Metric Distortion of Random Sets

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Random Measures

- Let $X_1, \ldots X_n$ and $Y_1, \ldots, Y_n$ be independent uniform random points on the circle S^1.
- In Bobkov's talk (about the his work with Ledoux) we defined random measures

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{X_k},\ \nu_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{Y_k},$$

  and looked at the Wasserstein distance $W_p(\mu, \nu)$.
- The optimal solution is of the same order as passing thorough the evenly spaced (average) case

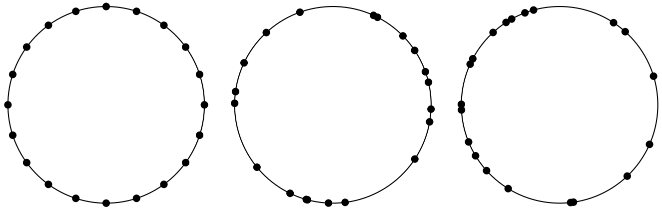
$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{e^{2\pi ik/n}}.$$

The Setup

- The metric distortion of $A, B \subseteq S^1$ with cardinality n is

$$\mathrm{dist}(A, B) = \inf\{\|f\|_{Lip}\|f^{-1}\|_{Lip};\ f : A \to B \text{ bijection}\}.$$

- Assume that A, B is defined by n i.i.d uniform points.
  - What is the "typical" $\mathrm{dist}(A, B)$?
  - Let $C \subseteq S^1$ be n evenly spaced points. How does $\mathrm{dist}(A, C)$ compare to $\mathrm{dist}(A, B)$?



Remarks on the Question

- Similarly, one may ask if the following algorithm is good:
  - Choose starting points in both A and B.
  - Move the $i-$th point in A to the $i-$th point in B.
  - Optimize over all choices of starting points.
- This question does not make sense in a non probabilistic setting (worst case scenario).

Structure of Random Sets

Let $\{X_1, \ldots, X_n\} \subseteq S^1$ be a random set, ordered (by choosing a starting point).

- There are no unusual accumulations or empty stretches. For any $s > 10$, with probability at least $1 - Cn^{1-s/8}$ we have

$$\frac{j-i}{4n} \leq |X_j - X_i| \leq \frac{4(j-i)}{n},$$

  for any i, j such that $\max\{|j-i|, |n+1-j-i|\} > s \log n$
- Let $Y_i = |X_{i+1} - X_i|$ then $(Y_1, \ldots, Y_{n-1})$ is a uniform random vector on the boundary of the $n-1$ dimensional simplex.
- By standard arguments the distribution of Y_i is the same as $Z_i/(Z_1 + \cdots + Z_{n-1})$ where $Z_1, \ldots, Z_{n-1}$ are i.i.d exponential random variables. The sum $Z_1 + \cdots + Z_{n-1}$ is concentrated around n.

Distortion to Evenly Spaced Set

- Since the distances in a random sets are comparable to exponent random variables:
  - $$\mathbb{P}\left(\frac{1}{n^2 \log n} \leq \min |X_{i+1} - X_i| \leq \frac{\log n}{n^2}\right) \to 1.$$
  - $$\mathbb{P}\left(\frac{1}{n} \leq \max |X_{i+1} - X_i| \leq \frac{10 \log n}{n}\right) \to 1.$$
- By these bounds, with high probability the metric, the metric distortion is $\Theta^*(n)$.

No Expectation

- Let $\beta(A) = \min |X_{i+1} - X_i|$, then $\mathrm{dist}(A, B) \geq \beta(A)/\beta(B)$.
- Let $Y_1, \ldots, Y_n$ be the points in B. Then,

$$\mathbb{P}(\beta(B) \leq t) \geq \mathbb{P}(|Y_2 - Y_1| \leq t) \geq Ct.$$

- $$\mathbb{P}\left(\beta(A) \geq \frac{1}{2n^2}\right) \geq \frac{1}{2}.$$

- $$\mathbb{P}\left(\mathrm{dist}(A, B) \geq t\right) \geq \mathbb{P}\left(\beta(A) \geq \frac{1}{2n^2} \text{ and } \beta(B) \leq \frac{1}{2tn^2}\right) \geq \frac{C}{tn^2}.$$

- $$\mathbb{E}\mathrm{dist}(A, B) \geq \frac{C}{n^2} \int_1^{\infty} \frac{1}{t} dt = \infty.$$

Main Results

Theorem

For n large enough, we have

$$\mathbb{P}\left(\exists f : A \to B,\ dist(f) \leq n^{2/3-\varepsilon}\right) \leq \frac{3}{8}.$$

Theorem

For n large enough, we have

$$\mathbb{P}\left(\exists f : A \to B,\ dist(f) \leq n^{2/3+\varepsilon}\right) \geq \frac{5}{8}.$$

Lower Bound - Small Permutations

- By choosing a starting point, we associate a function $f : A \to B$ with a permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$.
- We define (reverse) permutation distortion by

$$s(f) := \max_{i=1,\dots,n} |i - \pi(i)|$$
$$s_r(f) := \max_{i=1,\dots,n} |i - (n + 1 - \pi(i))|.$$

- Let $0 < \alpha < 1/2$. With high probability, no map $f : A \to B$ with $\min\{s(f), s_r(f)\} \leq n^\alpha$ has $\|f\|_{Lip} \leq n^{1-\alpha}/\log^2 n$. Sketch of the proof:
  - Consider the shortest segment. It is of length smaller than $\log n/n^2$.
  - It is mapped to one of $2n^\alpha$ consecutive segment.
  - With high probability, the shortest segment there is bigger than $n^{1-\alpha}/\log n$.

