On the Isoperimetric Profile of the Hypercube

Thm: There's a function $g : (0,1) \to [0,\infty)$, positive on $(0,1)\setminus\{\frac{1}{2}\}$ so that

$$I_{[0,1]^d} \geqslant \sqrt{2\pi}\, I_\gamma + g$$

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Isoperimetric Problem

X space with two concepts: $(E \subseteq X)$

Volume $|E|$

Perimeter $\mathrm{Per}(E)$

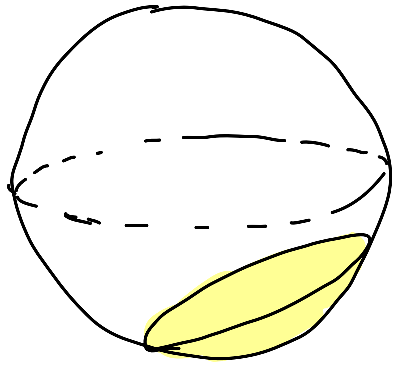
$$I_X(\lambda) := \inf_{E \subseteq X} \{ \mathrm{Per}(E) : |E| = \lambda \} \qquad (0 < \lambda < |X|)$$

A set E so that $\mathrm{Per}(E) = I_X(|E|)$ is called isoperimetric set.

Examples:

$X = \mathbb{R}^d$: Isoperimetric sets are balls.

$X = \mathbb{S}^d$: Isoperimetric sets are (metric) balls.



$$X = (\mathbb{R}^d, \gamma_d) \qquad \varphi(t) := \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \quad \gamma_1 := \varphi \, d\mathcal{L}^1.$$

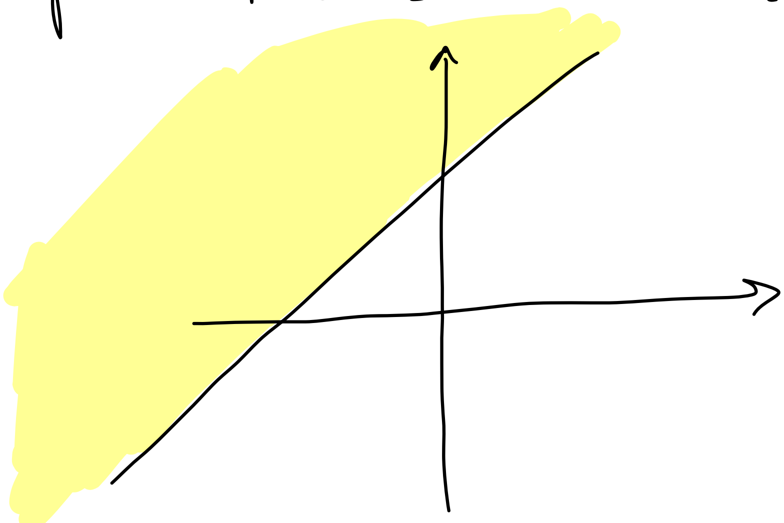
$|x|$

$$\gamma_d := \gamma_1 \otimes \cdots \otimes \gamma_1 .$$

Volume : $|E| := \gamma_d(E)$

Perimeter : $\mathrm{Per}_\gamma(E) := \int_{\partial^* E} \varphi_d \, d\mathcal{H}^{d-1}$

Isoperimetric sets are affine halfspaces.



