ANALYSIS I, HOME WORK 4, FALL 2018

Due October 3.

1. Limits.

1.1. **Example 1.** Prove that $\lim_{n\to\infty} \frac{n}{n+3} = 1$.

Proof. By the definition of the limit, we are required to show that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\left|\frac{n}{n+3} - 1\right| \leq \epsilon.$$

Select any $\epsilon > 0$. Let

$$N = N_\epsilon = [\frac{3}{\epsilon}]. \tag{1.1}$$

(It is recommended to leave the right hand side of the above blank, as you are working through the proof, and then fill it in, once it is clear what you want to write there.)

For any positive number n, we have

$$\left|\frac{n}{n+3} - 1\right| = \left|\frac{n-n-3}{n+3}\right| = \frac{3}{n+3}. \tag{1.2}$$

Next, for any integer $n \geq N_\epsilon$, we estimate

$$\frac{3}{n+3} \leq \frac{3}{N_\epsilon + 3}. \tag{1.3}$$

By the definition of integer part, we have, for any number a,

$$a - 1 \leq [a] \leq a \tag{1.4}$$

In view of (1.4), applied with $a = \frac{3}{\epsilon}$, and recalling the definition of N_ϵ given in (1.1), we write

$$\frac{3}{N_\epsilon + 3} = \frac{3}{[\frac{3}{\epsilon}] + 3} \leq \frac{3}{\frac{3}{\epsilon} + 2} < \epsilon. \tag{1.5}$$

Combining (1.2), (1.3) and (1.5), we see that we showed, that for every $\epsilon > 0$ there exists a positive integer $N = N_\epsilon$, such that for all $n \geq N_\epsilon$, we have $|\frac{n}{n+3} - 1| \leq \epsilon$. Therefore, $\lim_{n\to\infty} \frac{n}{n+3} = 1$. The proof is complete. □

1.2. **Example 2.** Prove that $a_n = 2 \cdot (-1)^n$ does not have a limit.

Proof. Suppose, by contradiction, that a_n has a limit. Let us call this limit a. By the definition of the limit, we have, that for every $\epsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that for every $n \geq N$,

$$|a - 2 \cdot (-1)^n| \leq \epsilon. \tag{1.6}$$

Since we assume that this holds for every $\epsilon > 0$, it must hold for $\epsilon = 1$. Which means, that there exists a positive integer $N = N_1$ such that for all $n \geq N_1$ we have (1.6) with $\epsilon = 1$. In particular, plugging $n = N_1$ and $n = N_1 + 1$ correspondingly, we arrive to a pair of inequalities

$$|a - 2 \cdot (-1)^{N_1}| \leq 1; \tag{1.7}$$

$$|a - 2 \cdot (-1)^{N_1+1}| \leq 1. \tag{1.8}$$

One of the numbers N_1 and $N_1 + 1$ is necessarily even, while the other is necessarily odd. Hence we have, by (1.7) and (1.8), that

$$|a - 2| \leq 1, \tag{1.9}$$

and

$$|a + 2| \leq 1. \tag{1.10}$$

Therefore, combining (1.9) and (1.10), we have

$$a \in [1, 3] \cap [-3, -1]. \tag{1.11}$$

However, the set $[1, 3] \cap [-3, -1]$ is empty, and we arrive to the contradiction to our assumption that a_n has a limit. Therefore, a_n has no limit. The proof is complete. □

1.3. **Exercises. 1.** Show that $\lim_{n\to\infty} \frac{2n-1}{n} = 2$.

2. Show that $\lim_{n\to\infty} \frac{3}{n^2} = 0$.

3. Show that $\lim_{n\to\infty} \frac{(-1)^n}{n+2} = 0$.

4. Show that $a_n = -7 \cdot (-1)^n$ does NOT have a limit.

5. Show that 3 is NOT a limit of $a_n = \frac{n}{2n+3}$ (note: regardless of what you were told in calculus, we do not yet know whether or not a limit is always unique, and you are not allowed to make and use any such statements).

2. Cauchy sequences – exercizes

6. Show that $a_n = \frac{3n^2}{n^2+1}$ is a Cauchy sequence.

7. Show that $a_n = 2n - 1$ is NOT a Cauchy sequence.

3. Equivalent sequences, bounded sequences, $\mathbb{R}$ – exercises

8. Show that $a_n = \frac{1}{n}$ and $b_n = \frac{1}{2n+1}$ are equivalent sequences.

9. Finish showing that the multiplication of real numbers is well-defined: that is, show that if a_n and b_n are equivalent Cauchy sequences, and c_n and d_n are equivalent Cauchy sequences, than $a_n \cdot c_n$ and $b_n \cdot d_n$ are equivalent Cauchy sequences. (In class, we verified that they are Cauchy).

10. *Using directly the definition of the bounded sequence*, show that $a_n = \frac{3n^2-5}{n^2+1}$ is bounded.

11. Show that $\sqrt{5} \in \mathbb{R} \setminus \mathbb{Q}$.