HOME WORK 2, ANALYSIS II

Due February 13. Problems marked with asterisk are optional, but highly recommended. Please contact me if you have any questions!

1. Prove that $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ is a norm on $\mathbb{R}^n$ for $p \in [1, \infty)$, but is not a norm for $p \in (0, 1)$.

2. Prove that a metric limit in any metric space (X, d) is unique.

3. Prove that a convergent sequence in any metric space (X, d) is Cauchy.

4. Verify the statement discussed in class, that for any bounded open symmetric convex set K in $\mathbb{R}^n$, the Minkowski functional

$$\|x\|_K := \inf\{t > 0 : x \in tK\}$$

is a norm, and that K is the unit ball of that norm.

5. Show that the intersection of any collection of convex symmetric bounded sets is convex, symmetric and bounded. Conclude that for any finite collection $K_1, ..., K_m$ of convex symmetric bounded open sets, $\cap_{i=1}^{m} K_i$ is the unit ball of some norm, and find its expression. Can this conclusion be made for any infinite collection of such sets?

6. Prove that a countable intersection of nested compact sets is non-empty.

7. For a metric d on a space X, consider a ball

$$B(x_0, r) = \{y \in X : \ d(x_0, y) < r\},$$

and a "closed" ball

$$B^{cl}(x_0, r) = \{y \in X : \ d(x_0, y) \le r\}.$$

Show that $\overline{B(x_0, r)} \subset B^{cl}(x_0, r)$. Can it happen that this inclusion is proper, i.e. that

$$\overline{B(x_0, r)} \neq B^{cl}(x_0, r)?$$

8. Extend the notion of a limit point to the setting of metric spaces, and show that a is a limit point of a sequence $\{x_n\} \subset X$ if and only if it has a subsequence converging to a.

9. For $X = \mathbb{R}$ and metric $d(x, y) = |x - y|$, does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed set $A \subset \mathbb{R}$ such that $f(A)$ is not closed?