HOME WORK FOR THE GRADUATE COURSE IN HIGH-DIMENSIONAL PROBABILITY, SPRING 2024, GEORGIA TECH

GALYNA V. LIVSHYTS

Please upload solutions via Canvas in pdf anytime, any number of times. While one only needs 7 points total to pass the course with an A, interested students are encouraged to solve more problems. The deadline for the home work is April 20.

If you find typos or have questions, please let me know!

The problems will keep being added throughout the semester, I anticipate their number reach about a 100 or more, with potential 200 (or so) points available. The problems are always added in the end of each section, so the number of a problem stays the same; however, problems could be added to various sections at any time.

You are encouraged to take a note of the home work questions, since a chunk of worthwhile material will be left as a home work, in order to keep the course going with some pace.

1. Probabilistic method

Question 1.1 (1 point)**.** Let P be a polytope in $\mathbb{R}^n$ with N vertices and whose diameter is bounded by 1. Then P can be covered by at most $N^{[\frac{1}{\epsilon^2}]}$ Euclidean balls of radii $\epsilon > 0$.

Question 1.2 (1 point)**.** Improve the epsilon-net argument that we discussed in class to show that for any $\epsilon \in (0,1)$ the sphere $\mathbb{S}^{n-1}$ can be covered by at most $\left(\frac{C}{\epsilon}\right)^{n-1}$ balls of radius ϵ (note that the power is $n-1$ rather than n.)

Question 1.3 (1 point)**.** Show that the sequence $(1-\frac{1}{n})^n$ is increasing.

Question 1.4 (1 point)**.** Show that L_p-norm is indeed a norm on the space of measurable functions with finite L_p-norm.

Question 1.5 (1 point)**.** Show Hölder's inequality: for any $p, q \geq 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, one has

$$|\mathbb{E}XY| \leq \|X\|_p \cdot \|Y\|_q.$$

Question 1.6 (5 points)**.** Show that "hexagonal packing" (with centers at hexagonal lattice) corresponds to optimal packing in dimension 2.

Question 1.7 (1 point)**.** Prove Caratheodory's theorem that attests that any point in the convex hull of any set $A \subset \mathbb{R}^n$ belongs to a simplex with vertices in A (see also our class notes for the discussion).

Question 1.8 (1 point)**.** Let $n \in \mathbb{N}$. Show that indeed for any positive integer $R \geq n$ we have

$$\sum_{N=1}^{R} \binom{N+n-1}{N} \leq \left(\frac{CR}{n}\right)^n,$$

where is C is some absolute constant independent on R or n.

Question 1.9 (1 point)**.** Let A be any $N \times n$ matrix with entries a_{ij}. Show that

$$\sum_{i,j} a_{ij}^2 = \sigma_1^2(A) + ... + \sigma_n^2(A),$$

where $\sigma_i(A)$ are the singular values of A.

2. Concentration of sums of independent random variables

Question 2.1 (1 point)**.** Show that indeed

- $cosh(x) \leq e^{\frac{x^2}{2}}$;
- $\frac{1}{1-x} \leq e^{2x}$ for $x \in [0, \frac{1}{2}]$.

Question 2.2 (1 point)**.** Prove the general two-sided Hoeffding inequality for bounded random variables (Lemma 2.3):

Let $X_1, ..., X_n$ be independent random variables taking values in $[m_i, M_i]$, $i = 1, ..., n$. Then for any $\beta > 0$,

$$P\left(\left|\sum_{i=1}^{n} X_i - \mathbb{E}X_i\right| \geq \beta\right) \leq 2e^{-\frac{c\beta^2}{\sum_{i=1}^{n}(M_i - m_i)^2}},$$

where $c > 0$ is an absolute constant.

Question 2.3 (1 point)**.** Show that Theorem from the Exercise 2.2 is not sharp when X_i are non-symmetric Bernoullis (taking value 1 with probability p, and, say, 0 otherwise, where p is close to 1, say).

