Floating bodies and random polytopes.

Olivier Guédon

LAMA, Université Gustave Eiffel

May 19, 2020

Random polytopes

Geometry of convex bodies

Let X be a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X. We study

$$\mathrm{absconv}(X_1, \ldots, X_N) = AB_1^N \subset \mathbb{R}^n$$

where A is a matrix which columns are the vectors X_i.

Random polytopes

Geometry of convex bodies

Let X be a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X. We study

$$\mathrm{absconv}(X_1, \ldots, X_N) = AB_1^N \subset \mathbb{R}^n$$

where A is a matrix which columns are the vectors X_i.

1. combinatorics : asymptotic computation as N goes to ∞ and n is fixed : of the number of vertices, facets or volume.

Random polytopes

Geometry of convex bodies

Let X be a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X. We study

$$\mathrm{absconv}(X_1, \ldots, X_N) = AB_1^N \subset \mathbb{R}^n$$

where A is a matrix which columns are the vectors X_i.

1. combinatorics : asymptotic computation as N goes to ∞ and n is fixed : of the number of vertices, facets or volume.
2. geometry : asymptotic behavior as $N \geq n$ et $n \to \infty$: extremal properties of the volume of the polytope or its polar.

Random polytopes

Geometry of convex bodies

Let X be a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X. We study

$$\mathrm{absconv}(X_1, \ldots, X_N) = AB_1^N \subset \mathbb{R}^n$$

where A is a matrix which columns are the vectors X_i.

1. combinatorics : asymptotic computation as N goes to ∞ and n is fixed : of the number of vertices, facets or volume.
2. geometry : asymptotic behavior as $N \geq n$ et $n \to \infty$: extremal properties of the volume of the polytope or its polar.
3. probability : geometric properties of the polytope according to the law of the random vector which generates the polytope, properties of the operator norm of A.

Random polytopes

1. A key result in the local theory of Banach spaces (due to Gluskin in 1981) : the Banach Mazur distance between 2 such random polytopes is "extremal" : $X \sim \mathcal{N}(0, \mathrm{Id})$,with high probability, for $N \geq 10n$

$$d(P_N, P'_N) \geq cn$$

Random polytopes

1. A key result in the local theory of Banach spaces (due to Gluskin in 1981) : the Banach Mazur distance between 2 such random polytopes is "extremal" : $X \sim \mathcal{N}(0, \mathrm{Id})$,with high probability, for $N \geq 10n$
$$d(P_N, P'_N) \geq cn$$

2. Szarek (1990), Tikhomirov (2019). With $N = n^3$,
$$d(P_N, B_1^n) \geq n^{5/9} \log^{-C}(n)$$

Random polytopes

1. A key result in the local theory of Banach spaces (due to Gluskin in 1981) : the Banach Mazur distance between 2 such random polytopes is "extremal" : $X \sim \mathcal{N}(0, \mathrm{Id})$,with high probability, for $N \geq 10n$

$$d(P_N, P'_N) \geq cn$$

2. Szarek (1990), Tikhomirov (2019). With $N = n^3$,

$$d(P_N, B_1^n) \geq n^{5/9} \log^{-C}(n)$$

3. Extremal properties of such random polytopes : (CFPP 2015) if X has a bounded density (by 1) then

$$\mathbb{E}\mathrm{Vol}\left(\mathrm{absconv}(X_1, \ldots, X_N)\right)^o$$

is maximal when $X \sim \mathcal{U}_{B_2^n}$.

Random matrices : $\Gamma = (X_1, \ldots, X_N)^T : \ell_2^n \to \ell_2^N$

Study of the extreme singular values of the matrix :

$$s_1(\Gamma) = \sup_{|x|_2=1} |\Gamma x|_2 = \sup_{|x|_2=1} \left(\sum_{j=1}^{N} \langle X_j, x \rangle^2 \right)^{1/2}$$

$$s_N(\Gamma) = \inf_{|x|_2=1} |\Gamma x|_2 = \inf_{|x|_2=1} \left(\sum_{j=1}^{N} \langle X_j, x \rangle^2 \right)^{1/2}$$

By duality, showing that $s_N \geq \alpha\sqrt{N}$ is equivalent to

$$\alpha\sqrt{N} B_2^n \subset A B_2^N (\subset \sqrt{N} P_N)$$

LPRT (2005), LPRTV (2006) : good hypotheses on the random vector X. Net arguments. In all these arguments, they need a good bound on s_1 to get a lower bound on s_N.
Kolesnikov, Mendelson (2014)

Floating body and geometry of the polytopes.