Big Permutations

Lemma

Let $\varepsilon > 0$ *and let* $0 < \alpha < 1$. *Let* $f : A \to B$ *be a bi-Lipschitz function such that* $\|f\|_{Lip}\|f^{-1}\|_{Lip} \leq n^{\alpha-\varepsilon}$ *and* $\min\{s(f), s_r(f)\} \geq n^{\alpha}$. *Then,*

$$\mathbb{P}\left(\exists i;\ \min\{|i-\pi(i)|, |n+1-i-\pi(i)|\} > n^{\alpha-\varepsilon/2}\right) \to 1.$$

- We partition the indices to left $\{i \leq n/2 - n^{\alpha-\varepsilon/2}\}$, center and right $\{i \geq n/2 + n^{\alpha-\varepsilon/2}\}$.
- An index is projected if it is mapped $n^{\alpha-\varepsilon/4}$ close to itself and reflected if to $n+1-i$.
- Either all the left (right) is projected or reflected. Otherwise there is a pair that drift too far apart. A contradiction to $\min\{|i-\pi(i)|, |n+1-i-\pi(i)|\}$.

Lower Bound - Big permutations

Theorem (Benjamini and Shamov)

Let $\pi : \mathbb{Z} \to \mathbb{Z}$ be a bi-Lipschitz bijection. Then,

$$\pi(x) = \pm x + const + r(x),$$

where $|r(x)| \leq \|\pi\|_{Lip} \|\pi^{-1}\|_{Lip}$.

Lemma

Let $0 < \alpha < 1$ and let $\varepsilon > 0$. Then, with high probability there is no $f : A \to B$ bi-Lipschitz such that

$$\|f\|_{Lip} \|f^{-1}\|_{Lip} < n^{\alpha - \varepsilon} \ and \ \min\{s(f), s_r(f)\} \geq n^{\alpha}.$$

Lower Bound

Define the events

$$E_1 = \left\{\exists f : A \to B \quad ; \|f\|_{Lip} < n^{1/3-\varepsilon/2} \ \& \ \min\{s(f), s_r(f)\} < n^{2/3}\right\}$$
$$E_2 = \left\{\exists g : B \to A \quad ; \|g\|_{Lip} < n^{1/3-\varepsilon/2} \ \& \ \min\{s(g), s_r(g)\} < n^{2/3}\right\}$$
$$E_3 = \left\{\exists f : A \to B \quad ; \|f\|_{Lip}\|f^{-1}\|_{Lip} < n^{2/3-\varepsilon/2} \ \& \ \min\{s(g), s_r(g)\} \geq n^{2/3}\right\}$$

Then,

$$\left\{dist(A, B) \leq n^{2/3-\varepsilon}\right\} \subseteq E_1 \cup E_2 \cup E_3,$$

and each of them occur with probability less then $1/8$ (when n is big enough).

Upper Bound - The Setup

- We partition our sets to segments of $n^{1/3}$ points.
- We group together consecutive $n^{1/3}$ segments.
- We condition of the high probability events:
  - Up to a constant, each segment is of length $n^{-2/3}$.
  - The shortest distance between any two points is bigger than $1/n^2$.
  - The longest distance between any two points is smaller than $1/n^{1+\varepsilon}$.
- Different segments are “almost” independent.
- In each segment we are interested in two scales:
  - Short intervals, of length less than $n^{-4/3-\varepsilon}$.
  - Long intervals, of length bigger than $n^{-4/3-\varepsilon}$.

Upper Bound - The Mapping

We base the map between A and B on the following observations:

- If two segments have the same sequences of short intervals they are mapped to each other with distortion less than $n^{2/3+2\varepsilon}$.
- If we increase the short scale to $n^{-4/3+\varepsilon}$, then the distortion is still bounded by $n^{2/3+2\varepsilon}$.
- If each segments ends and start with a long interval, and two grouping have the same sequences of segments of the previous observation, then they are mapped to each other with the same distortion bound.
- If we end and start each grouping with a segment without short intervals, and all previous conditions are fulfilled then A is mapped to B with distortion less than $n^{2/3+2\varepsilon}$.

Upper Bound - Conditioning

In order to have the conditions to be able to map A to B, it is enough to use the previous slide.

- The probability for a segment to have short intervals is bounded by $C/n^{-\varepsilon}$.
- By rotations, we can ensure that we begin and end with long intervals, and that each grouping begin and end with a segment free of short intervals.
- We can count the sequences of short intervals as a *bins and balls* problem. Note that each of them is a concentrated binomial random variable.
- In order to cover all sequences, we allow scaling up by $n^{-1/\varepsilon}$. By the concentration inequalities, it is enough (we just add the missing pieces).
- We use a similar argument for the groupings.

Further Questions

- How does the metric distortion change with dimension? I find three cases that interest me the most:
  - The d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$.
  - The d-dimensional sphere S^d.
  - Fractals, such as Cantor sets.
- How the result would change if we are allowed to throw away n^β points?

Thank You