Question 2.4 (1 point)**.** Exercise 2.3.2, Vershynin

Question 2.5 (1 point)**.** Exercise 2.3.3, Vershynin

Question 2.6 (1 point)**.** Exercise 2.3.5, Vershynin

Question 2.7 (1 point)**.** Exercise 2.3.6, Vershynin

Question 2.8 (1 point)**.** Exercise 2.4.2, Vershynin

Question 2.9 (1 point)**.** Exercise 2.4.3, Vershynin

Question 2.10 (2 points)**.** Exercise 2.4.4, Vershynin

Question 2.11 (1 point)**.** Exercise 2.4.5, Vershynin

Question 2.12 (1 point)**.** Exercise 2.5.1, Vershynin

Question 2.13 (1 point)**.** Exercise 2.5.4, Vershynin

Question 2.14 (1 point)**.** Exercise 2.5.7, Vershynin

Question 2.15 (1 point)**.** Show that:

- Gaussian, $p-$Bernoulli and bounded random variables are sub-Gaussian.
- Exponential, Poisson and Cauchy distributions are not sub-Gaussian.

Question 2.16 (2 points)**.** Exercise 2.5.10, Vershynin

Question 2.17 (1 point)**.** Exercise 2.5.11, Vershynin

Question 2.18 (1 point)**.** Complete the following:

- Exercise 2.6.6, Vershynin
- Exercise 2.6.7, Vershynin

Question 2.19 (1 point)**.** Prove the equivalence of the sub-exponential properties (Proposition 2.7.1 in Vershynin).

Question 2.20 (1 point)**.** Exercise 2.7.10, Vershynin

Question 2.21 (1 point)**.** Let Z be the standard Gaussian random variable. Prove that for any $t \geq 1$,

$$P(Z > t) \geq \frac{1}{\sqrt{2\pi}}(\frac{1}{t} - \frac{1}{t^3})e^{-\frac{t^2}{2}}.$$

Question 2.22 (2 points)**.** Exercises 2.8.5, 2.8.6, Verhsynin.

Question 2.23 (2 points)**.** Exercise 3.1.4, Vershynin

Question 2.24 (2 points)**.** Exercise 3.1.5, Vershynin

Question 2.25 (2 points)**.** Exercise 3.1.6, Vershynin

Question 2.26 (1 point)**.** Exercise 3.1.7, Vershynin

Question 2.27 (1 point)**.** Exercise 3.3.7, Vershynin

Question 2.28 (1 point)**.** Exercise 3.4.3, Vershynin

Question 2.29 (1 point)**.** Exercise 3.4.7, Vershynin

Question 2.30 (2 points)**.** Exercise 3.4.9, Vershynin

Question 2.31 (1 point)**.** Exercise 3.4.10, Vershynin

Question 2.32 (2 points)**.** Exercise 3.5.3, Vershynin

Question 2.33 (3 points)**.** Solve exercise 3.5.5, prove Theorem 3.5.6, exercise 3.5.7, Vershynin (applications of Grothendieck's inequality to semi-definite programming.)

Question 2.34 (3 points)**.** Prove Proposition 3.6.3, solve exercise 3.6.4, Vershynin (applications of Grothendieck's inequality to MAX-CUT algorithm.)

Question 2.35 (1 point)**.** Prove the lower bound in Khinchine's inequality: for a mean zero variance 1 independent random variables $X_1, ..., X_N$, and any $a \in \mathbb{R}^N$, letting $X = (X_1, ..., X_N)$ one has, for all $p \geq 2$:

$$|a| \leq (\mathbb{E}|\langle a, X\rangle|^p)^{\frac{1}{p}} .$$

Question 2.36 (1 point)**.** Let $X = (X_1, ..., X_n)$ be a standard Gaussian random vector (that is, a random vector in $\mathbb{R}^n$ such that its coordinates are independent, and each X_i is a $N(0, 1)$ random variable.) How to estimate from above

$$P(\|X\|_p \geq t),$$

where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$, and $p \geq 1$? Note that the answer may depend on p.

