Inequalities on the Derivative of the Radon Transform on Convex Bodies

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A slicing inequality for functions

Let K be an origin-symmetric convex body of volume 1 in $\mathbb{R}^n$, and let f be any non-negative measurable function on K with $\int_K f = 1$. Does there exist a constant c_n, depending only on n, so that for any such K and f there exists a direction $\xi \in S^{n-1}$ with $\int_{K\cap\xi^\perp} f \geq c_n$?

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If $f \equiv 1$, this is Bourgain's slicing problem. In this case, we can take $c_n > o(n^{-\epsilon})$ for any $\epsilon > 0$, which was shown by Chen in 2021.

A slicing inequality for functions, continued

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Koldobsky, Pajor, 2017: This also holds for unit balls of subspaces of L_p, $p > 2$, where the constant is of the order $p^{-1/2}$.

For a function f on $\mathbb{R}^n$, denote by

$$Rf(\xi,t) = \int_{K\cap\{x\in\mathbb{R}^n:\ (x,\xi)=t\}} f(x)dx, \quad \xi \in S^{n-1}, t \in \mathbb{R}$$

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The result described above means that if f is a probability density on an origin-symmetric convex body in $\mathbb{R}^n$, the sup-norm of its Radon transform is bounded from below by a positive constant depending only on the dimension n.

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We will prove a similar estimate for the derivatives of the Radon transform. If h is an even, continuous function which is m-smooth in a neighborhood of 0, and $q\in\mathbb{C}$ with $-1<\text{Re } q<m$ and $q\neq 0,1,\dots,m-1$, we define:

$$h^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^1 t^{-1-q}\left(h(t)-h(0)-\dots-h^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!}\right)dt$$
$$+\frac{1}{\Gamma(-q)}\int_1^\infty t^{-1-q}h(t)dt+\frac{1}{\Gamma(-q)}\sum_{k=0}^{m-1}\frac{h^{(k)}(0)}{k!(k-q)}.$$

Theorem 1 There exists an absolute constant $c > 0$ so that for any infinitely smooth origin-symmetric convex body K of volume 1 in $\mathbb{R}^n$, any even infinitely smooth probability density f on K, and any $q \in \mathbb{R}$, $0 \le q \le n-2$, which is not an odd integer, there exists a direction $\xi \in S^{n-1}$ so that

$$\left(c\frac{q+1}{\sqrt{n\log^3\left(\frac{ne}{q+1}\right)}}\right)^{q+1} \le \frac{1}{\cos\left(\frac{\pi q}{2}\right)}\left(Rf(\xi,t)\right)_t^{(q)}(0).$$

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If $q = 2k$ is an even integer, then we are talking about the usual derivative:

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If $q = 2k-1$ is an odd integer, then computing the limit as $q \to 2k-1$, we find there is a $\xi \in S^{n-1}$ so that

$$\left(\frac{2kc}{\sqrt{n\log^3(\frac{ne}{2k})}}\right)^{2k} \leq (-1)^k (2k-1)! \times$$

$$\int_0^\infty t^{-2k}\left(Rf(\xi,t) - \sum_{j=0}^{k-1} \frac{t^{2j}}{(2j)!}(Rf(\xi,t))_t^{(2j)}(0)\right) dt.$$

If $\mathcal{A}$ is a class of compact sets in $\mathbb{R}^n$, the outer volume ratio distance from K to $\mathcal{A}$ is:

$$d_{\text{ovr}}(K,\mathcal{A}) = \inf\left\{ \left(\frac{|D|}{|K|}\right)^{1/n} : K \subset D, D \in \mathcal{A} \right\}.$$

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For a star body D in $\mathbb{R}^n$, we say that $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-p}, $0 < p < n$, if the function $\|\cdot\|_D^{-p}$ represents a positive definite distribution. We denote the class of such star bodies by L_{-p}^n.

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Theorem 2: Suppose K is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, f is a non-negative even infinitely smooth function on K and $-1 < q < n-1$ is not an odd integer. Then

$$\int_K f \le \frac{n}{(n-q-1)2^q \pi^{\frac{q-1}{2}} \Gamma(\frac{q+1}{2})} |K|^{\frac{q+1}{n}} (d_{\text{ovr}}(K, L_{-1-q}^n))^{q+1}$$

$$\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0).$$

For $0 \le q \le n-2$, we use the estimate $n/(n-q-1) < e^{q+1}$ and the Stirling formula to obtain:

$$\max_{\xi \in S^{n-1}} \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0) \ge \left(\frac{c(q+1)}{d_{\mathrm{ovr}}(K, L^n_{-1-q})} \right)^{\frac{q+1}{2}},$$

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For q a real number with $-1 < q < n-1$, we have:

1. For $0 < p \leq 2$, if K is the unit ball of an n-dimensional subspace of L_p, then $K \in L_{-1-q}^n$, so $d_{\mathrm{ovr}}(K, L_{-1-q}^n)$ is 1.

