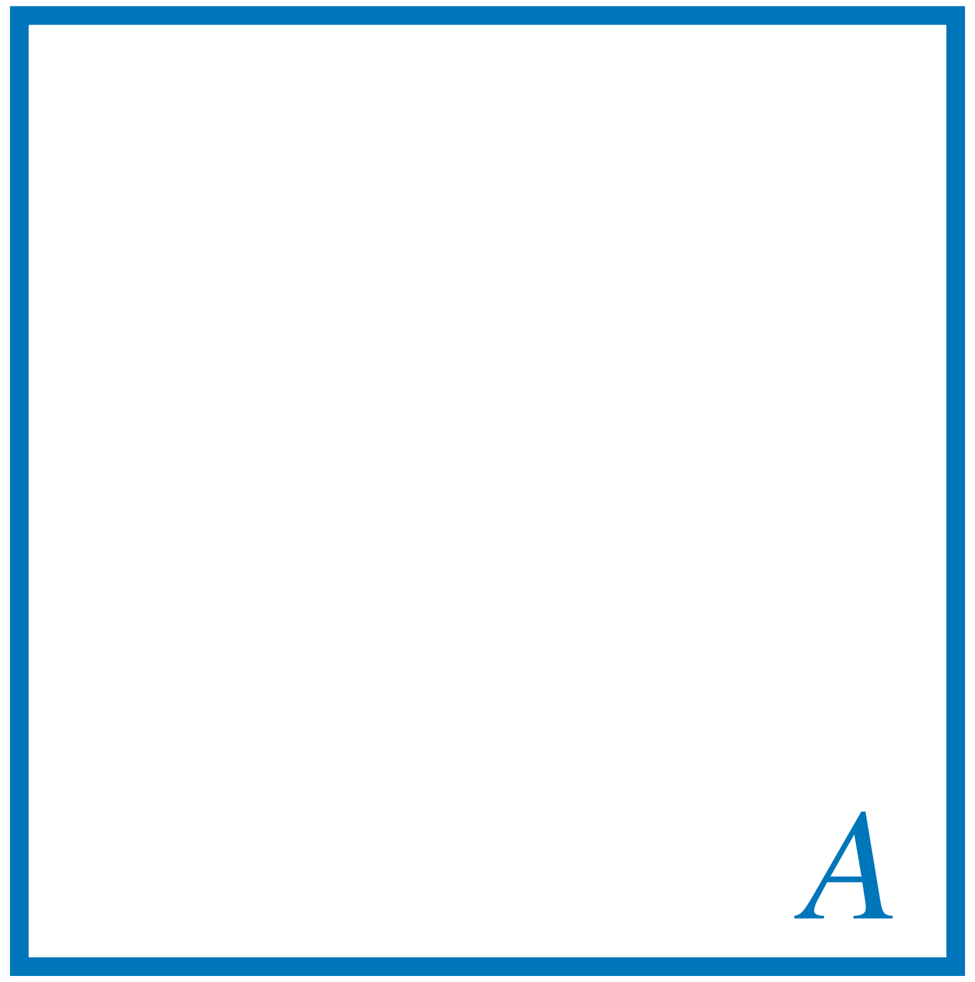

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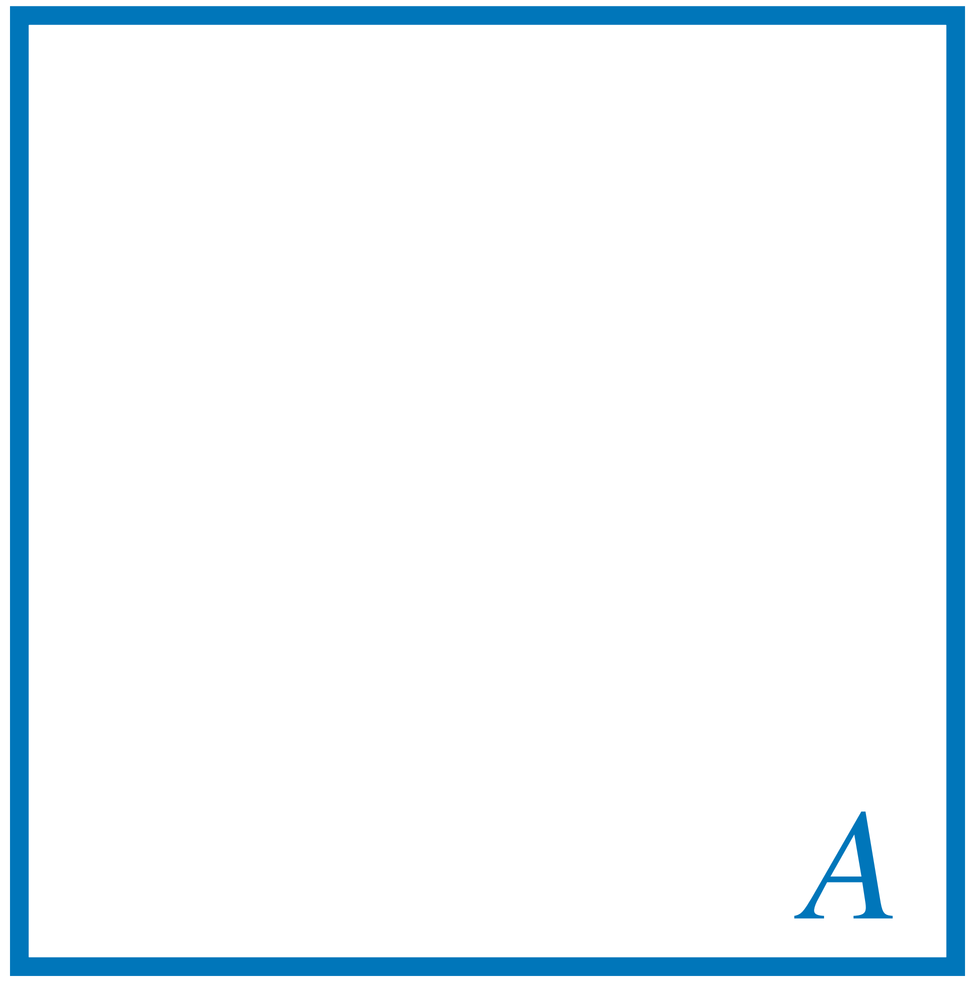

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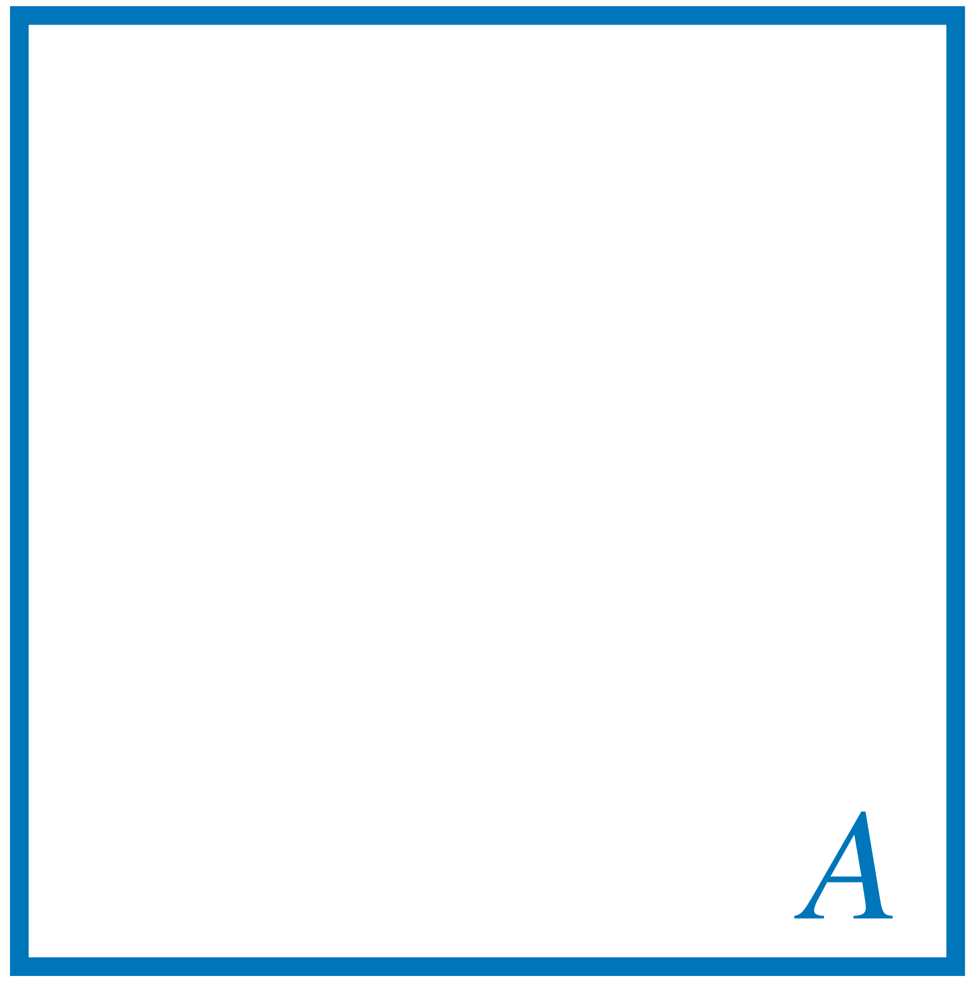

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Can find vector orthogonal to all columns besides A_i

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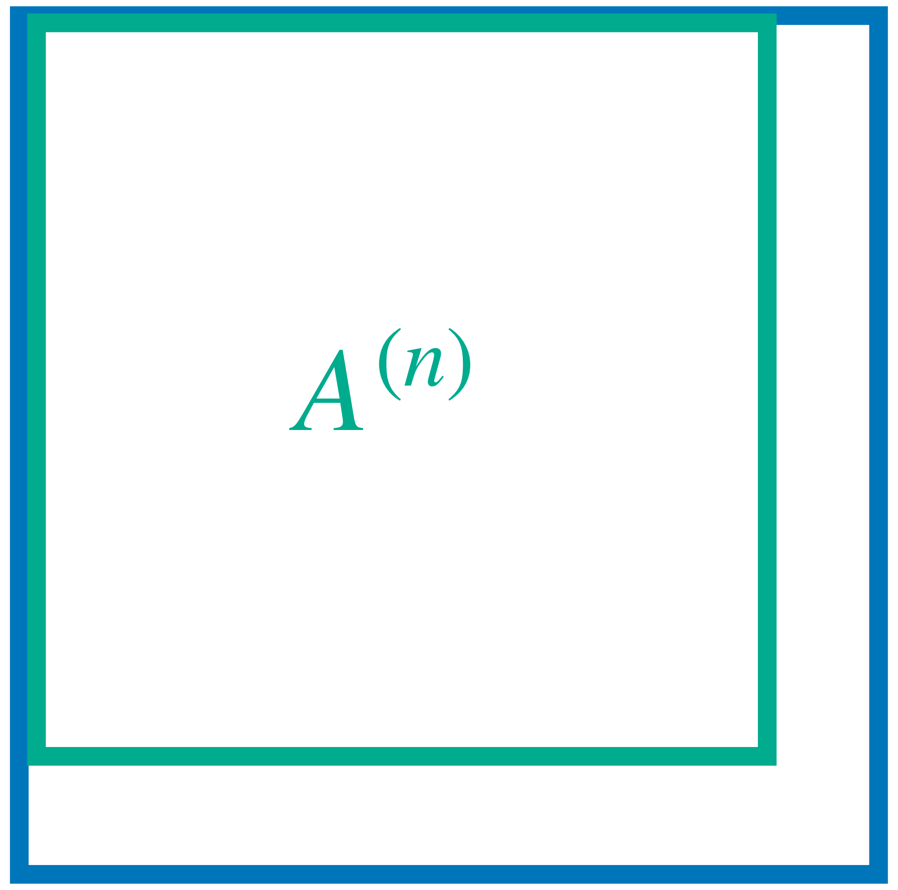