Question 2.37 (1 point)**.** Find additional proofs of the fact that a product of sub-Gaussian random variables is sub-exponential, which involve:

a) The implication of the sub-Gaussian property (i) into sub-exponential property (a);
b) The implication of the sub-Gaussian property (iI) into sub-exponential property (b);
c) The implication of the sub-Gaussian property (iii) into sub-exponential property (c).

Question 2.38 (1 point)**.** Prove Lemma 2.6.8 from Vershynin.

Question 2.39 (1 point)**.** Check that $\|\cdot\|_{\psi_1}$ is a norm.

Question 2.40 (1 point)**.** Show that for a sub-Gaussian random variable X we have $\|\mathbb{E}X\|_{\psi_2} \leq C\|X\|_{\psi_2}$.

Question 2.41 (1 point)**.** Show that if g is the standard Gaussian random vector on $\mathbb{R}^n$ then $\frac{g}{|g|}$ is uniformly distributed on $\mathbb{S}^{n-1}$.

Question 2.42 (1 point)**.** Let g be the standard Gaussian random vector on $\mathbb{R}^n$, and let $u, v \in \mathbb{S}^{n-1}$. Consider $U = \langle g, u\rangle$ and $V = \langle g, v\rangle$. Show that

$$\mathbb{E}(UV) = \langle u, v\rangle.$$

Question 2.43 (1 point)**.** Let U be the random variable defined above in Question 2.42, let $R > 0$ and let $U^+ = 1_{\{U \geq R\}}U$. Show that

$$\mathbb{E}U^2 \leq \frac{4}{R^2}.$$

3. Random matrices

Question 3.1 (1 point)**.** Prove that for an $n \times n$ symmetric random matrix with upper corner elements being mean zero independent K-sub-Gaussian we have for all $t > 0$,

$$\|A\| \leq CK(\sqrt{n} + t)$$

with probability $1 - 4e^{-t^2}$.

Question 3.2 (1 point)**.** Let A be any $N \times n$ matrix and pick $\delta > 0$. Suppose $\|A^T A - \mathrm{Id_n}\| \leq \max(\delta, \delta^2)$, then for all $x \in \mathbb{S}^{n-1}$,

$$|Ax| \in [1 - \delta, 1 + \delta].$$

Question 3.3 (2 points)**.** Show that the bound on the norm of a random matrix with independent sub-Gaussian mean zero coordinates which we deduced in class is optimal when $\mathbb{E}a_{ij}^2 = 1$ while K is an absolute constant.

Question 3.4 (7 points)**.** Let A be a random matrix with independent mean zero coordinates a_{ij} such that $\mathbb{E}a_{ij}^2 = 1$.
a) Show that when $\mathbb{E}a_{ij}^4 \leq K < \infty$ one has $\mathbb{E}\|A\| \leq C\sqrt{n}$, where C only depends on K.
b) Show that it can happen that $\mathbb{E}\|A\| >> \sqrt{n}$ when $\mathbb{E}a_{ij}^4 = \infty$ for all i, j. Here we use notation $a_n >> b_n$ to mean that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.
c) How large could $\|A\|$ be in b)?
d) Is it true that necessarily $\mathbb{E}\|A\| >> \sqrt{n}$ whenever $\mathbb{E}a_{ij}^4 = \infty$ for some of the i, j?

Question 3.5 (1 point)**.** Show that if X is a mean zero sub-Gaussian random variable then it has a bounded concentration function (and a small ball estimate).

Question 3.6 (1 point)**.** Exercise 4.5.4 from Vershynin.

Question 3.7 (1 point)**.** Exercise 4.6.3 from Vershynin.