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2. If K is an unconditional body, $d_{\text{ovr}}(K, L_{-1-q}^n) \le e$.
3. For $p > 2$, if K is a unit ball of an n-dimensional subspace of L_p, then $d_{\text{ovr}}(K, L_{-1-q}^n) \le c\sqrt{p}$.

Busemann-Petty problem

The well-known Busemann-Petty problem asks the following: If K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ with $|K \cap \xi^\perp| \leq |L \cap \xi^\perp|$ for each direction $\xi \in S^{n-1}$, does it follow that $|K| \leq |L|$?

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Koldobsky, 1999: The answer becomes affirmative if we compare the derivatives (of high enough orders) of the parallel section function:

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For $n \ge 4$, $q \in [n-4, n-1)$ not odd, and K, L infinitely smooth origin-symmetric convex bodies such that for every $\xi \in S^{n-1}$,

$$\frac{1}{\cos(\frac{\pi q}{2})} A^{(q)}_{K,\xi}(0) \le \frac{1}{\cos(\frac{\pi q}{2})} A^{(q)}_{L,\xi}(0),$$

then $|K| \le |L|$. For $-1 < q < n-4$ this is no longer true.

Zvavitch, 2005: Let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$, and f an even continuous strictly positive function on $\mathbb{R}^n$. Suppose that

$$R(f|_K)(\xi, t)(0) \leq R(f|_L)(\xi, t)(0)$$

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We extend these results to derivatives of the Radon transform.

Theorem 3. Let K, L be infinitely-smooth, origin-symmetric convex bodies in $\mathbb{R}^n$, f, g infinitely differentiable functions on K and L respectively, $\|g\|_\infty = g(0) = 1$, and $q \in (-1, n-1)$ not an odd integer. Then if for every $\xi \in S^{n-1}$,

$$\frac{1}{\cos(\frac{\pi q}{2})}(Rf(\xi,t))_t^{(q)}(0) \leq \frac{1}{\cos(\frac{\pi q}{2})}(Rg(\xi,t))_t^{(q)}(0),$$

then we have:

$$\int_K f \leq \frac{n}{n-q-1}\left(d_{\mathrm{ovr}}(K, L^n_{-1-q})\right)^{q+1}\left(\int_L g\right)^{\frac{n-q-1}{n}}|K|^{\frac{q+1}{n}}.$$

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As $q \to -1$, we get that $\int_K f \leq \int_L g$.

An expression for the Radon derivative

Lemma Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, let f be an even infinitely smooth function on K, and let $q \in (-1, n-1)$. Then for every fixed $\xi \in S^{n-1}$,

$$(Rf(\xi,t))_t^{(q)}(0) = \frac{\cos(\pi q/2)}{\pi} \left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f(r\frac{x}{|x|_2}) dx \right) \right)_x^{\wedge} (\xi).$$

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The latter is homogeneous of degree $-1-q$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$. □

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$$= \frac{\cos(q\pi/2)}{\pi}\int_{S^{n-1}}\left(\int_0^{\infty} t^q\hat{\phi}(t\theta)dt\right)\left(\int_0^{\|\theta\|_K^{-1}} r^{n-q-2} f(r\theta)dr\right)d\theta.$$

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$$(Rf(\xi,t))_t^{(q)}(0) = \frac{\cos(\pi q/2)}{\pi} \left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f(r\frac{x}{|x|_2}) dx \right) \right)_x^\wedge (\xi).$$

Proof.

On the other hand:

$$\left\langle \frac{\cos(\pi q/2)}{\pi} \left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f(r\frac{x}{|x|_2}) dr \right) \right)_x^\wedge (\xi), \phi(\xi) \right\rangle$$

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Theorem 2 Suppose K is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, f is a non-negative even infinitely smooth function on K and $-1 < q < n-1$ is not an odd integer. Then

$$\int_K f \le \frac{n}{(n-q-1)2^q \pi^{\frac{q-1}{2}} \Gamma(\frac{q+1}{2})} |K|^{\frac{q+1}{n}} (d_{\mathrm{ovr}}(K, L^n_{-1-q}))^{q+1}$$
$$\times \max_{\xi \in S^{n-1}} \frac{1}{\cos(\frac{\pi q}{2})} (Rf(\xi,t))_t^{(q)}(0).$$

Proof.

