HOME WORK VII, DIFFERENTIAL GEOMETRY: JACOBI FIELDS, CONJUGATE POINTS, ISOMETRIC IMMERSIONS.

Due April 10. The Home Work must be uploaded on Canvas as a pdf. To complete the home work, use the lecture notes, as well as the DoCarmo's book Riemannian Geometry, Chapters 5 and 6. Please contact me if you have any questions!

1. Show that in the flat case ($\mathbb{R}^n$ with euclidean metric), no point p has any conjugate points.

2. Prove that on every compact manifold M for which the exponential map is injective, every point has a conjugate point. *Hint: if this is not the case for some point, what can be said about the exponential map at that point? could this be true?*

3. Let $N \subset K \subset M$ be sub-manifolds. Suppose that K is totally geodesic in M and N is totally geodesic in K. Show that N is totally geodesic in M.

4. Let $\overline{M}$ be an n-dimensional manifold. Let $f : \overline{M} \to \mathbb{R}$ be a differentiable function. For a regular value $a \in \mathbb{R}$ of f, consider a sub-manifold $M_a \subset \overline{M}$ (of dimension $n-1$) given by

$$M_a = \{p \in \overline{M} : f(p) = a\}.$$

Show that the mean curvature H of M_a is given by

$$H = -\frac{1}{n-1} \operatorname{div}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right),$$

at every point $p \in M$.