Question 3.8 (1 point)**.** Exercise 4.6.4 from Vershynin.

Question 3.9 (5 points)**.** Let A be an $N \times n$ random matrix, $N \geq n$, such that all of its entries a_{ij} are independent mean zero, and $\mathbb{E}a_{ij}^2 = 1$. Suppose

$$P(\|A\| \geq C\sqrt{N}) \leq e^{-cn},$$

for some constants $C, c > 0$. Does this mean that at least some of the entries of A are sub-Gaussian? If yes, "how many" of the entries are sub-Gaussian?

Question 3.10 (1 point)**.** Let A be the symmetric $n \times n$ matrix with mean zero sub-Gaussian entries, such that the upper-triangular entries are independent. What bound on the large deviation of the norm of A do you get from the matrix Bernstein's inequality applied to X_{ij}, the matrices all of whose entries are zero except a_{ij} and a_{ij} (some of them only have one non-zero diagonal entry)? Note that we can write $A = \sum_{i,j} X_{ij}$.

Question 3.11 (1 point)**.** Prove Lieb's inequality when $n = 1$ using things that you know.

Question 3.12 (1 point)**.** Prove Lemma 5.4.10 from Vershynin.

Question 3.13 (1 point)**.** Conclude the proof of the Matrix Bernstein's inequality using all the facts we deduced in class.

Question 3.14 (1 point)**.** Suppose ξ is $K-$sub-Gaussian and mean zero. Show that there exists $\epsilon_0 > 0$ which depends only on K and $C > 0$ which depends only on K, such that for any $\epsilon \geq \epsilon_0$,

$$P(|\xi| \leq \epsilon) \leq C\epsilon.$$

Question 3.15 (1 point)**.** Show (using Rogozin's theorem): suppose X in $\mathbb{R}^n$ is a random vector with independent coordinates X_i which satisfy $P(|X_i| < a) < b$ for some $a > 0$ and $b \in (0,1)$. Then for any $\theta \in \mathbb{S}^{n-1}$ one has

$$P(|\langle \theta, X\rangle| \leq a_1) \leq b_1,$$

where a_1 and b_1 only depend on a and b.

Question 3.16 (1 point)**.** Let $P = [0,\alpha_1] \times ... \times [0,\alpha_n]$ be a "coordinate box" in $\mathbb{R}^n$ with sides $\alpha_1, ..., \alpha_n > 0$ such that $\alpha_i \in [0,1]$ for all i, and $\prod_{i=1}^n \alpha_i \geq \kappa^{-n}$ for some $\kappa \geq 1$. Then there exists a covering of $\mathbb{S}^{n-1}$ by the copies of ϵP of the size $\left(\frac{C\kappa}{\epsilon}\right)^n$, or, in other words, there is a finite set $\mathcal{N} \subset B_2^n$ with

$$\mathbb{S}^{n-1} \subset \cup_{y \in \mathcal{N}}(y + \epsilon P),$$

and $\#\mathcal{N} \leq \left(\frac{C\kappa}{\epsilon}\right)^n$.

Question 3.17 (1 point)**.** Fix $y_1, ..., y_n > 0$ and find the minimum value of the expression

$$\sum_{i=1}^n a_i y_i$$

among all a_i such that $\prod a_i = C$ and $a_i > 0$, $i = 1, ..., n$.

Question 3.18. With $\mathcal{B}_\kappa(A)$ as was defined in class (on February 19), and A, an $n \times n$ random matrix with independent columns, find an upper bound $F(\kappa, t)$ such that

$$P(\mathcal{B}_\kappa(A) \geq t\mathbb{E}\|A\|_{HS}^2) \leq F(\kappa, t),$$

where $t \geq 10$ and $F(\kappa, t) \leq (C\kappa)^{-n}$, and such that $F(\kappa, t) \to_{t\to\infty} 0$.