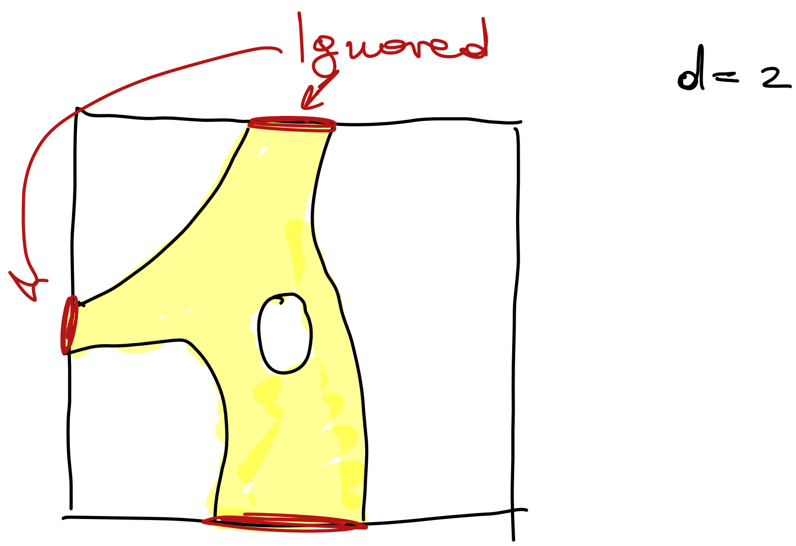
Gaussian Isoperimetric profile $I_\gamma = \varphi \circ \Phi^{-1}$, where $\Phi(t) := \int_{-\infty}^{t} \varphi$.

REMARK: Let I_{S^d} be the isoperimetric profile of the d-dimensional sphere rescaled so that it has volume 1. Then $I_{S^d} \xrightarrow{d\to\infty} I_\gamma$.

ISOPERIMETRIC PROBLEM IN $(0,1)^d$

As volume we use the Lebesgue measure. We shall consider the relative perimeter.

$$\mathrm{Per}(E) = \mathcal{H}^{d-1}(\partial^* E \cap (0,1)^d).$$

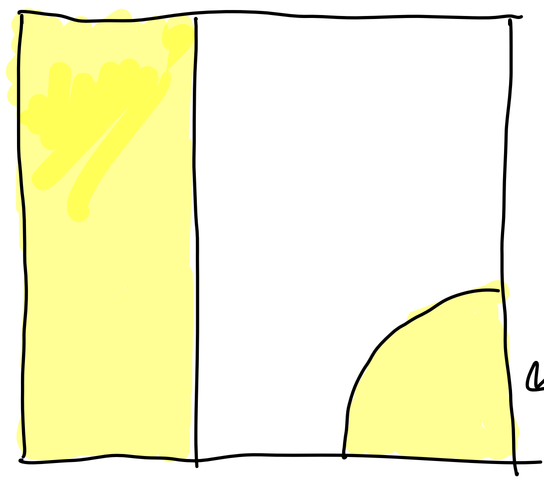


The problem in $(0,1)^d$ is equivalent

to the problem in $\mathbb{T}^d$.

Let's study $d=2$ (square).

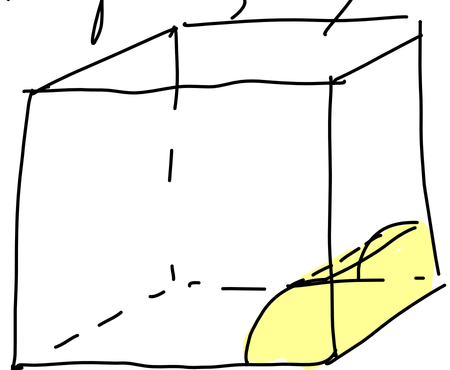
Thm (Bresis-Bruckstein):
The isoperimetric sets in $(0,1)^2$ are half spaces or balls.



Its complement is not convex.

Rmk: In dim 2, CMC implies that the (boundary) set is either a circle or a line.

Conjecture: (Ros) In $(0,1)^3$ the isoperimetric sets are half-spaces, cylinders, or balls.

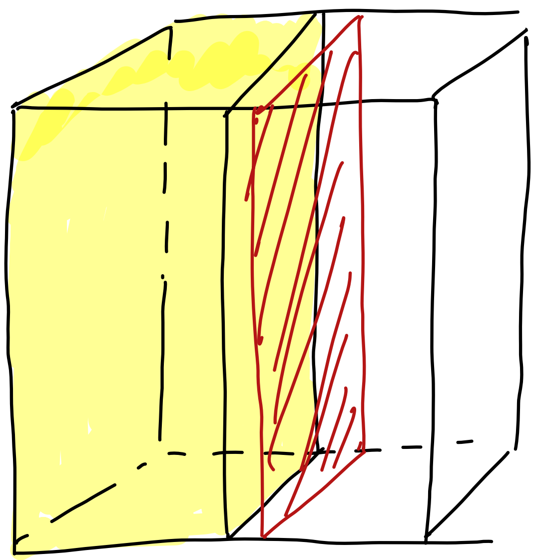


Connection to the Gaussian isoperimetric problem.