Question

X a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X.

Floating body and geometry of the polytopes.

Question

X a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X.

1. Define a ”natural set” K associated to X such that with high probability

$$K \subset \text{absconv}(X_1, \ldots, X_N)$$

Floating body and geometry of the polytopes.

Question

X a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X.

1. Define a "natural set" K associated to X such that with high probability
$$K \subset \mathrm{absconv}(X_1, \ldots, X_N)$$
2. Give some kind of precise description of K ?

Floating body and geometry of the polytopes.

Question

X a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of X.

1. Define a "natural set" K associated to X such that with high probability
$$K \subset \operatorname{absconv}(X_1, \ldots, X_N)$$
2. Give some kind of precise description of K ?

Definition

A family of floating bodies. Let X be a symmetric random vector, for every $p \geq 1$, set

$$K_p(X) = \{t \in \mathbb{R}^n, \mathbb{P}(\langle X, t\rangle \geq 1) \leq e^{-p}\}$$

I) Floating body - Examples.

Definition

A family of floating bodies. Let X be a symmetric random vector, for every $p \geq 1$, set

$$K_p(X) = \{t \in \mathbb{R}^n, \mathbb{P}(\langle X, t \rangle \geq 1) \leq e^{-p}\}$$

Assume X is reasonnably "nice"

I) Floating body - Examples.

Definition

A family of floating bodies. Let X be a symmetric random vector, for every $p \geq 1$, set

$$K_p(X) = \{t \in \mathbb{R}^n, \mathbb{P}(\langle X, t\rangle \geq 1) \leq e^{-p}\}$$

Assume X is reasonnably ”nice”

- For every $t \in \mathbb{R}^n$, $\langle X, t\rangle$ has moments of all order. And define

$$B(L_p(X)) = \left\{t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^p\right)^{1/p} \leq 1\right\}$$

Then by Chebychev inequality

$$\frac{1}{e} B(L_p(X)) \subset K_p(X)$$

I) Floating body - Examples.

Definition

A family of floating bodies. Let X be a symmetric random vector, for every $p \geq 1$, set

$$K_p(X) = \{t \in \mathbb{R}^n, \mathbb{P}(\langle X, t \rangle \geq 1) \leq e^{-p}\}$$

Assume X is reasonnably "nice"

- For every $t \in \mathbb{R}^n$, $\langle X, t \rangle$ has moments of all order. And define

$$B(L_p(X)) = \left\{ t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t \rangle|^p\right)^{1/p} \leq 1 \right\}$$

- Assume also that there exists $D \geq 1$ such that

$$\forall q \geq 2, \forall t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t \rangle|^{2q}\right)^{1/2q} \leq D \left(\mathbb{E}|\langle X, t \rangle|^q\right)^{1/q}$$

Then by Paley-Zygmund,

$$K_p(X) \subset 2B(L_{c_1 p}(X))$$

where c_1 depends only on D.

Floating body - X is reasonnably "nice" .

- For every $t \in \mathbb{R}^n$, $\langle X, t \rangle$ has moments of all order. And define

$$B(L_p(X)) = \left\{ t \in \mathbb{R}^n, \left(\mathbb{E} |\langle X, t \rangle|^p \right)^{1/p} \leq 1 \right\}$$

- Assume also that there exists $D \geq 1$ such that

$$\forall q \geq 2, \forall t \in \mathbb{R}^n, \left(\mathbb{E} |\langle X, t \rangle|^{2q} \right)^{1/2q} \leq D \left(\mathbb{E} |\langle X, t \rangle|^q \right)^{1/q}$$

Conclusion

$$\exists C_1(D) \geq 1, \forall p \geq 2, \quad \frac{1}{e} B(L_p(X)) \subset K_p(X) \subset C_1 B(L_p(X))$$

Floating body - X is reasonnably "nice" .