Question 3.19 (7 points)**.** Prove Rogozin's theorem which we stated in class.

4. Gaussian random processes

Question 4.1 (1 point)**.** Let X be any mean zero Gaussian random vector on $\mathbb{R}^n$ and g be the standard Gaussian random vector on $\mathbb{R}^n$. Then there exist vectors $v_1, ..., v_n$ in $\mathbb{R}^n$ such that $(\langle g, v_1\rangle, ..., \langle g, v_n\rangle)$ has the same distribution as X.

Question 4.2 (3 points)**.** a) Show that a Gaussian random vector is uniquely determined by its covariance matrix.
b) Conclude that a Gaussian random process is uniquely determined by its covariance function
c) Show that this characterizes the Gaussian distribution. Namely, show that for a non-Gaussian random vector X in $\mathbb{R}^n$, one may find at least two different (not just up to symmetries) collections of vectors $u_1, ..., u_n$ and $v_1, ..., v_n$ in $\mathbb{R}^n$ such that the random vectors $(\langle X, u_1\rangle, ..., \langle X, u_n\rangle)$ and $(\langle X, v_1\rangle, ..., \langle X, v_n\rangle)$ have the same covariance matrices. (Note: it is enough to consider the case $n = 2$.)

Question 4.3 (1 point)**.** Let g be the standard Gaussian random vector on $\mathbb{R}^n$ and T be any set in $\mathbb{R}^n$. Let $X_t = \langle g, t\rangle$, for $t \in T$. Show that

$$\mathbb{E}|X_t - X_s|^2 = |t - s|^2$$

for any $t, s \in T$.

Question 4.4 (4 points)**.** Show that the assumption of the random processes X_t and Y_t being Gaussian is necessary for the validity of Slepian's inequality.

Question 4.5 (1 point)**.** Prove the multi-dimensional first order Gaussian integration by parts (Lemma 7.2.5 from Vershynin).

Question 4.6 (1 point)**.** Show that $\Sigma(Z(u)) = u\Sigma(X) + (1-u)\Sigma(Y)$ if $Z(u) = \sqrt{u}X + \sqrt{1-u}Y$ and X, Y are independent Gaussian random vectors.

Question 4.7 (1 point)**.** Show that the processes in Slepian's inequality can indeed be assumed to be independent.

Question 4.8 (1 point)**.** Show that it is enough to prove Slepian's inequality for random vectors in order to deduce it for processes.

Question 4.9 (1 point)**.** Complete the details in the proof of the Sudakov-Fernique inequality which we left out in class.

Question 4.10 (2 points)**.** Let u, v, w, z be unit vectors in $\mathbb{R}^n$. Show that
$$\|u \otimes v - w \otimes z\|_{HS}^2 \leq |u-w|^2 + |v-z|^2.$$

Question 4.11 (1 point)**.** Exercise 7.1.8 from Vershynin.

Question 4.12 (2 points)**.** Exercise 7.1.9 from Vershynin.

Question 4.13 (1 point)**.** Exercise 7.1.13 from Vershynin.

Question 4.14 (1 point)**.** Exercise 7.2.2 from Vershynin.

Question 4.15 (1 point)**.** Exercise 7.2.13 from Vershynin.

Question 4.16 (2 points)**.** Exercise 7.2.14 from Vershynin.

Question 4.17 (2 points)**.** Exercise 7.3.4 from Vershynin.

Question 4.18 (2 points)**.** Exercise 7.3.5 from Vershynin.

Question 4.19 (3 points)**.** Would it be possible to use the tools we learned in order to give the estimate on the large deviation for the norm of the standard Gaussian random matrix, rather than just provide a bound on the expected value of the norm?

Question 4.20 (1 point)**.** Fix $n, N \in \mathbb{N}$. For a collection of i.i.d. random vectors $X_1, ..., X_N$ on $\{-1,1\}^n$, estimate $\mathbb{E}\sup_{j=1,...,N} \#\{i : X_i^j = 1\}$.