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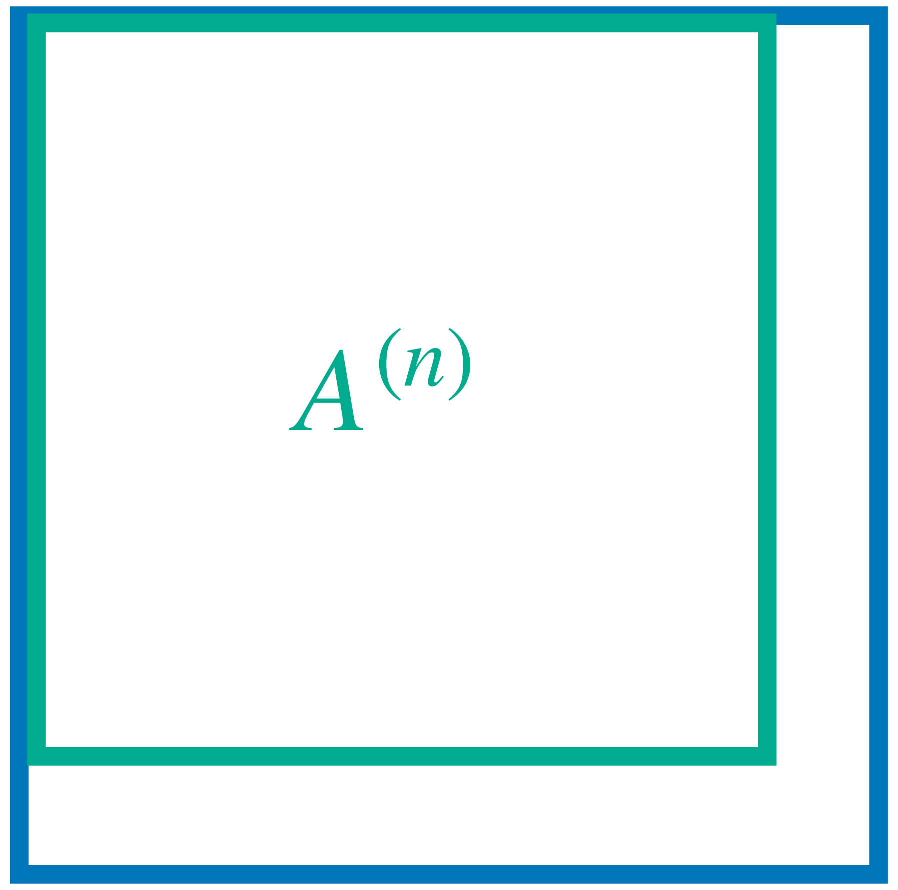

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Case 1: $A^{(n)}$ **has kernel vector with large support**

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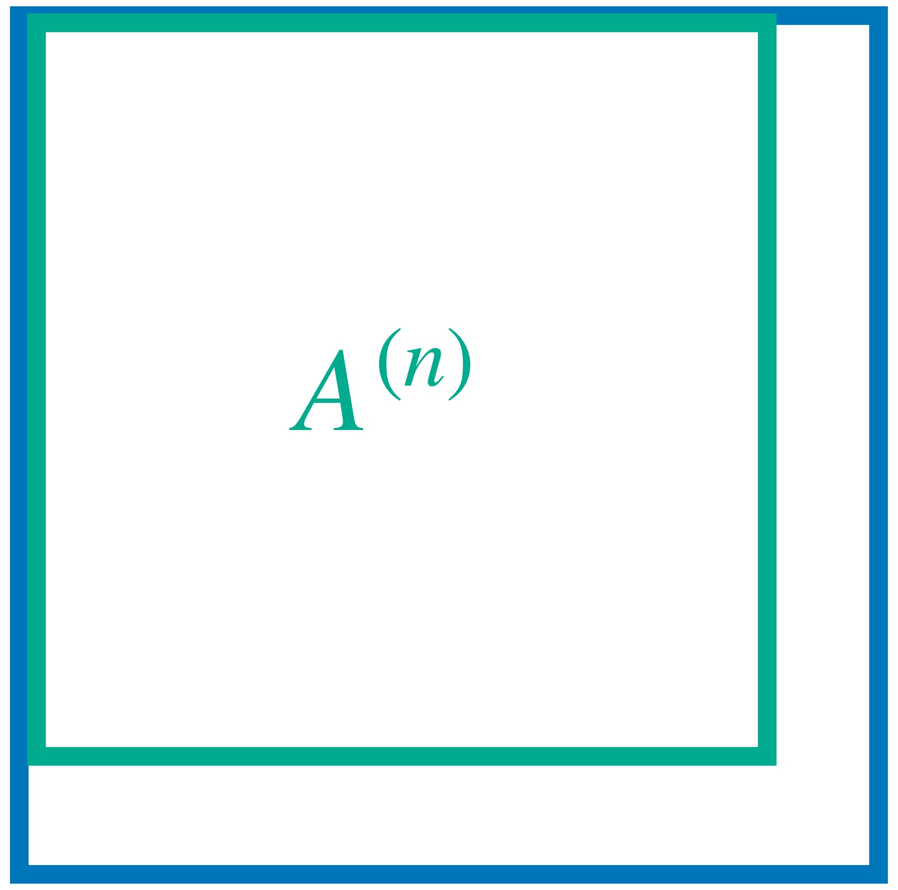

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Case 1: $A^{(n)}$ **has kernel vector with large support**

Case 2: $A^{(n)}$ **has no kernel vector with large support**

Case 1: $A^{(n)}$ has kernel vector v with large support

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Goal: Bound

$$\Pr[A_n \in \text{Span}(\{A_i\}_{i<n})]$$

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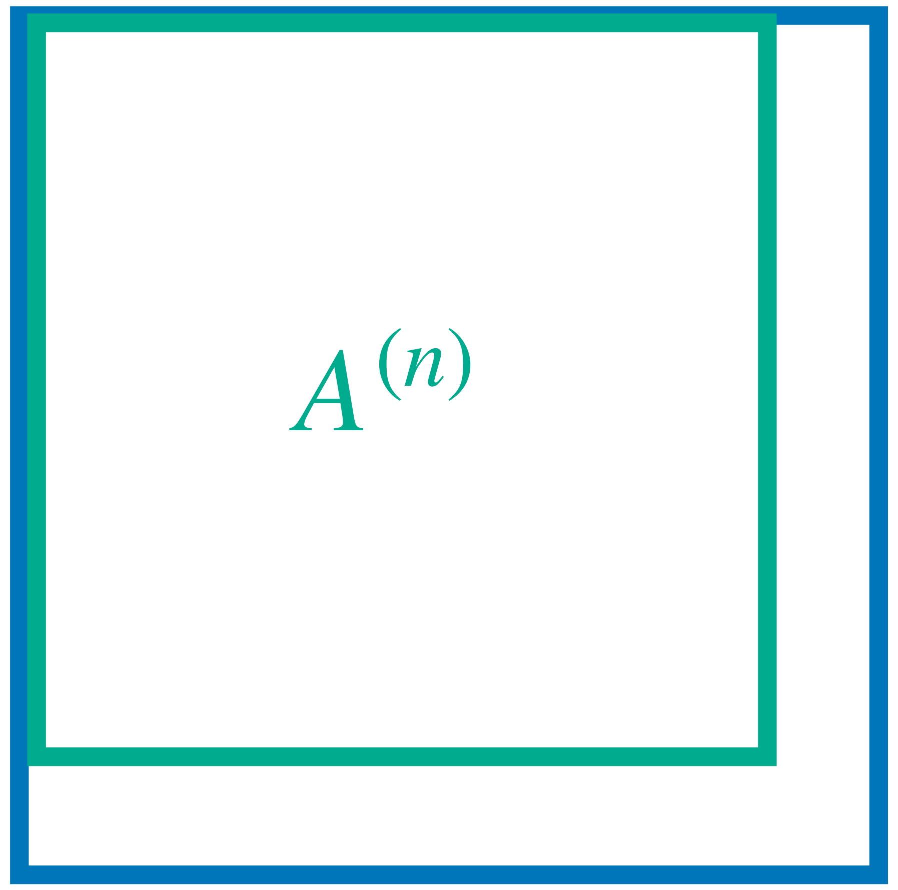

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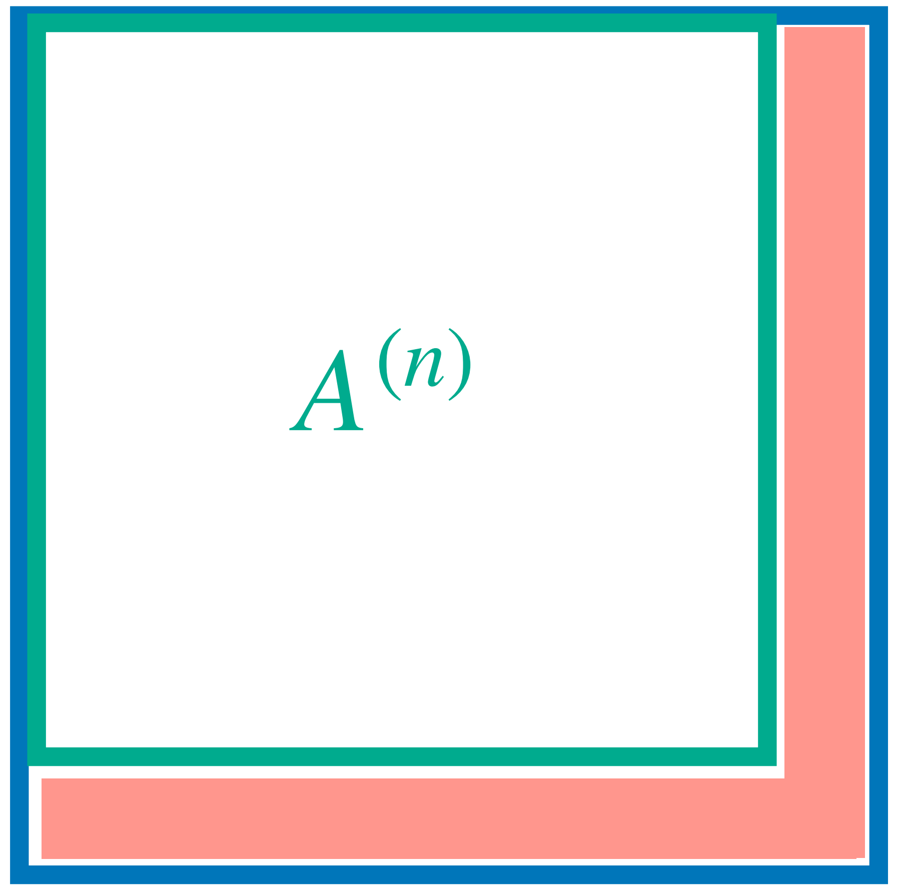

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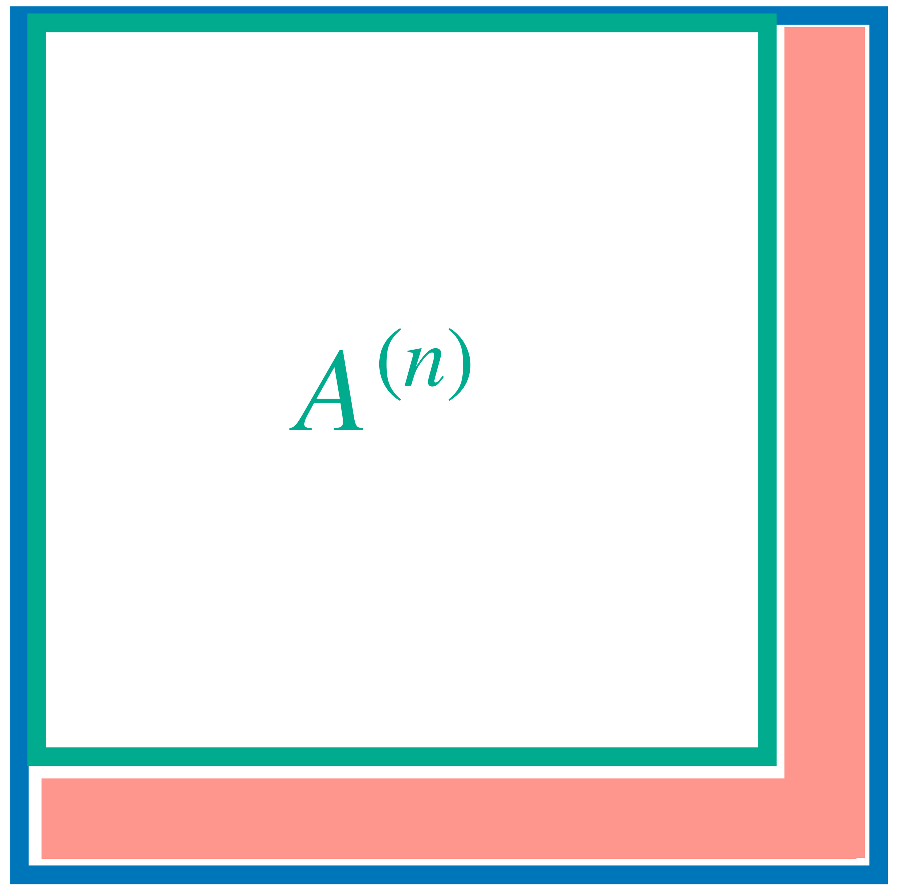