- For every $t \in \mathbb{R}^n$, $\langle X, t\rangle$ has moments of all order. And define

$$B(L_p(X)) = \left\{ t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^p\right)^{1/p} \leq 1 \right\}$$

- Assume also that there exists $D \geq 1$ such that

$$\forall q \geq 2, \forall t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^{2q}\right)^{1/2q} \leq D \left(\mathbb{E}|\langle X, t\rangle|^q\right)^{1/q}$$

Conclusion

$$\exists C_1(D) \geq 1, \forall p \geq 2, \quad \frac{1}{e} B(L_p(X)) \subset K_p(X) \subset C_1 B(L_p(X))$$

Remark

The polar $B(L_p(X))^o$ is called the Z_p-centroid body of X

$$Z_p(X) = B(L_p(X))^o$$

and is well studied in the geometry of log-concave measures.

Floating body - Various examples.

- Set $G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$K_p(G) \approx \frac{1}{\sqrt{p}} B_2^n \text{ and } K_p(G)^o \approx \sqrt{p} B_2^n$$

Floating body - Various examples.

- Set $G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$K_p(G) \approx \frac{1}{\sqrt{p}} B_2^n \text{ and } K_p(G)^o \approx \sqrt{p} B_2^n$$

- Set X uniformly distributed on a symmetric convex body K then

$$K_p(X)^o \approx Z_p(X)$$

where $h_{Z_p(X)}(\theta) = (\mathbb{E}\langle X, \theta\rangle^p)^{1/p}$.

Floating body - Various examples.

- Set $G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$K_p(G) \approx \frac{1}{\sqrt{p}} B_2^n \quad \text{and} \quad K_p(G)^o \approx \sqrt{p} B_2^n$$

- Set X uniformly distributed on a symmetric convex body K then

$$K_p(X)^o \approx Z_p(X)$$

where $h_{Z_p(X)}(\theta) = \left(\mathbb{E}\langle X, \theta\rangle^p\right)^{1/p}$.

Theorem of Montgomery-Smith (1990)

$$\mathbb{P}\left(\sum_{i=1}^n x_i \varepsilon_i > K_{1,2}(x,t)\right) \approx e^{-ct^2}$$

where

$$K_{1,2}(x, \sqrt{p}) = \sum_{i=1}^p x_i^* + \frac{1}{\sqrt{p}} \left(\sum_{i=p+1}^n x_i^{*2}\right)^{1/2}$$

Floating body - Various examples.

- Set $G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$K_p(G) \approx \frac{1}{\sqrt{p}} B_2^n \quad \text{and} \quad K_p(G)^o \approx \sqrt{p} B_2^n$$

- Set X uniformly distributed on a symmetric convex body K then

$$K_p(X)^o \approx Z_p(X)$$

where $h_{Z_p(X)}(\theta) = (\mathbb{E}\langle X, \theta\rangle^p)^{1/p}$.

- Set $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ where ε_i are iid Rademacher r.v. then

$$K_p(\mathcal{E}) \approx \mathrm{conv}\left(B_1^n \cup \frac{1}{\sqrt{p}} B_2^n\right) \quad \text{and} \quad K_p(\mathcal{E})^o \approx B_\infty^n \cap \sqrt{p} B_2^n$$

The result (GKKMR 2019).

The result (GKKMR 2019).

Few hypotheses on X :

1. $\|\cdot\|$ a norm

The result (GKKMR 2019).

Few hypotheses on X :

1. $\|\cdot\|$ a norm
2. Small ball property : there exist $\gamma > 0$ and $\delta > 0$ such that

$$\forall t \in \mathbb{R}^n, \mathbb{P}(|\langle X, t\rangle| \geq \gamma\|t\|) \geq \delta$$

The result (GKKMR 2019).

Few hypotheses on X :

1. $\|\cdot\|$ a norm
2. Small ball property : there exist $\gamma > 0$ and $\delta > 0$ such that
$$\forall t \in \mathbb{R}^n, \mathbb{P}(|\langle X, t\rangle| \geq \gamma\|t\|) \geq \delta$$
3. Moment assumption : for some $r > 0$ and $L > 0$, we have
$$\forall t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^r\right)^{1/r} \leq L\|t\|$$

The result (GKKMR 2019).