5. Markov semigroups

Question 5.1 (1 point)**.** Give an example of a Stochastic process which is not a Markov process.

Question 5.2 (1 point)**.** Let L be a second order linear elliptic operator on $\mathbb{R}^n$ with the measure μ such that for any functions f, g in the domain of the operator, $\int fLg d\mu = -\int \langle \nabla f, \nabla g \rangle d\mu$. Let $P_t f$ be the semigroup with the generator L (and therefore, the stationary measure μ). Prove (using PDE methods, and without using the conditional expectation), that
a) P_t is a linear operator;
b) $P_t P_s f = P_{t+s} f$;
c) $P_t 1 = 1$;
d) $\|P_t f\|_{L_p(\mu)} \leq \|f\|_{L_p(\mu)}$.

Question 5.3 (1 point)**.** Prove that the heat semigroup on $\mathbb{R}^n$ is reversible.

Question 5.4 (2 points)**.** Show that linear functions provide the only equality cases in the Gaussian Poincare inequality.

Question 5.5 (2 points)**.** Using the heat semigroup on the circle, prove the p-Beckner inequality for periodic functions, with $p \in [1,2]$: for any continuously twice differentiable 2π-periodic function $\psi \geq 0$ on $\mathbb{R}$ one has
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi^2 - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \psi^p\right)^{\frac{2}{p}} \leq \frac{2-p}{2\pi}\int_{-\pi}^{\pi} \dot{\psi}^2.$$
Here $\dot{\psi}$ stands for the derivative of ψ.

Question 5.6 (3 points)**.** Let γ be the standard Gaussian measure on $\mathbb{R}^n$ and $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a non-decreasing twice differentiable function. Find some general necessary conditions on Φ so that for every measurable $f \in C^2(\mathbb{R}^n)$ such the integrals in question exist, one has
$$\int \Phi(f) d\gamma - \Phi\left(\int d\gamma\right) \leq C \int |\nabla f|^2 \Phi''(f) d\gamma,$$
for some $C > 0$ (the conditions might involve C.)

Question 5.7 (1 point)**.** Show that when $d\mu = e^{-V}dx$ on $\mathbb{R}^n$ where $V \in C^2(\mathbb{R}^n)$ and $\int d\mu = 1$, and if μ obeys some Poincare inequality, and there exists a C^1 function which saturates it, then the Poincare constant of μ is inverse of the first eigenvalue of the operator $Lu = \Delta u - \langle \nabla u, \nabla V \rangle$.

Question 5.8 (4 points)**.** Let $d\mu = \frac{1}{2}e^{-|t|}dt$ on $\mathbb{R}$. Show that it does obey some Poincare inequality, but the equality case is not attained.

Question 5.9 (3 points)**.** Provide a proof of the Gaussian Poincare inequality using Hermit polynomial decomposition.

Question 5.10 (2 points)**.** Provide a proof of the Poincare inequality on the circle for periodic functions using heat semigroup on the circle (which we briefly discussed in class).

Question 5.11 (1 point)**.** Show that $\log \varepsilon(P_t f, P_t f)$ is convex in t for a reversible Markov semigroup P_t.

Question 5.12 (1 point)**.** Finish the proof of the theorem about the Poincare inequality for a general ergodic reversible Markov semigroup by proving the remaining implications $2 \implies 3$, $4 \implies 2$, $5 \implies 3$.

Question 5.13 (1 point)**.** Prove Beckner's inequalities using the semigroup method and the Ornstein-Uhlenbeck semigroup: for $p \in [0, 1]$ and any measurable non-negative $f \in C^1(\mathbb{R}^n)$ such the integrals below exist, show that

$$\int f^2 d\gamma - \left(\int f^p d\gamma \right)^{\frac{2}{p}} \le (2-p) \int |\nabla f|^2 d\gamma.$$

Hint: maybe you want to consider the equivalent form of this inequality which corresponds to the $\Phi-$Sobolev inequality with $\Phi(s) = s^p$, $p \in [1, 2]$.