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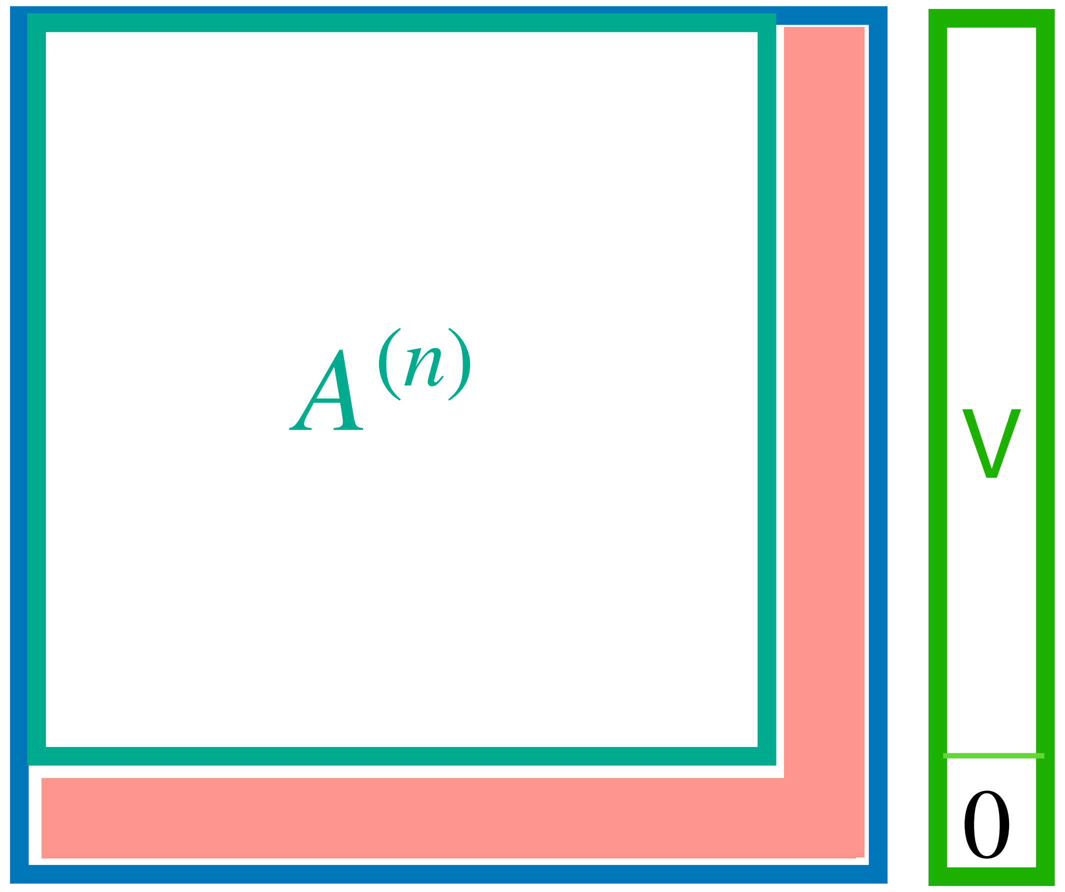

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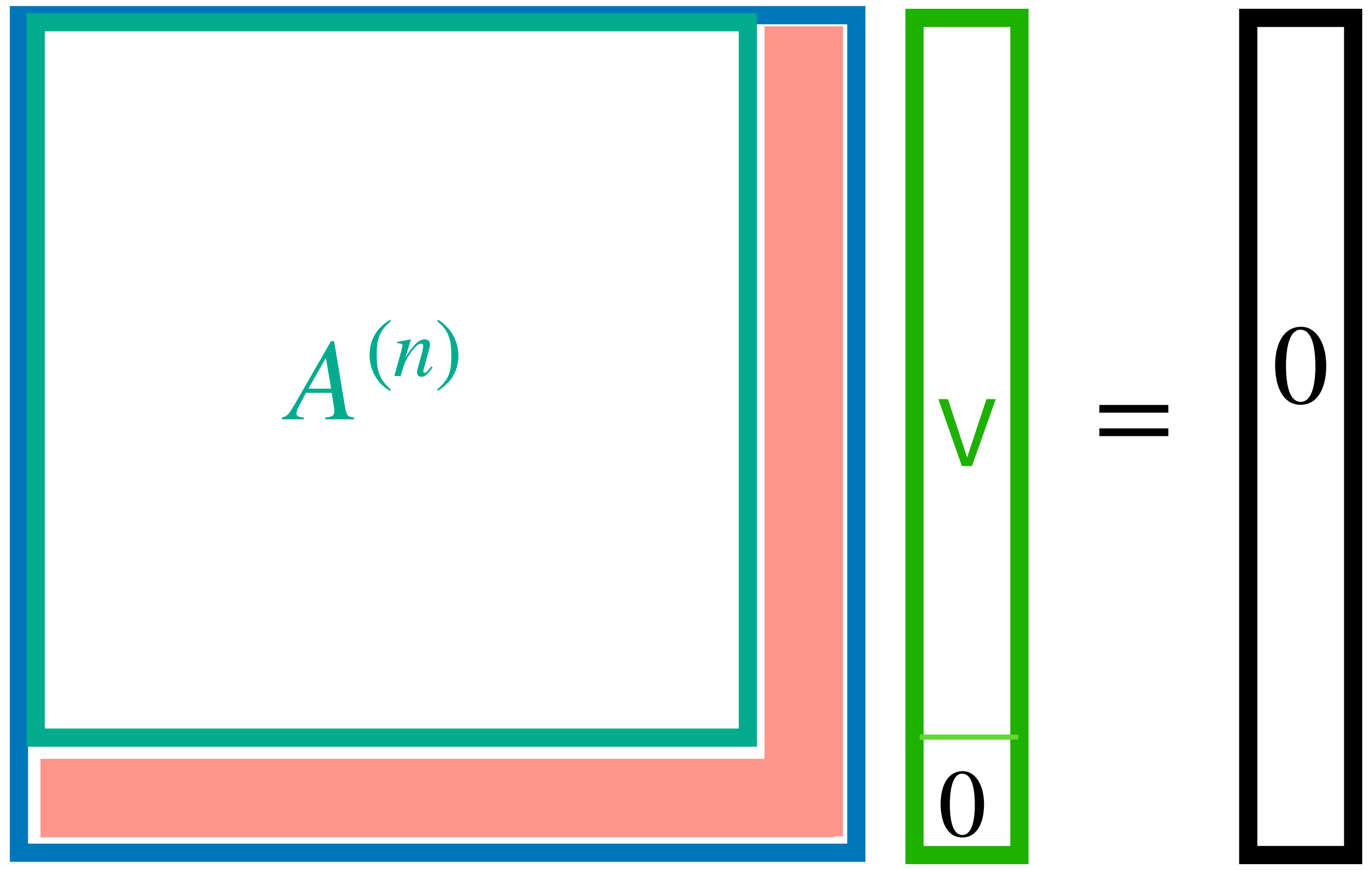

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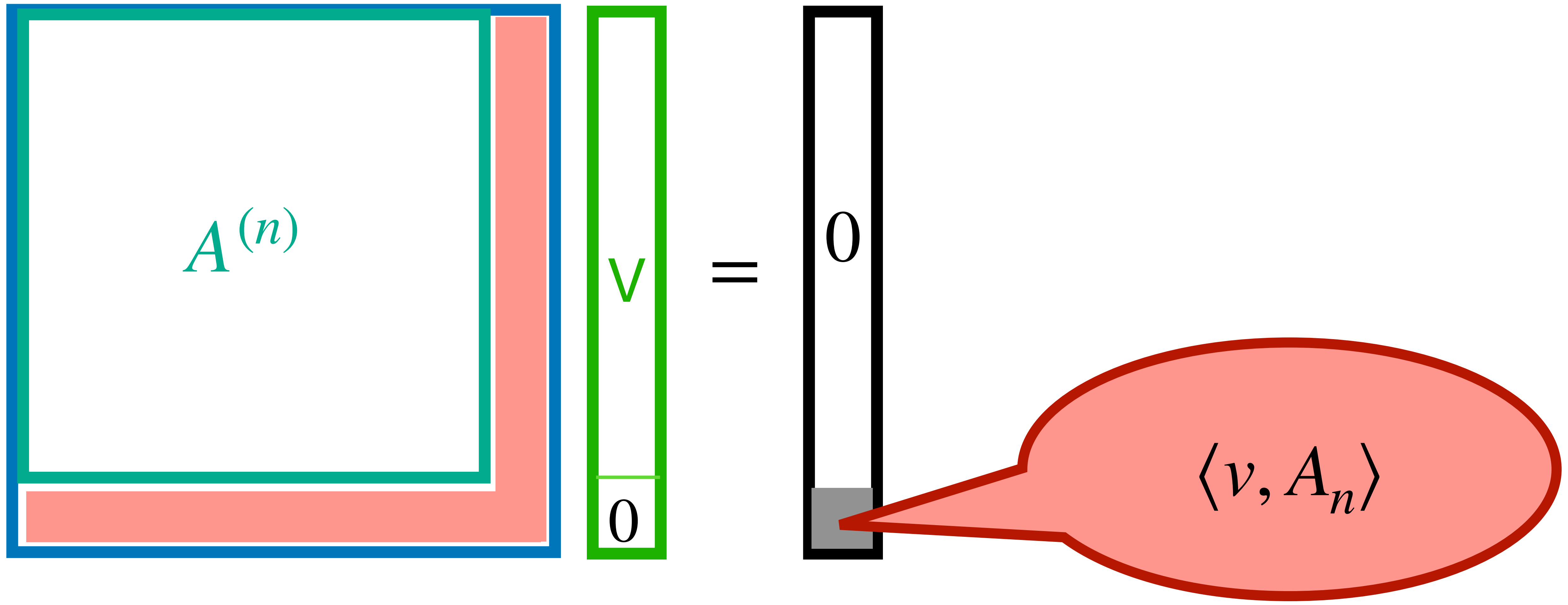