Few hypotheses on X :

1. $\|\cdot\|$ a norm
2. Small ball property : there exist $\gamma > 0$ and $\delta > 0$ such that

$$\forall t \in \mathbb{R}^n, \mathbb{P}(|\langle X, t\rangle| \geq \gamma\|t\|) \geq \delta$$

3. Moment assumption : for some $r > 0$ and $L > 0$, we have

$$\forall t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^r\right)^{1/r} \leq L\|t\|$$

Theorem

Let $0 < \alpha < 1$, $p = \alpha \log\left(\frac{eN}{n}\right)$ and $N \geq c_0(\alpha, r, \delta, L/\gamma)\, n$. Therefore, with probability $\geq 1 - 2\exp(-C_1 N^{1-\alpha} n^{\alpha})$,

$$\frac{1}{2} K_p^o \subset \mathrm{absconv}(X_1, \dots, X_N)$$

Various older results

With probability $\geq 1 - 2\exp(-C_1 N^{1-\alpha} n^{\alpha})$

Various older results

With probability $\geq 1 - 2\exp(-C_1 N^{1-\alpha} n^\alpha)$

- $X = G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$c_2 \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \subset \mathrm{absconv}(X_1, \dots, X_N)$$

Various older results

With probability $\geq 1 - 2\exp(-C_1 N^{1-\alpha} n^\alpha)$

- $X = G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$c_2 \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \subset \mathrm{absconv}(X_1, \ldots, X_N)$$

- (LPRT 2005) $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ where ε_i are iid Rademacher r.v. then

$$c_2 \left(B_\infty^n \cap \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \right) \subset \mathrm{absconv}(X_1, \ldots, X_N)$$

Various older results

With probability $\geq 1 - 2\exp(-C_1 N^{1-\alpha} n^{\alpha})$

- $X = G \sim \mathcal{N}(0, \mathrm{Id})$ then

$$c_2 \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \subset \mathrm{absconv}(X_1, \ldots, X_N)$$

- (LPRT 2005) $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ where ε_i are iid Rademacher r.v. then

$$c_2 \left(B_\infty^n \cap \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \right) \subset \mathrm{absconv}(X_1, \ldots, X_N)$$

- (DGT 2009) X uniformly distributed on a symmetric convex body K then

$$c_2 Z_p(X) \subset \mathrm{absconv}(X_1, \ldots, X_N)$$

The case of q-stable random vector.

$X = (\xi_1, \dots, \xi_n)$ with ξ_i iid q-stable : $\mathbb{E}\exp(itX) = \exp(-|t|^q/2)$
Observe that $\langle X, t\rangle = \sum t_i \xi_i \sim |t|_q \, \xi$ and remember that for every large enough u

$$\mathbb{P}(\xi \geq u) \approx 1/u^q$$

The case of q-stable random vector.

$X = (\xi_1, \dots, \xi_n)$ with ξ_i iid q-stable : $\mathbb{E}\exp(itX) = \exp(-|t|^q/2)$
Observe that $\langle X, t\rangle = \sum t_i \xi_i \sim |t|_q\, \xi$ and remember that for every large enough u

$$\mathbb{P}(\xi \geq u) \approx 1/u^q$$

1. $\|\cdot\| = |\cdot|_q$ a norm

The case of q-stable random vector.

$X = (\xi_1, \ldots, \xi_n)$ with ξ_i iid q-stable : $\mathbb{E}\exp(itX) = \exp(-|t|^q/2)$
Observe that $\langle X, t\rangle = \sum t_i \xi_i \sim |t|_q \, \xi$ and remember that for every large enough u

$$\mathbb{P}(\xi \geq u) \approx 1/u^q$$

1. $\|\cdot\| = |\cdot|_q$ a norm
2. Small ball property : there exist $\gamma > 0$ and $\delta > 0$ such that

$$\forall t \in \mathbb{R}^n, \mathbb{P}(|\langle X, t\rangle| \geq \gamma |t|_q) \geq \delta$$

The case of q-stable random vector.

$X = (\xi_1, \dots, \xi_n)$ with ξ_i iid q-stable : $\mathbb{E}\exp(itX) = \exp(-|t|^q/2)$
Observe that $\langle X, t\rangle = \sum t_i \xi_i \sim |t|_q\, \xi$ and remember that for every large enough u

$$\mathbb{P}(\xi \geq u) \approx 1/u^q$$

1. $\|\cdot\| = |\cdot|_q$ a norm
2. Small ball property : there exist $\gamma > 0$ and $\delta > 0$ such that

$$\forall t \in \mathbb{R}^n, \mathbb{P}(|\langle X, t\rangle| \geq \gamma |t|_q) \geq \delta$$

3. Moment assumption : for $r = q/2 > 0$ there is $L > 0$ such that

$$\forall t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^r\right)^{1/r} \leq L|t|_q$$

The case of q-stable random vector.

