HOME WORK I, ANALYSIS I

Due September 5. Problems marked with asterisk are optional, but highly recommended. Please contact me if you have any questions!

1. Let $g(x) = x^2$ and $f(x) = x + 2$ for $x \in \mathbb{R}$ and let $h = g \circ f$.
a) Find the direct image $h(E)$ of $E = \{x \in \mathbb{R} : x \in [0, 1]\}$.
b) Find the inverse image $h^{-1}(G)$ of $G = \{x \in \mathbb{R} : x \in [0, 4]\}$.

2. Show that if $f : A \to B$ and G, H are subsets of B, then
a) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(G) \cup f^{-1}(H)$;
b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

3. a) Show that if $f : A \to B$ is injective and $E \subset A$, then $f^{-1}(f(E)) = E$. Give an example to show that the equality need not hold if f is not injective.
b) Show that if $f : A \to B$ is surjective and $H \subset B$, then $f(f^{-1}(H)) = H$. Give an example to show that the equality need not hold if f is not surjective.

4. Try and guess a formula for $1 + 3 + 5 + ... + (2n - 1)$, and prove the formula using induction.

5. Prove that for all natural numbers n such that $n \geq 2$, one has

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

6*. Given the set of 51 integers between 1 and 100 (inclusive), show that at least one member of the set must divide another member of the set.
Hint: use induction

7. Let n be a natural number. Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + ... + \frac{1}{n^3} < \frac{3}{2}.$$

8^*. Let $n \geq 3$ be a natural number. Let S denote an $n \times n$ lattice square, that is

$$S = \{(k, m) : \ k, m \in \mathbb{N}, \ k \in [1, n], \ m \in [1, n]\}.$$

Show that it is possible to draw a polygonal path consisting of $2n - 2$ segments which will pass through all of the n^2 lattice points of S.