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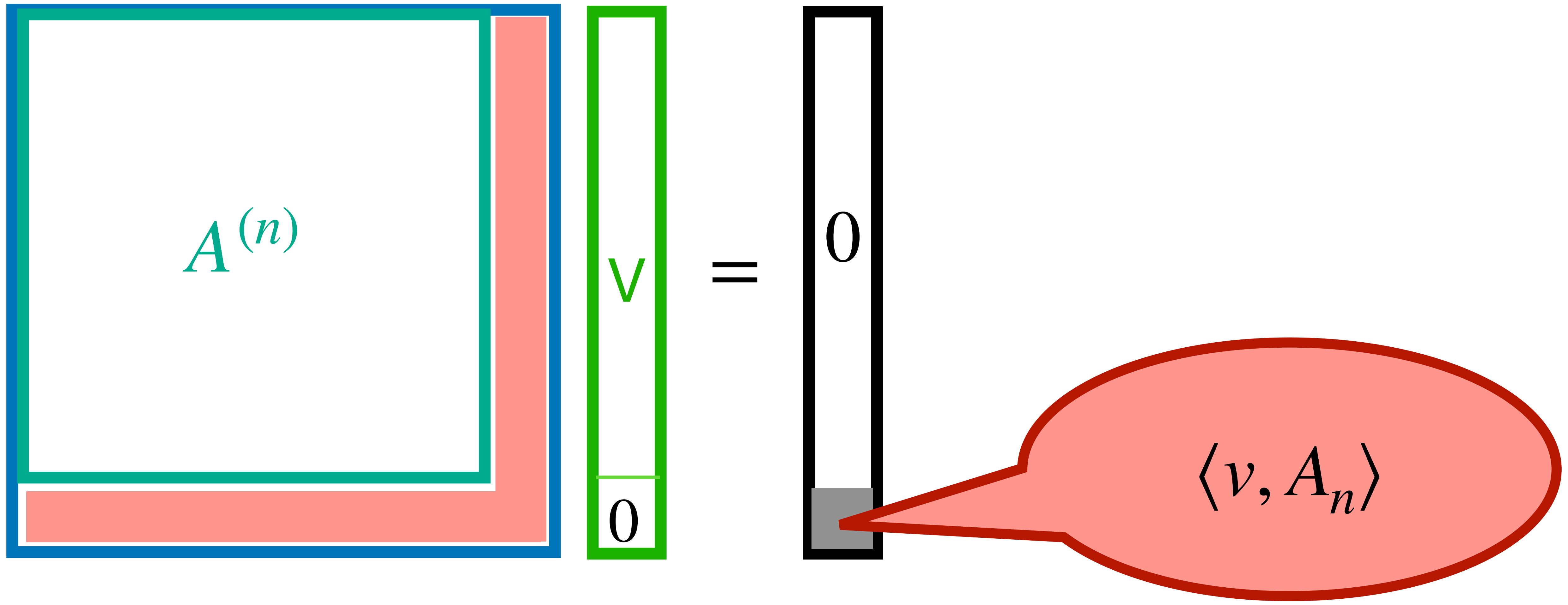

Sparse Littlewood-Offord:

Then $\Pr[A_n^T v = 0] \leq O\left(1/\sqrt{d}\right)$

Conclusion

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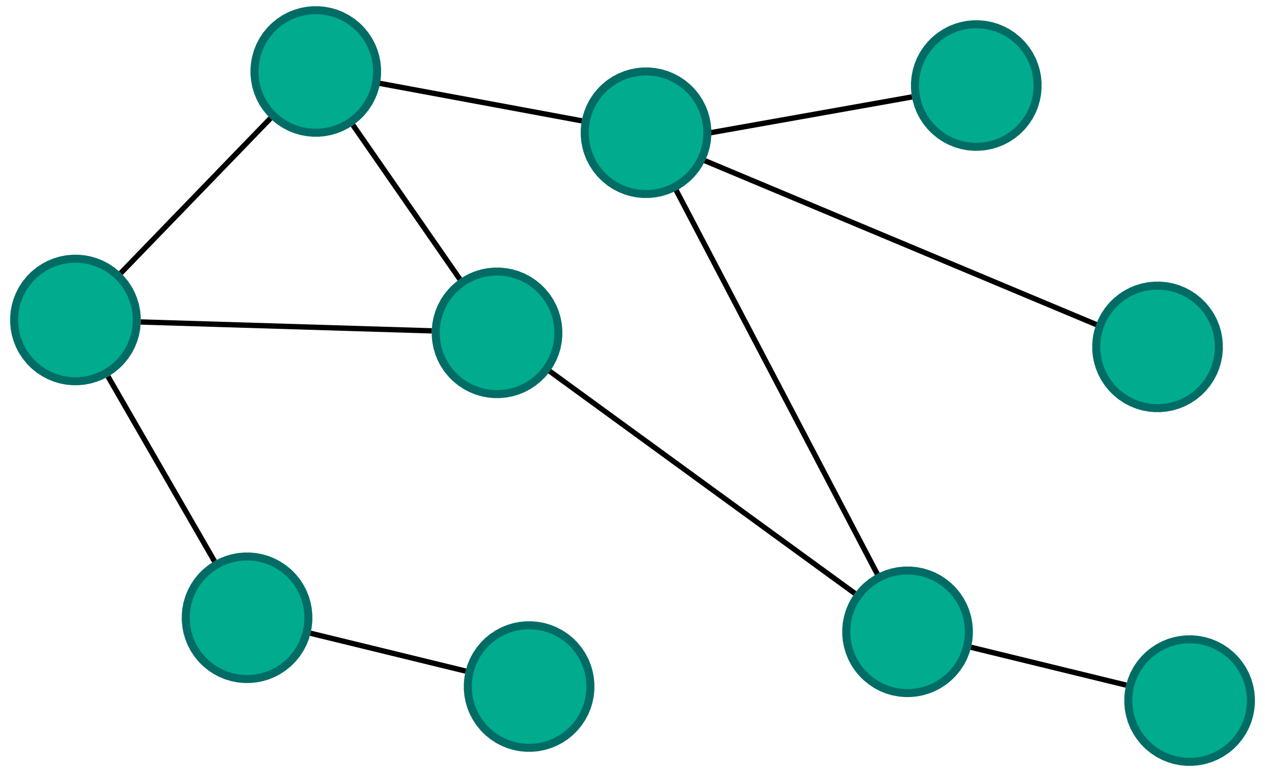

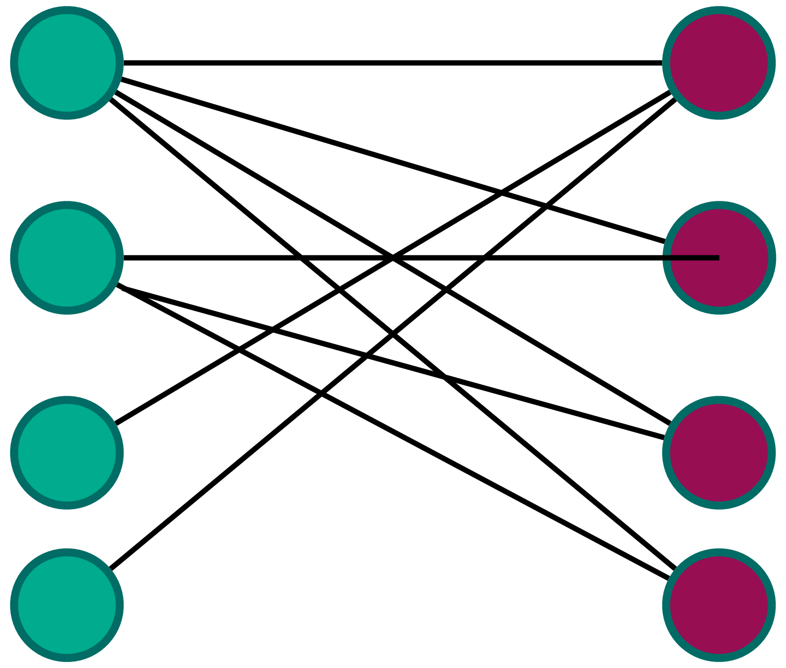

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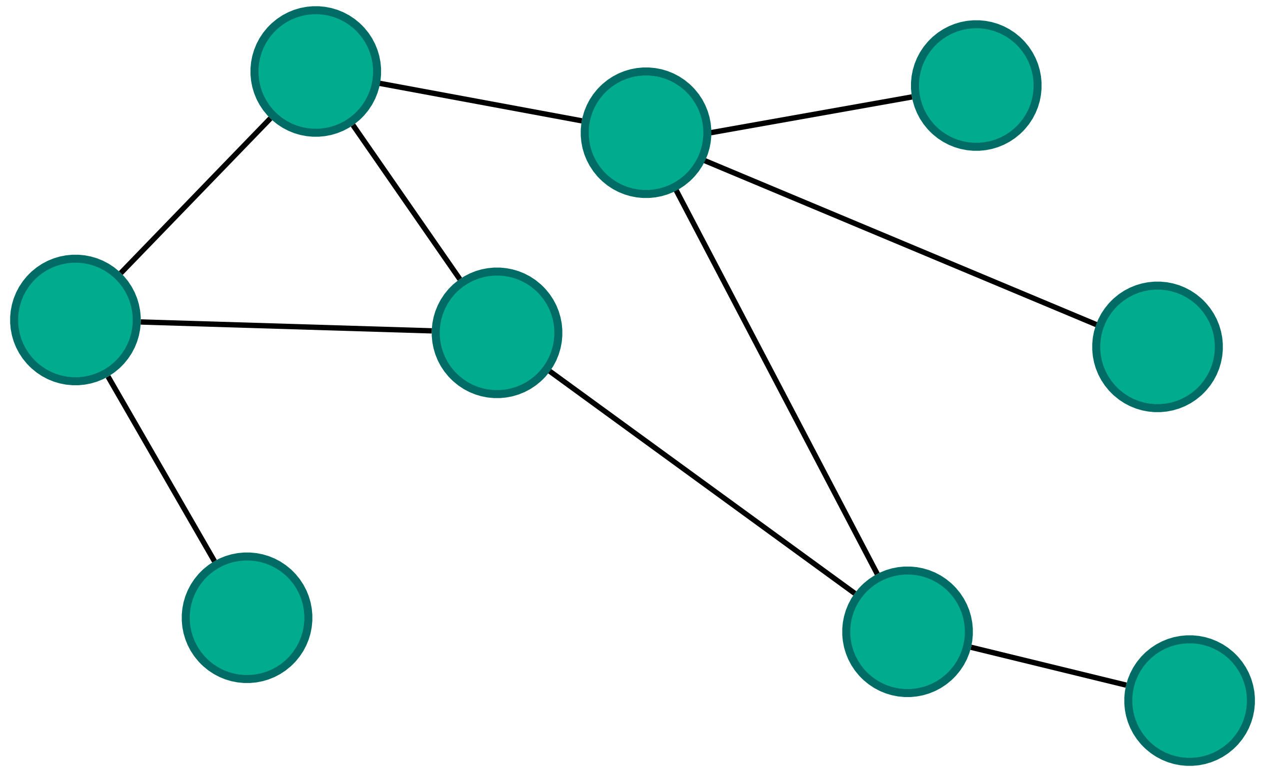