Question 5.14 (2 points)**.** Show that $p-$Beckner inequality gets stronger as p increases, and that this fact implies Log-Sobolev inequality when $p \to 2$.

Question 5.15 (1 point)**.** Show the $p-$Beckner inequalities for $p \in [1, 2]$ with the Gaussian measure replaced by the uniform measure on the circle for periodic functions.

Hint: use the heat semigroup.

Question 5.16 (6 points)**.** a) Prove that the $p-$Beckner inequality on the circle for periodic functions holds with $p = -2$, using whichever method you might find (this is hard without additional tools!).

b) Show that part a) is impossible to do via the semigroup method using the heat semigroup on the circle (this is possible by outlining the inequality which the method would require and finding a numeric counter-example).

Question 5.17 (1 point)**.** Show that the Gaussian Log-Sobolev inequality of Gross is equivalent to the following inequality, under the assumption that f is measurable and $f \ge 0$ and $\int f^2 dx = 1$, and all the integrals below exist:

$$\int f^2 \log f dx + \frac{n}{4} \log(2\pi e^2) \le \int |\nabla f|^2 dx.$$

Question 5.18 (1 point)**.** Show that the inequality from Question 5.17 (and therefore, the Gaussian Log-Sobolev inequality) follows from the Sobolev inequality:

$$\|f\|_{\frac{n}{n-1}} \le C_n \int |\nabla f| dx$$

for any measurable f for which the above integrals make sense, and

$$C_n = \frac{|B_2^n|^{\frac{n}{n-1}}}{|\mathbb{S}^{n-1}|} = \frac{|B_2^n|^{\frac{1}{n-1}}}{n}.$$

Question 5.19 (1 point)**.** Deduce the isoperimetric inequality from Sobolev's inequality (the one from Question 5.18).

Question 5.20 (1 point)**.** Show that the Log-Sobolev inequality for a probability measure μ, that is,

$$\int f\log f d\mu - \int f d\mu \log \int f d\mu \leq C(\mu)\int \frac{|\nabla f|^2}{f}d\mu$$

for all $f \geq 0$ for which the integrals make sense, fails to hold when $d\mu = \frac{1}{2}e^{-|t|}dt$ on $\mathbb{R}$ (in other words, $C(\mu) = \infty$ in this case).

6. Concentration of measure

Question 6.1 (1 point)**.** Show that for any $t \geq 0$,

$$\frac{1}{\sqrt{2\pi}}\int_t^{\infty} e^{-\frac{s^2}{2}}ds \leq \frac{1}{2}e^{-\frac{t^2}{2}}.$$

Question 6.2 (2 points)**.** Iron out the details of the implication of the concentration of measure on the sphere via the Gaussian concentration of measure (that we discussed in class), and replace the assumption $\gamma(A) \geq 0.51$ with $\gamma(A) \geq 0.5$.

Question 6.3 (1 point)**.** Show that the median can be replaced with the mean in the statement of the concentration of measure estimate for Lipschitz functions on the sphere (possibly, with a different constant).

Question 6.4 (3 points)**.** Prove the concentration of measure on the Hamming cube – Theorem 5.2.5 in Vershynin.

Question 6.5 (3 points)**.** Exercise 5.3.3 in Vershynin.

Question 6.6 (3 points)**.** Exercise 5.3.4 in Vershynin.

Question 6.7 (2 points)**.** Show that for any integers m and n with $1 \leq m \leq n$, we have

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \max_{j\in\{1,\dots,m\}} |x_j|dx \geq c\left(\frac{\log m}{b}\right)^{\frac{1}{2}}.$$

Here dx is the uniform measure on the sphere.