Thm (Barthe–Maurey) (Hadwiger)

$$I_{(0,1)^d} \geq \sqrt{2\pi}\, I_\gamma .$$

Observe that this is sharp for $\lambda = \frac{1}{2}$.



Question (by Brezis): Is it true that

$I_{(0,1)^d}(\lambda) = 1$ if λ is close to $\frac{1}{2}$.

Thm (Acerbi-Fusco-Morini, G.): For any $d \geq 2$, there is $\varepsilon_d > 0$, so that $I_{(0,1)^d}(\lambda) = 1$ if $\lambda \in \left(\frac{1}{2} - \varepsilon_d, \frac{1}{2} + \varepsilon_d\right)$.

<u>Quick sketch</u>: Let $|E_\lambda| = \lambda$ be isoperimetric.

As $\lambda \to \frac{1}{2}$, $\underbrace{\partial^* E_\lambda}$ $\to$ $\underbrace{\partial^* E_{\frac{1}{2}}}$.

Nice CMC objects

Thus the convergence is <u>graphical</u>:

$\partial^* E_\lambda$ is a graph over $\partial^* E_{\frac{1}{2}}$ for $\lambda \sim \frac{1}{2}$.

Then it's easy to conclude that $\partial^* E_\lambda$ is flat.

- $I_{(0,1)^d} \geq \sqrt{2\pi}\, I_\gamma$ ⟵ NOT SHARP.

- $I_{S^d} \longrightarrow I_\gamma$

- $I_{(0,1)^2}$ and $\sqrt{2\pi}\, I_\gamma$ look very close.

One may believe that $I_{(0,1)^d} \to \sqrt{2\pi}\, I_\gamma$.

<u>This is false</u>.

Thm: There's a function $g : (0,1) \to [0,\infty)$, positive on $(0,1) \setminus \{\frac{1}{2}\}$ so that

$$I_{(0,1)^d} \geq \sqrt{2\pi}\, I_\gamma + g$$

[g does not depend on d]

Quick sketch:

$$\Phi_d : \mathbb{R}^d \to (0,1)^d$$

$$(x_1, \dots, x_d) \longmapsto (\Phi(x_1), \dots, \Phi(x_d))$$

$\boxed{1}$ $\frac{1}{\sqrt{2\pi}}$ - Lip

$\boxed{2}$ $(\Phi_d)_\# \gamma_d = \mathcal{L}^d\big|_{(0,1)^d}$ $\rightsquigarrow$ $I_{(0,1)^d} \geq \sqrt{2\pi}\, I_\gamma$.

For any $\lambda \in (0,1)$ there is a set $F \subseteq \mathbb{R}^d$ with $\gamma_d(F) = \lambda$ so that

$$\frac{1}{\sqrt{2\pi}} I_{(0,1)^d} - I_\gamma \geq \mathrm{Per}_{\gamma_d}(F) - I_\gamma(|F|) + \int_{\partial^* F} \left(\sqrt{\underbrace{\sum_{i=1}^d (\nu_{\partial F})_i^2 \, e^{x_i^2}}_{\sum \nu_i^2 = 1}} - 1 \right) (\varphi_d \, d\mathcal{H}^{d-1})$$

If this is small $\overset{\text{stability}}{\Longrightarrow}$ F is very close to an affine halfspace $\Rightarrow$ the other term is large

$$\geq \underbrace{c(\lambda)}_{\underline{\text{explicit}}}$$

OPEN QUESTION: What is $I_{(0,1)^\infty}$

$$\left(:= \lim_{d\to\infty} I_{(0,1)^d}\right).$$

Symmetric isop. problem in the Gaussian space.