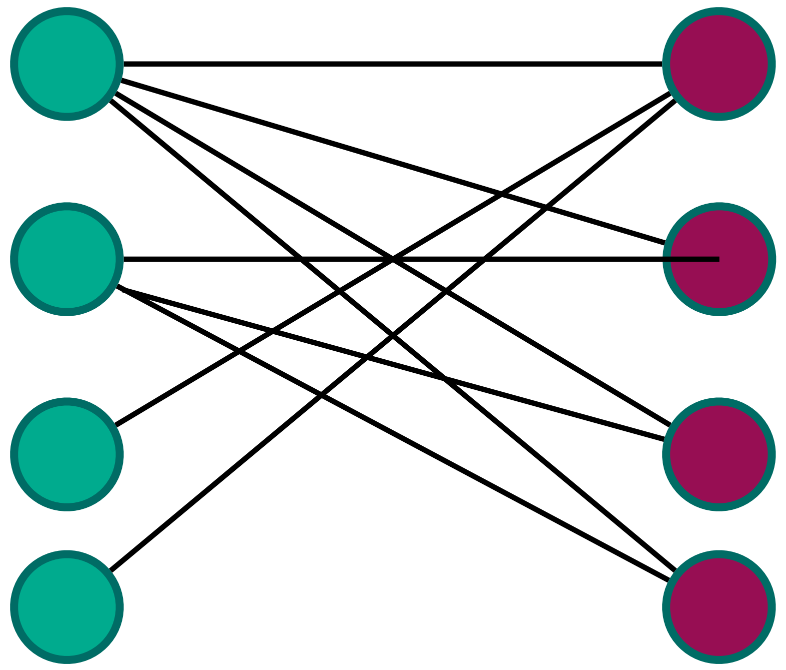

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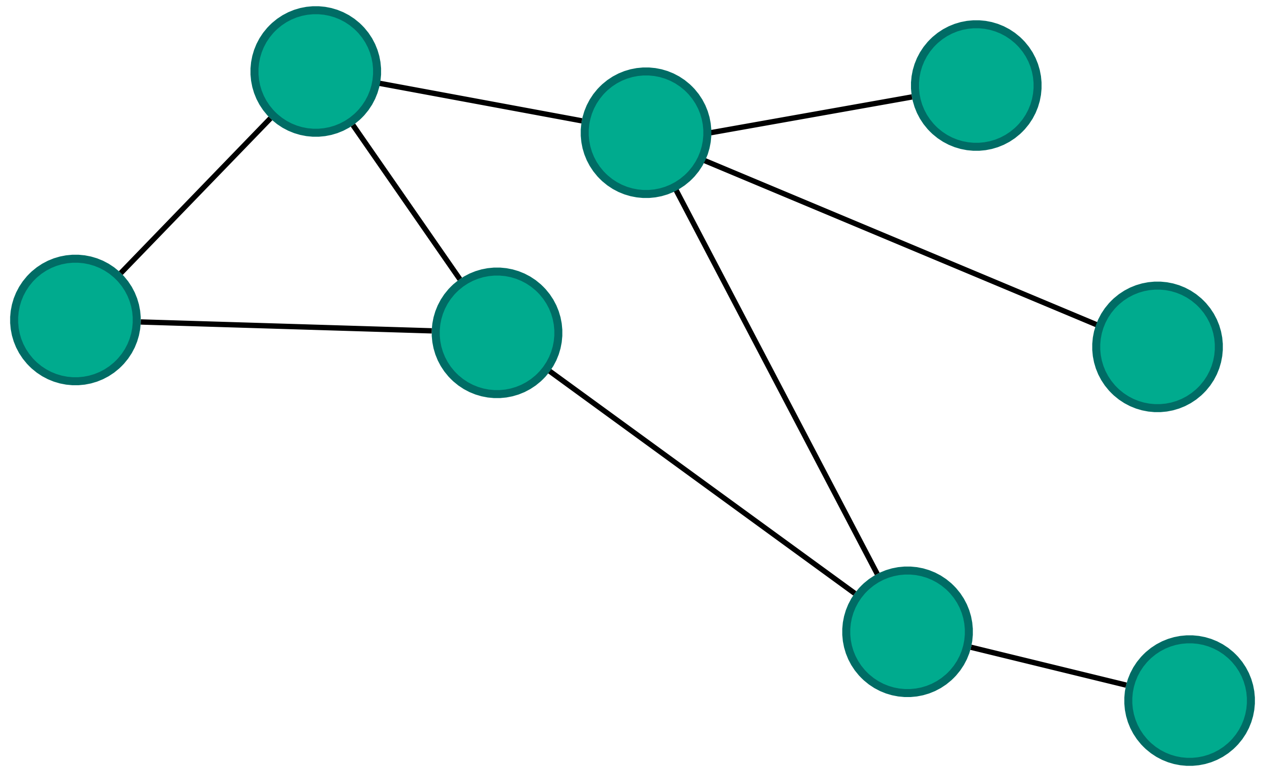

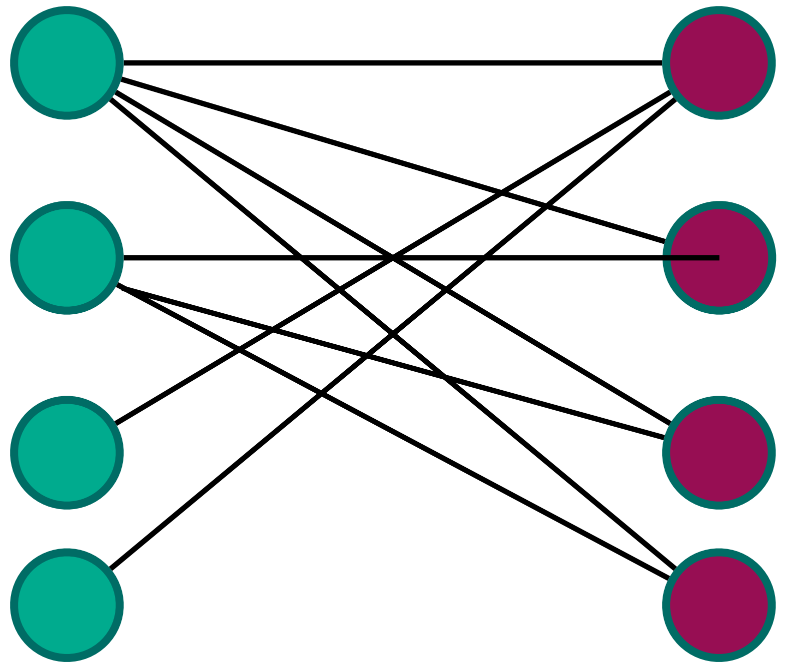

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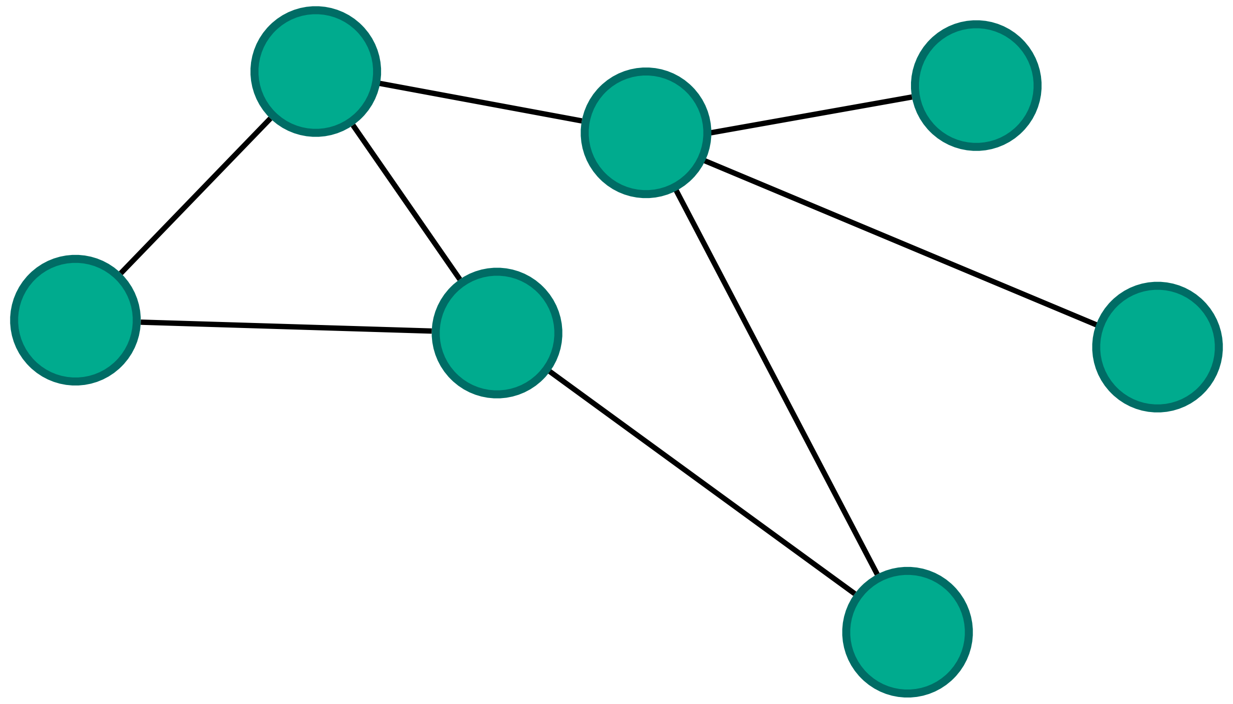

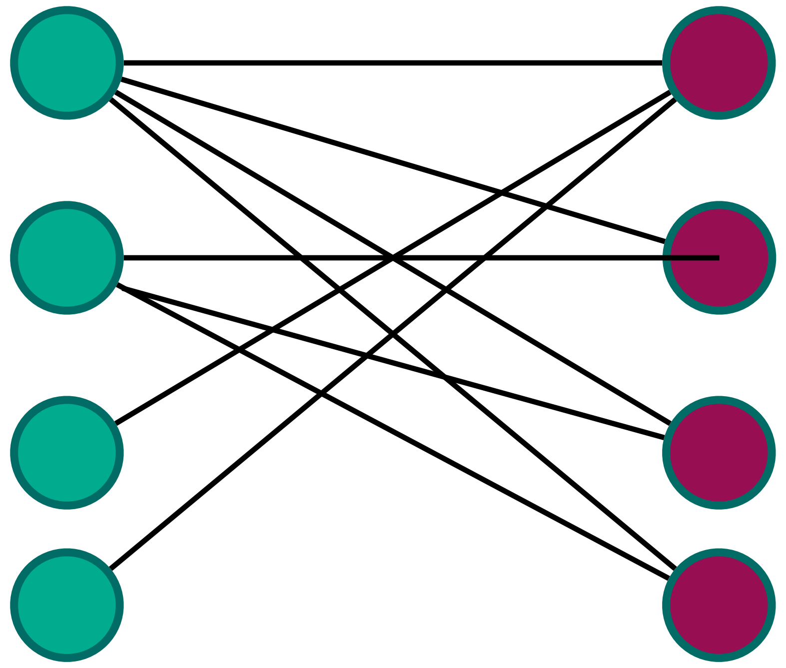

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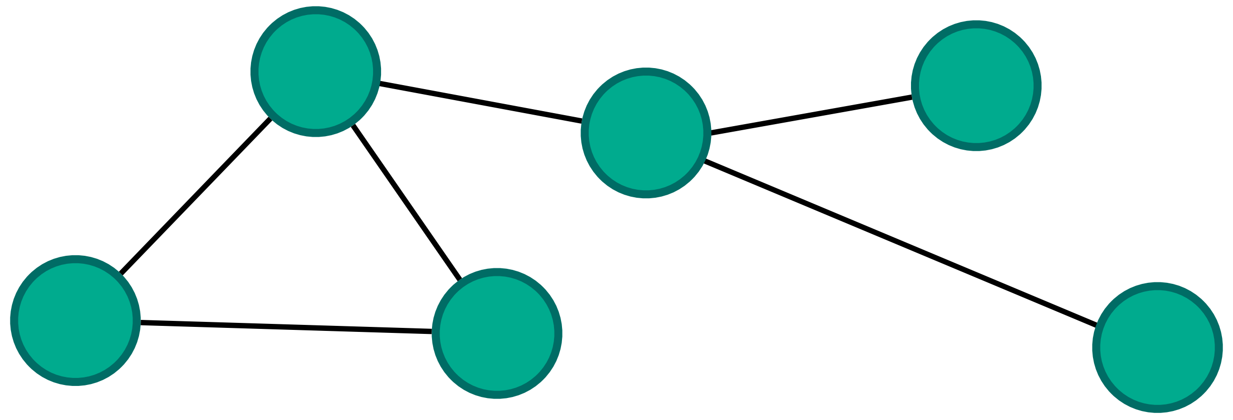