Consider a number $\epsilon > 0$ such that for every $\xi \in S^{n-1}$,

$$\frac{1}{\cos(\pi q/2)} (Rf(\xi,t))_t^{(q)}(0) \le \frac{\epsilon}{\cos(\pi q/2)} (R(\chi_{B_2^n})(\xi,t))_t^{(q)}(0).$$

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By the Lemma, for every $\xi \in S^{n-1}$,

$$\left(|x|_2^{-n+q+1} \left(\int_0^{\frac{|x|_2}{\|x\|_K}} r^{n-q-2} f(r \frac{x}{|x|_2}) dr \right) \right)_x^\wedge (\xi) \leq \frac{\epsilon}{n-q-1} \left(|x|_2^{-n+q+1} \right)^\wedge (\xi).$$

$\square$

Proof of Theorem 2 (continued)

Proof.

Let $D \in L^n_{-1-q}$ be such that $|D|^{1/n} \leq (1+\delta) d_{\mathrm{ovr}}(K, L^n_{-1-q})$, $K \subseteq D$, $\delta > 0$. $(\|x\|_D^{-1-q})^\wedge$ is positive on the sphere, so we can multiply by it on both sides, integrate on the sphere, and use Parseval's, we get:

$$\int_K \|x\|_D^{-1-q} f(x)dx \leq \frac{\epsilon}{n-q-1} \int_{S^{n-1}} \|x\|_D^{-1-q} dx.$$

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On the other hand, we can use Hölder's inequality and the polar formula for volume to get

$$\int_{S^{n-1}} \|x\|_D^{-1-q} dx \leq |S^{n-1}|^{\frac{n-q-1}{n}} n^{\frac{q+1}{n}} |D|^{\frac{q+1}{n}}.$$

□

Proof.

Now, put

$$\epsilon = \max_{\xi \in S^{n-1}} \frac{\frac{1}{\cos(\pi q/2)}(Rf(\xi,t))_t^{(q)}(0)}{\frac{1}{\cos(\pi q/2)}(R(\chi_{B_2^n})(\xi,t))_t^{(q)}(0)}.$$

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Then we have that

$$\int_K f(x)dx \le c(n,q)\left(\frac{|D|}{|K|}\right)^{\frac{q+1}{n}} |K|^{\frac{q+1}{n}} \max_{\xi \in S^{n-1}} \frac{1}{\cos(\pi q/2)}(Rf(\xi,t))_t^{(q)}(0).$$

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$$c(n,q) = \frac{n\Gamma\left(\frac{n-q-1}{2}+1\right)}{2^q \pi^{\frac{q-1}{2}} \Gamma\left(\frac{q+1}{2}\right)(n-q-1)\left(\Gamma\left(\frac{n}{2}+1\right)\right)^{\frac{n-q-1}{n}}}$$

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$$\int_K f(x)dx \leq c(n,q)\left(\frac{|D|}{|K|}\right)^{\frac{q+1}{n}} |K|^{\frac{q+1}{n}} \max_{\xi\in S^{n-1}} \frac{1}{\cos(\pi q/2)}(Rf(\xi,t))_t^{(q)}(0).$$

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$$\leq \frac{n}{2^q \pi^{\frac{q-1}{2}} \Gamma\left(\frac{q+1}{2}\right)(n-q-1)}.$$

We obtain the result as $\delta \to 0$. □

References

- A. Koldobsky, *A $\sqrt{n}$ estimate for measures of hyperplane sections of convex bodies,* Adv. Math. 254 (2014), 33–40.
- G. Chasapis, A. Giannopoulos, D. Liakopoulos, *Estimates for measures of lower dimensional sections of convex bodies,* Adv. Math. 306 (2017), 880-904.
- B. Klartag and A. Koldobsky, *An example related to the slicing inequality for general measures*, J. Funct. Anal. 274 (2018), 2089–2112.
- B. Klartag and G. Livshyts, *The lower bound for Koldobsky's slicing inequality via random rounding*, to appear in GAFA Seminar Notes, arXiv:1810.06189.
- Y. Chen, *An almost constant lower bound of the isoperimetric coefficient in the KLS Conjecture. Geom. Funct. Anal.*. **31,** 34-61 (2021).
- A. Koldobsky, *Slicing inequalities for measures on convex bodies*, Adv. Math. 283 (2015), 473-488.
- A. Koldobsky and A. Pajor, *A remark on measures of sections of L_p-balls*, GAFA Seminar Notes, Lect. Notes in Math. 2169 (2017), 213–220.
- A. Koldobsky, *Fourier analysis in convex geometry*, Amer. Math. Soc., Providence RI, 2005.