$X = (\xi_1, \dots, \xi_n)$ with ξ_i iid q-stable : $\mathbb{E}\exp(itX) = \exp(-|t|^q/2)$
Observe that $\langle X, t\rangle = \sum t_i\xi_i \sim |t|_q\,\xi$ and remember that for every large enough u

$$\mathbb{P}(\xi \geq u) \approx 1/u^q$$

1. $\|\cdot\| = |\cdot|_q$ a norm
2. Small ball property : there exist $\gamma > 0$ and $\delta > 0$ such that
$$\forall t \in \mathbb{R}^n, \mathbb{P}(|\langle X, t\rangle| \geq \gamma |t|_q) \geq \delta$$
3. Moment assumption : for $r = q/2 > 0$ there is $L > 0$ such that
$$\forall t \in \mathbb{R}^n, \left(\mathbb{E}|\langle X, t\rangle|^r\right)^{1/r} \leq L|t|_q$$

Theorem

For all $q \geq 1$, taking $p = \alpha \log(eN/n)$, we have

$$c_2(q)\left(\frac{N}{n}\right)^{\alpha/q} B^n_{q'} \subset K_p(X)^o \subset 2\mathrm{absconv}(X_1, \dots, X_N)$$

II) Stochastic domination and floating bodies.

Definition

Let X and Y be two centered random vectors in $\mathbb{R}^n$. We say that X dominates Y when there exist λ_1 and λ_2 such that

$$\forall t \in \mathbb{R}^n, \forall u \in \mathbb{R}, \quad \mathbb{P}(\langle X, t\rangle \geq u) \geq \lambda_1 \mathbb{P}(\langle Y, t\rangle \geq \lambda_2 u)$$

This gives

$$K_p(X) \subset \lambda_2 K_{p'}(Y)$$

with $p' = p - \log(1/\lambda_1)$.

This property is stable by tensorization : if x and y are symmetric r.v. such that for every $u > 0$, $\mathbb{P}(x > u) \leq \lambda_1 \mathbb{P}(y > \lambda_2 u)$ then $X = (x_1, \ldots, x_n)$ dominates $Y = (y_1, \ldots, y_n)$ with constants $c_1\lambda_1$ and $c_2\lambda_2$, where $x_1, \ldots, x_n$ are iid copies of x and $y_1, \ldots, y_n$ are iid copies of y.

Let $X = (\xi_1, \ldots, \xi_n)$ with ξ_i independent copies of a symmetric r.v. ξ.
Assume $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$.

Let $X = (\xi_1, \ldots, \xi_n)$ with ξ_i independent copies of a symmetric r.v. ξ. Assume $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$. Then for every $u \in \mathbb{R}$

$$\mathbb{P}(\xi \geq u) \geq \delta_0 \mathbb{P}(\varepsilon \geq \frac{u}{\gamma_0})$$

Let $X = (\xi_1, \dots, \xi_n)$ with ξ_i independent copies of a symmetric r.v. ξ. Assume $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$. Then for every $u \in \mathbb{R}$

$$\mathbb{P}(\xi \geq u) \geq \delta_0 \mathbb{P}(\varepsilon \geq \frac{u}{\gamma_0})$$

By tensorisation, for $X = (\xi_1, \dots, \xi_n)$ and $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_n)$, we get that there exist λ_1, λ_2 such that for every $t \in \mathbb{R}^n$

$$\forall u \in \mathbb{R}, \quad \mathbb{P}(\langle X, t\rangle \geq u) \geq \lambda_1 \mathbb{P}(\langle \mathcal{E}, t\rangle \geq \lambda_2 u)$$

Let $X = (\xi_1, \dots, \xi_n)$ with ξ_i independent copies of a symmetric r.v. ξ. Assume $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$. Then for every $u \in \mathbb{R}$

$$\mathbb{P}(\xi \geq u) \geq \delta_0 \mathbb{P}(\varepsilon \geq \frac{u}{\gamma_0})$$

By tensorisation, for $X = (\xi_1, \dots, \xi_n)$ and $\mathcal{E} = (\varepsilon_1, \dots, \varepsilon_n)$, we get that there exist λ_1, λ_2 such that for every $t \in \mathbb{R}^n$

$$\forall u \in \mathbb{R}, \quad \mathbb{P}(\langle X, t \rangle \geq u) \geq \lambda_1 \mathbb{P}(\langle \mathcal{E}, t \rangle \geq \lambda_2 u)$$

In conclusion, $K_p(X) \subset \lambda_2 K_{p'}(\mathcal{E})$ and $K_p(X)^o \supset \lambda_2^{-1} K_{p'}(\mathcal{E})^o$ where $p' = p - \log(1/\lambda_1)$.