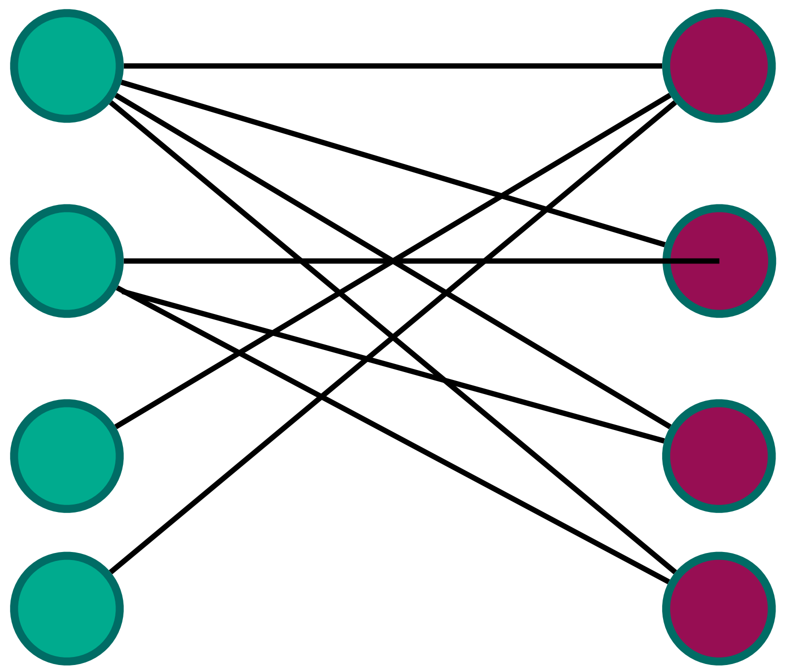

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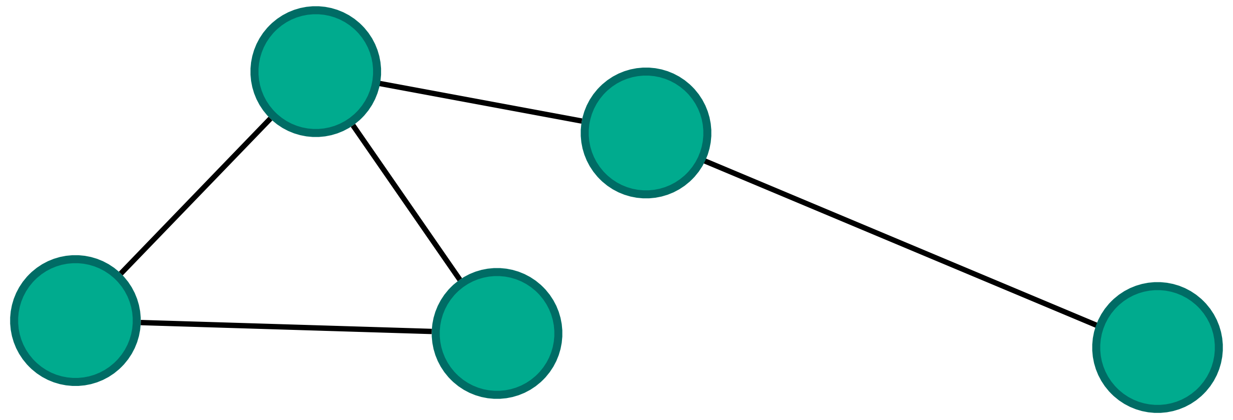

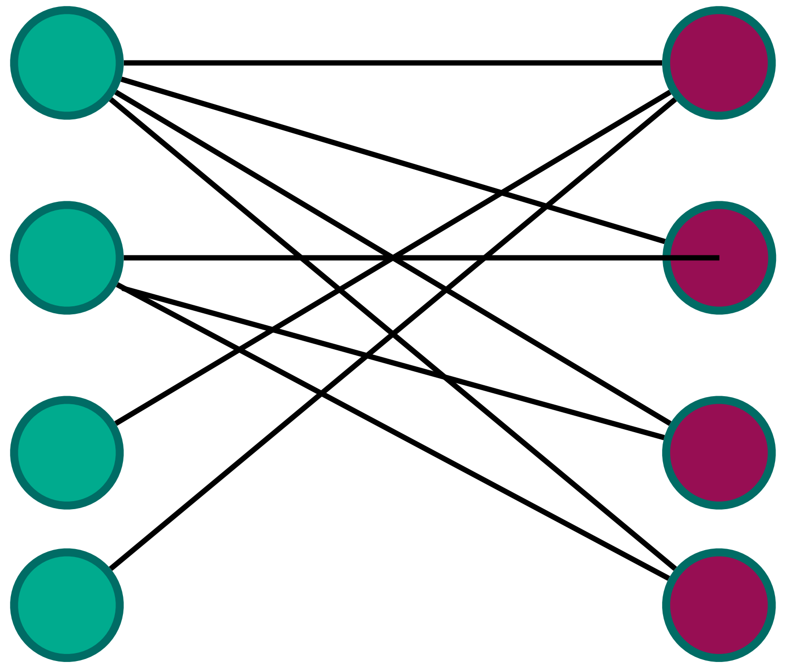

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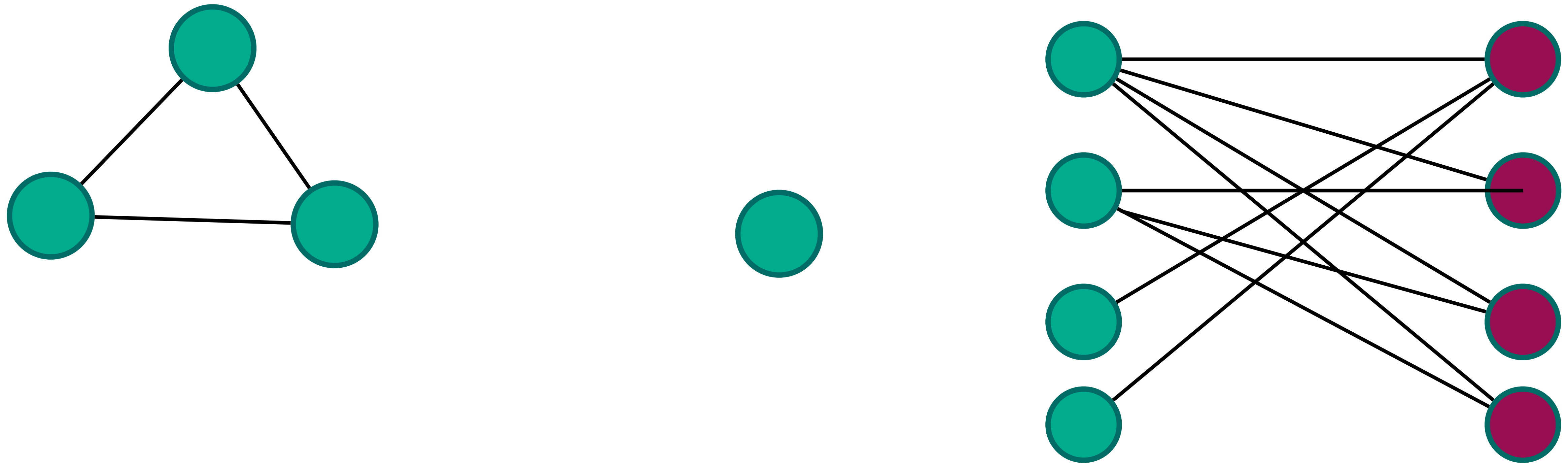

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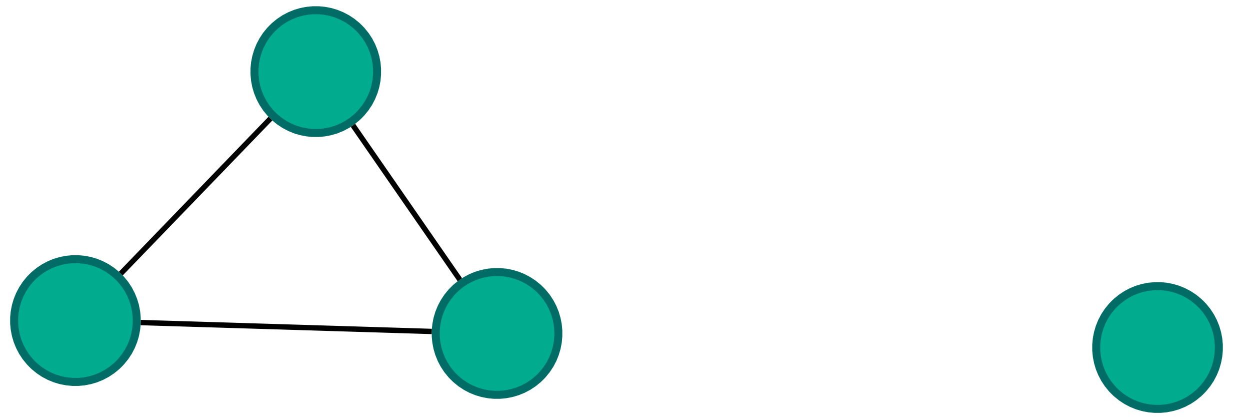
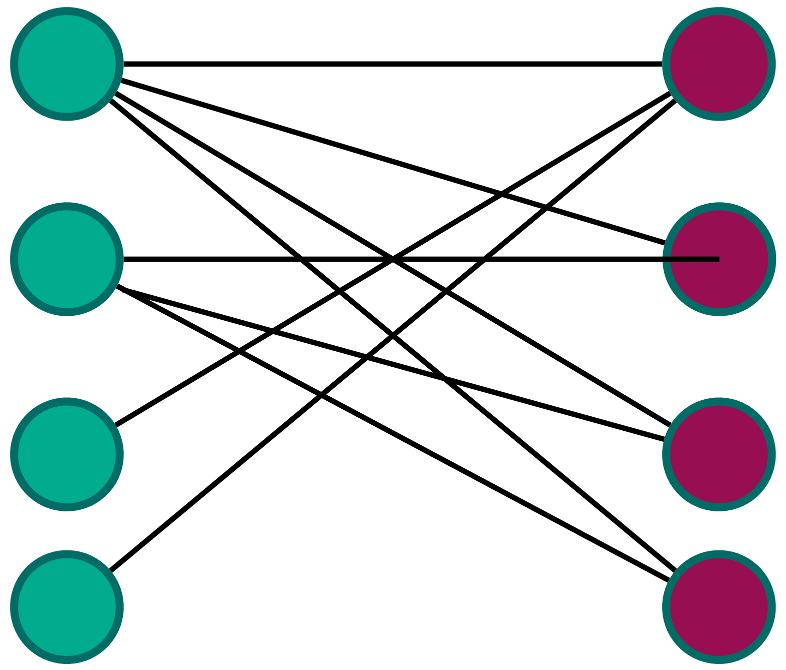
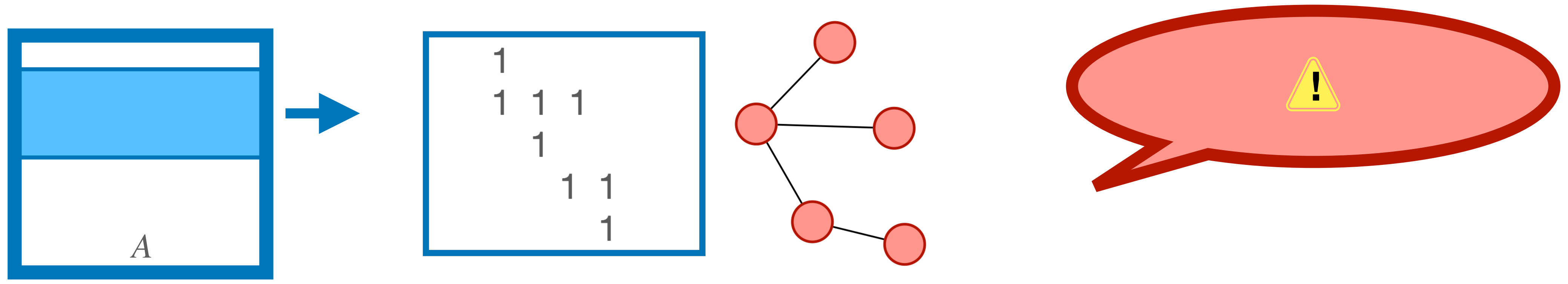