Question 6.8 (1 point)**.** In the setting of Dvoretzky's theorem, using the fact that $\frac{M}{b} \geq c\sqrt{\frac{\log n}{n}}$ in an appropriate affine position, deduce the Milman-Dvoretzky theorem (as stated in class, but with an extra $\log\frac{1}{\epsilon}$ factor) from our results in class.

Question 6.9 (2 points)**.** a) Show that the Milman-Dvoretzky theorem is sharp, up to an absolute constant, with $\epsilon = 0.01$, in the case when $\|\cdot\| = \|\cdot\|_\infty$.

b) But show that the lower inclusion alone is not sharp even in this case.

Question 6.10 (1 point)**.** Show that in the case of $\|\cdot\| = \|\cdot\|_\infty$ we have $\frac{M}{b} = c\sqrt{\frac{\log n}{n}}$ (in the notation we used in class when talking about Dvoretzky's theorem).

7. Mass transport

Question 7.1 (1 point)**.** Confirm that indeed, for $T_*\mu = \nu$, under the assumptions we made in class, the following two definitions are equivalent:
1. For all measurable $A \subset supp(\nu)$ one has $\nu(A) = \mu(T^{-1}A)$;
2. For all $\varphi \in L^1(\nu)$,

$$\int \varphi d\nu = \int \varphi\circ T d\mu.$$

Question 7.2 (1 point)**.** Outline the details of the proof for part 4) of the Claim in class about the Lipshitz transport map – the fact that $I_\nu \geq \frac{1}{L}I_\mu$ if there is an $L-$Lipshitz map T such that $T_*\mu = \nu$.

Question 7.3 (1 point)**.** Show that subgradient $\partial f(x)$ of a differentiable convex function consists of one point $\nabla f(x)$.

Question 7.4 (1 point)**.** Finish the proof of the Rockafellar theorem we stated in class on Monday April 15.

Question 7.5 (1 point)**.** Using the Rockafellar theorem, deduce part 2) of the discrete version of the Brenier theorem which we stated in class.

Question 7.6 (3 points)**.** Prove that the support of an optimal coupling between probability measures is a cyclically monotone set.

Question 7.7 (2 points)**.** Show that if a set $A \subset \mathbb{R} \times \mathbb{R}$ is a one-to-one map (i.e. if (x, y) and (x, z) being in A implies that $y = z$), then it is cyclically monotone if and only if it is contained in a graph of a non-decreasing function. (This follows from Rockafellar's theorem but do not use it, re-prove this elementary fact.)

Question 7.8 (1 point)**.** Conclude the proof of the Rockafellar's theorem which we started in class, by showing that the set S is contained in $\partial\varphi$, with φ that we defined in the proof in class.

Question 7.9 (1 point)**.** Show that if $\varphi \in C^1(\mathbb{R}^n)$, a non-infinite convex function, then for all $x \in \mathbb{R}^n$, we have $\varphi(x) + \varphi^*(\nabla\varphi(x)) = \langle \nabla\varphi(x), x\rangle$.

Question 7.10 (2 points)**.** Given a pair of absolutely continuous probability measures μ and ν, prove the uniqueness of the Brenier map T which transports μ into ν.

Question 7.11 (3 points)**.** Using an argument similar to the mass transport proof of Brunn-Minkowksi inequality (as discussed on April 17), provide a mass transport proof of the Prekopa-Leindler inequality (which was stated in class April 17).

Question 7.12 (1 point)**.** Show the equivalence between the multiplicative and the additive formulations of the Brunn-Minkowski inequality that we discussed in class.

Question 7.13 (3 points)**.** Prove (modulo regularity issues) the Gaussian Log-Sobolev inequality using Brenier's map and the transport equation, similarly to how we proved Talagrand's inequality and Brunn-Minkowski. Hint: when dealing with the log of the Hessian of the convex function, use the inequality $\log(1 + t) \leq t$ valid for all $t \geq 0$.