Theorem (GLT 2018)

Let $X = (\xi_1, \dots, \xi_n)$ with ξ_i indep. copies of ξ. Suppose that $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma) \geq \delta$. Then for $N \geq c_0(\alpha, \gamma, \delta) n$, we have with proba $\geq 1 - 2\exp(-c_1 N^{1-\alpha} n^\alpha)$,

$$\mathrm{absconv}(X_1, \dots, X_N) \supset c_2 \left(B_\infty^n \cap \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \right)$$

Stochastic domination and comparaison

Theorem

Let $X = (\xi_1, \dots, \xi_n)$ be an unconditional random vector in $\mathbb{R}^n$. Assume that there exist γ and $\delta > 0$ such that for any $i = 1, \dots, n$

$$\mathbb{P}(|\xi_i| \geq \gamma) \geq \delta$$

then

$$K_p(X) \subset \frac{c(\delta)}{\gamma} K_p(\mathcal{E})$$

Stochastic domination and comparaison

Theorem

Let $X = (\xi_1, \dots, \xi_n)$ be an unconditional random vector in $\mathbb{R}^n$. Assume that there exist γ and $\delta > 0$ such that for any $i = 1, \dots, n$

$$\mathbb{P}(|\xi_i| \geq \gamma) \geq \delta$$

then

$$K_p(X) \subset \frac{c(\delta)}{\gamma} K_p(\mathcal{E})$$

Stochastic domination and comparaison

Theorem

Let $X = (\xi_1, \dots, \xi_n)$ be an unconditional random vector in $\mathbb{R}^n$. Assume that there exist γ and $\delta > 0$ such that for any $i = 1, \dots, n$

$$\mathbb{P}(|\xi_i| \geq \gamma) \geq \delta$$

then

$$K_p(X) \subset \frac{c(\delta)}{\gamma} K_p(\mathcal{E})$$

Moreover if X satisfies the hypotheses of the main result then with proba $\geq 1 - 2\exp(-C_1 N^{1-\alpha} n^\alpha)$, we have

$$\frac{\gamma}{c_2(\delta)} c_2 \left(B_\infty^n \cap \sqrt{\alpha \log\left(\frac{eN}{n}\right)} B_2^n \right) \subset \mathrm{absconv}(X_1, \dots, X_N)$$

Proof

Set $\Gamma = (X_1, \ldots, X_N)^*$ the matrix whose rows are $X_1, \ldots, X_N$. We need to prove that

$$\mathbb{P}\left(\inf_{t \in \partial K_p(X)} |\Gamma t|_\infty \geq 1/2\right) \geq 1 - 2\exp(-c_1 N^{1-\alpha} n^\alpha)$$

We define the set

$$\mathcal{F} = \{f(\cdot) = \mathbf{1}_{|\langle \cdot, u\rangle| \geq 1/2}, \quad u \in \partial K_p\}$$

in such a way that

$$\frac{1}{N}\sum_{j=1}^{N} f(X_j) = \#\{j, |\langle X_j, u\rangle| \geq 1/2\}$$

Key tool - Concentration inequality

Theorem (Talagrand 1996)

Let $\mathcal{F}$ be a class of functions taking values in $\{0,1\}$ such that $VC(\mathcal{F}) \leq d$ and $\sup_{f\in\mathcal{F}} \mathbb{E}f^2 = \sigma^2$. The for every $x > 0$,

$$\mathbb{P}\left(\sup_{f\in\mathcal{F}}\left|\frac{1}{N}\sum_{j=1}^{N} f(X_j) - \mathbb{E}f\right| \geq R + x\right) \leq \exp\left(-N\frac{x^2/2}{\sigma^2 + 2R + x/3}\right)$$

where $R \simeq \frac{d}{N}\log(\frac{c}{\sigma^2}) + \sigma\sqrt{\frac{d}{N}\log(\frac{c}{\sigma^2})}$.

In our case, we have

$$\mathcal{F} = \{f(\cdot) = \mathbf{1}_{|\langle\cdot,u\rangle|\geq 1/2}, \quad u \in \partial K_p\}$$

so that $VC(\mathcal{F}) \leq 10(n+1)$