Main Results

- Whp, corank given by I_{KS}

- Characterization of minimal dependencies



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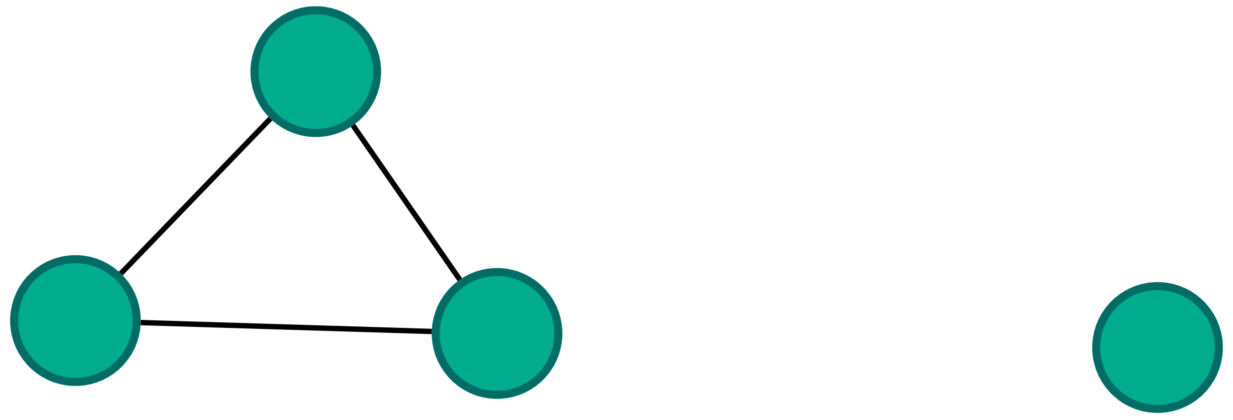

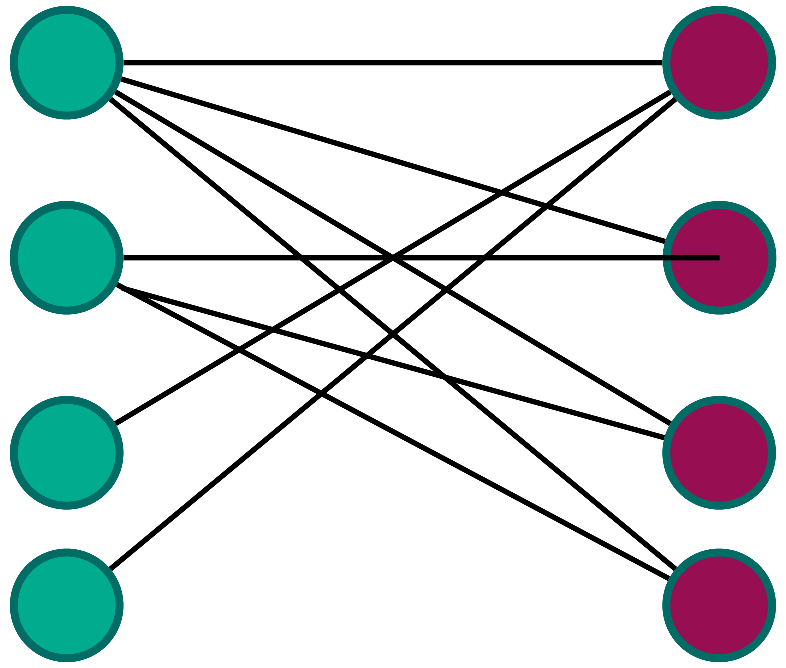

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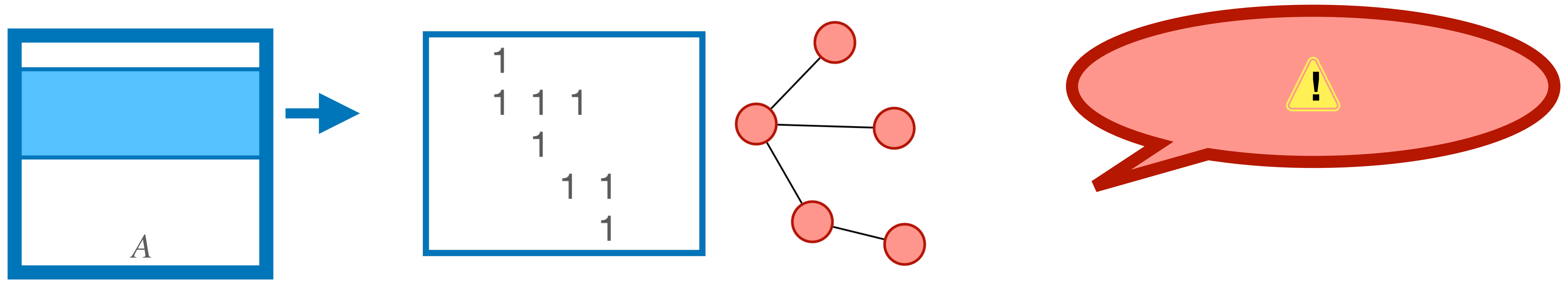
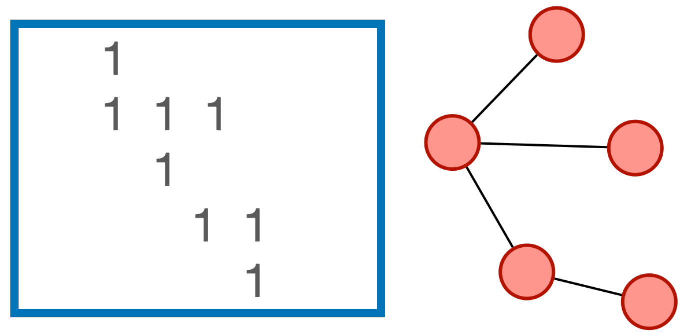
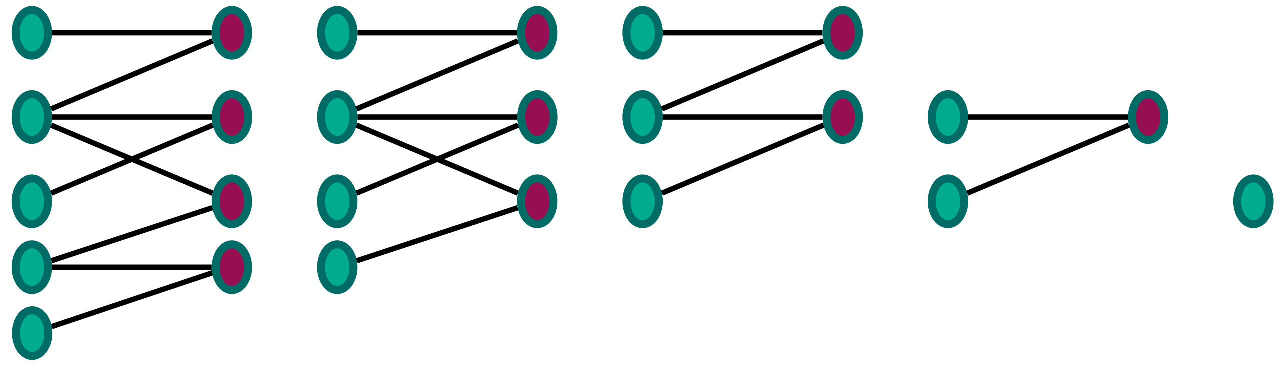


Tree Dependencies get Peeled!





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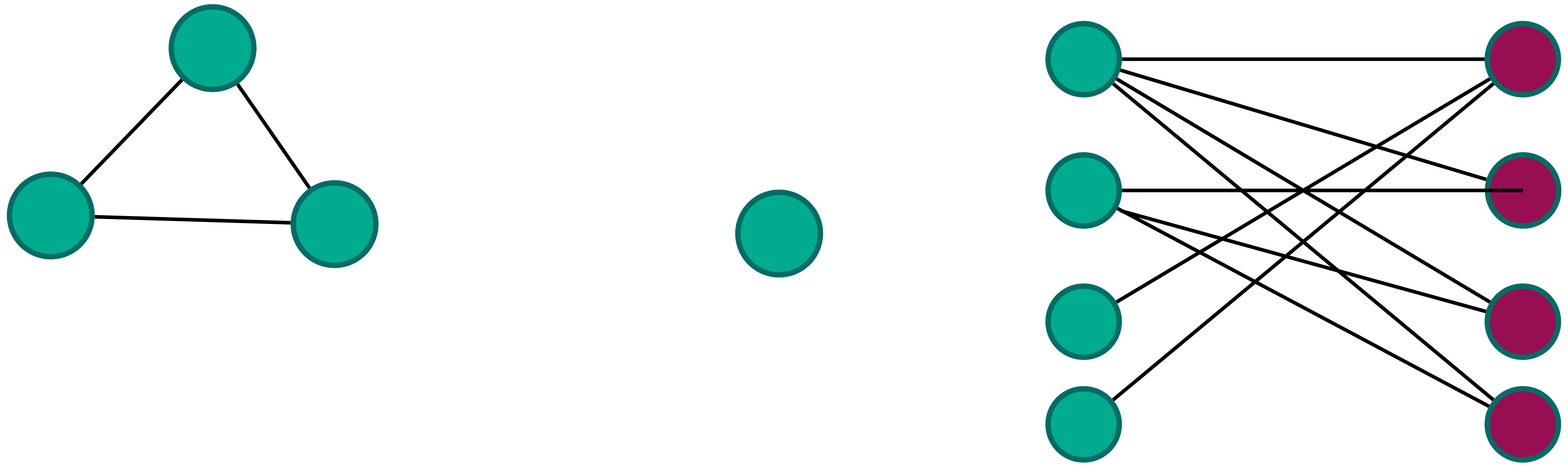

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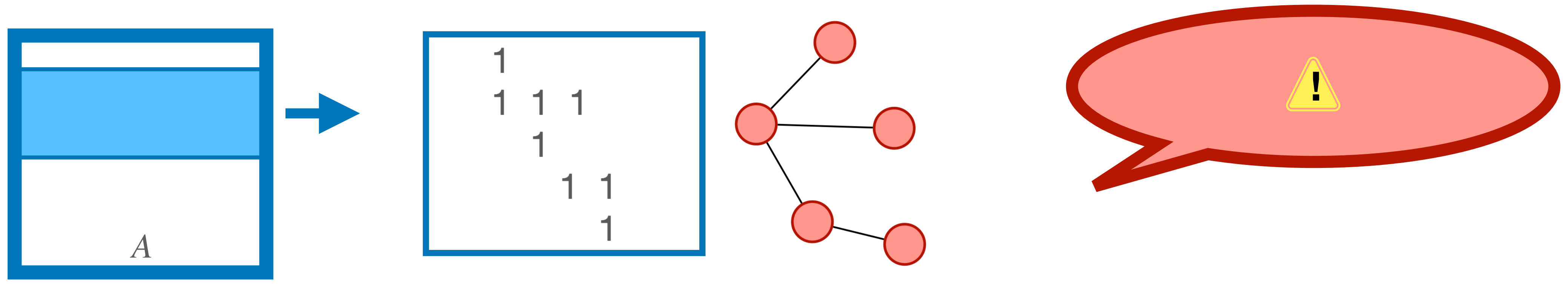

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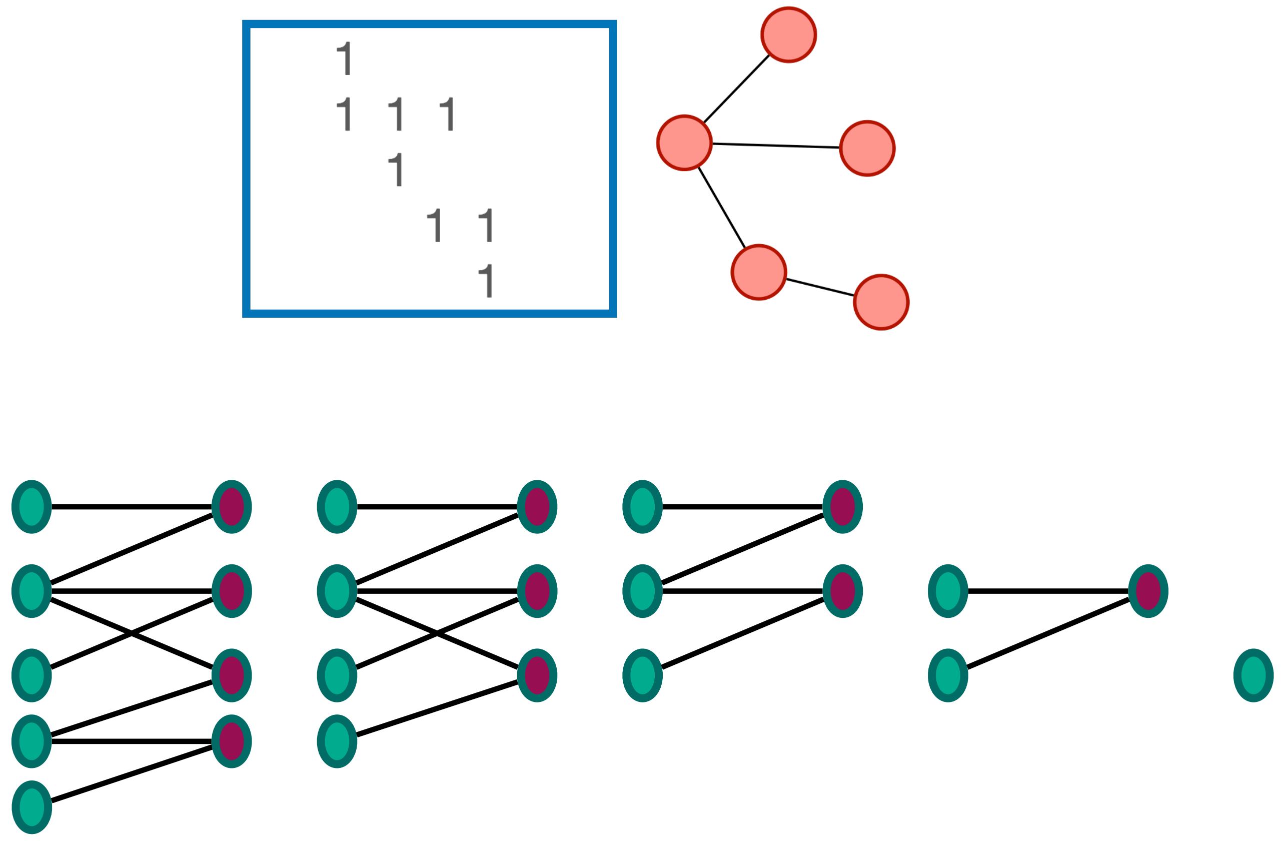
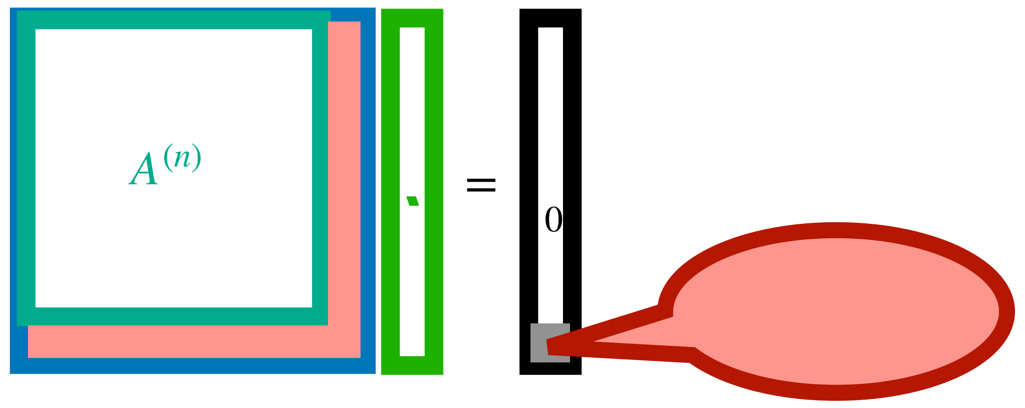


Key Proof Ideas for Characterization

- Union bound over small dependencies
- Anticoncentration for large dependencies



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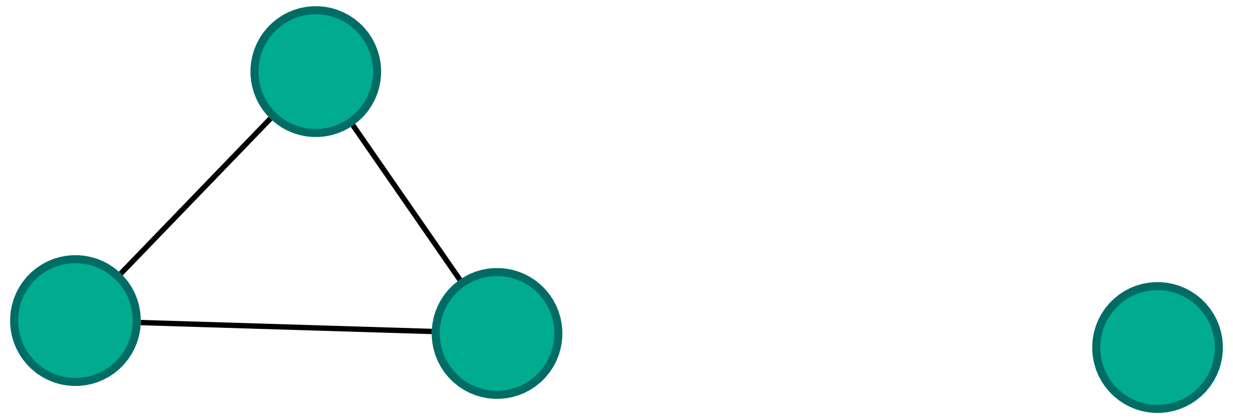

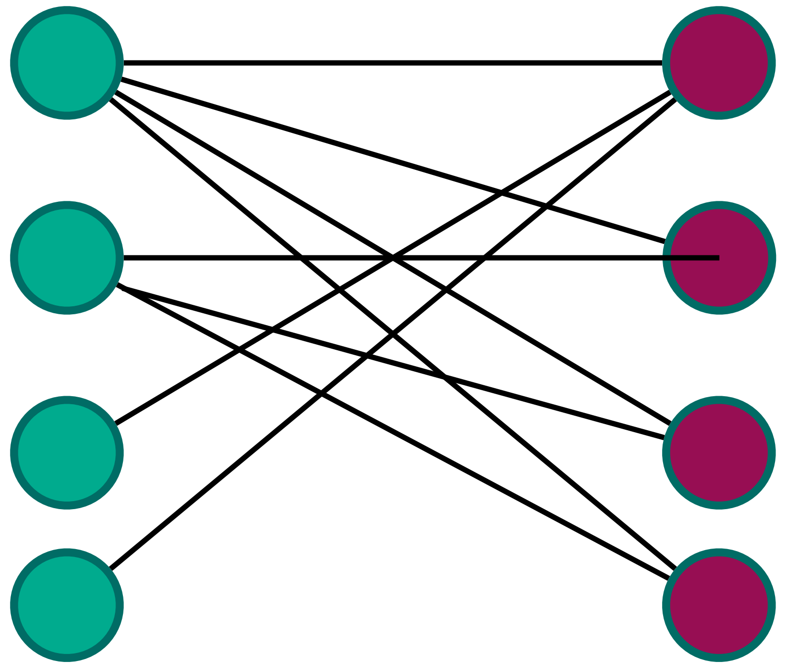

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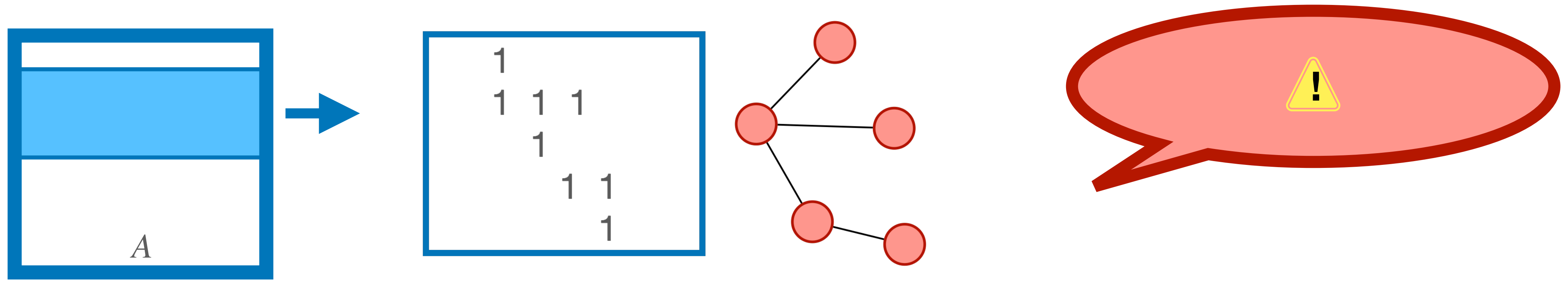

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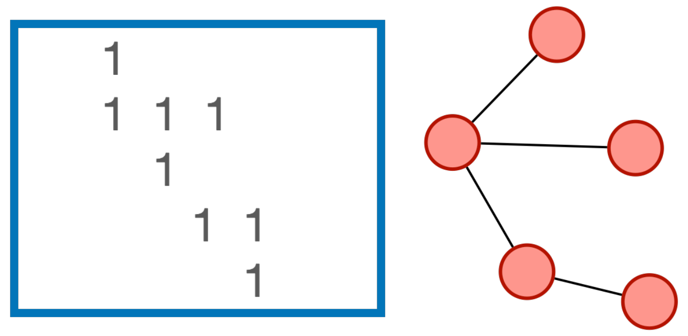

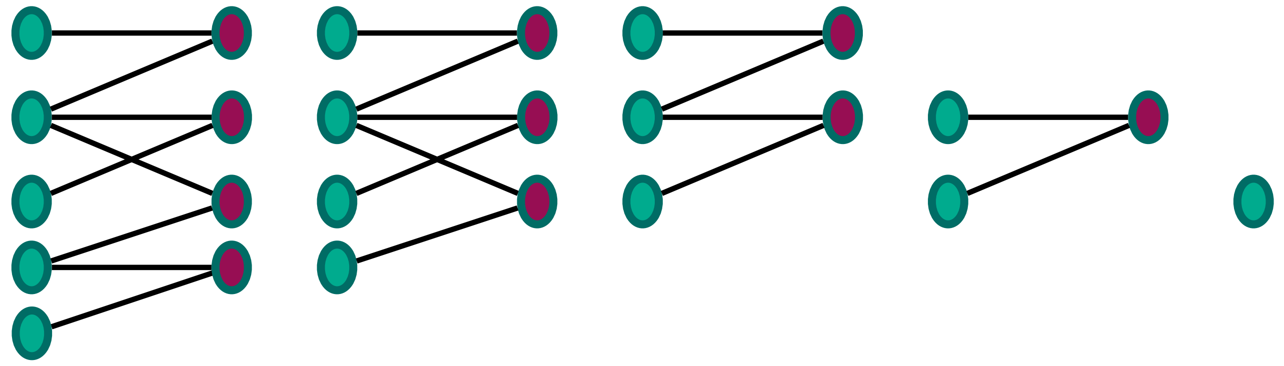

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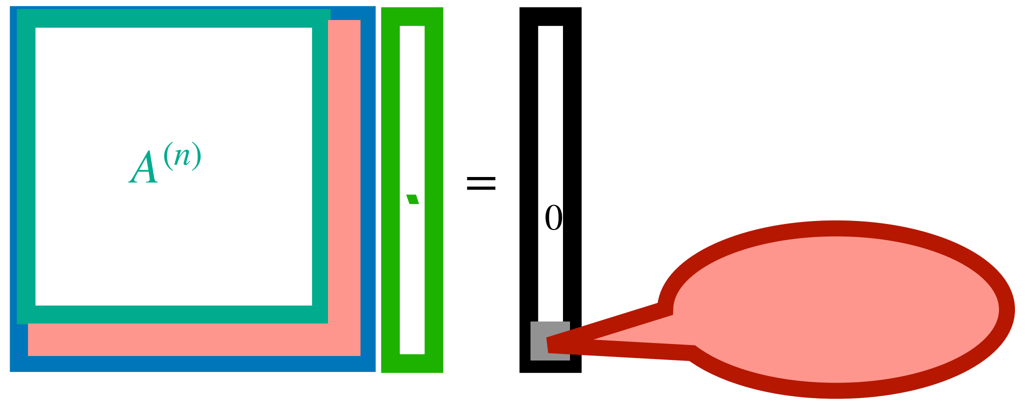

Limitations/Directions

Constant Average Degree?

Thanks!

Questions?