From affine Poincaré inequalities to affine spectral inequalities

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joint work with H. Jiménez and M. Montenegro

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Rayleigh quotients

For an open and bounded $\Omega \subset \mathbb{R}^n$ and $1 \leq p, q \leq \infty$, let

$$\lambda(\Omega) = \inf \left\{ \frac{\| |\nabla f| \|_p}{\|f\|_q} \middle| \, f : \overline{\Omega} \to \mathbb{R} \text{ smooth and } f = 0 \text{ in } \partial\Omega \right\}$$

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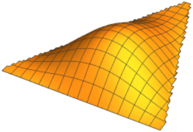
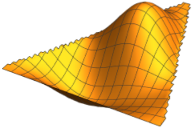
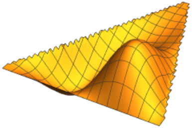
- There is a unique minimizer $f \in W_0^{1,2}(\Omega)$.
- It solves the differential equation

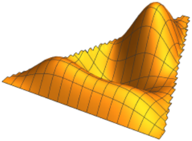
$$\begin{cases} \Delta f + \lambda_{2,\Omega}^2 f & = 0 \text{ in } \Omega \\ f & = 0 \text{ in } \partial\Omega. \end{cases}$$

- $f(x)\sin(\lambda_{2,\Omega} \cdot t)$ describes a vibrating membrane with the boundary fixed at $\partial\Omega$.









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Equivalently: $\lambda_{2,\Omega}$ is the best constant in the Poincaré inequality

$$\| |\nabla f| \|_2 \geq C \|f\|_2 \text{ for } f : \overline{\Omega} \to \mathbb{R} \text{ smooth with } f(\partial\Omega) = 0.$$

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Sobolev inequality: $\||\nabla f|\|_2 \geq C\|f\|_{\frac{2n}{n-2}}$ for $f : \mathbb{R}^n \to \mathbb{R}$

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Critical exponent: If $f_\varepsilon(x) = f(\varepsilon x)$ then

$$\frac{\||\nabla f_\varepsilon|\|_p}{\|f_\varepsilon\|_q} = \varepsilon^{n(\frac{1}{q} - \frac{1}{p} + \frac{1}{n})} \frac{\||\nabla f|\|_p}{\|f\|_q}$$

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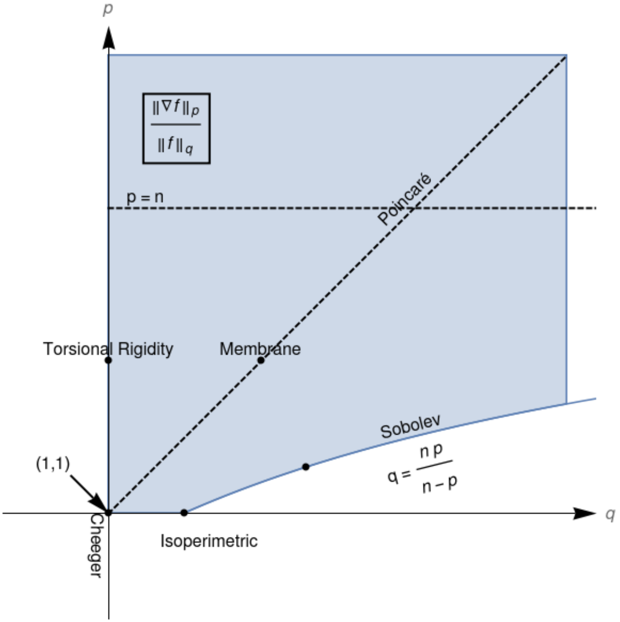

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which means

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The equality case is delicate (Brothers-Ziemer result)

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Affine

Definition

$$\mathcal{E}_p f = c_{n,p} \left(\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|_p^{-n} d\xi \right)^{-1/n}$$

'99, Zhang - The affine Sobolev Inequality

'03, Lutwak, Yang, Zhang - Sharp affine p-Sobolev Inequalities

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Equality case (Brothers-Ziemer result)

$$|||\nabla f|||_p = \left(\int_\Omega |\nabla f(x)|^p dx \right)^{1/p}$$

$$\mathcal{E}_p f = \left(\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|_p^{-n} d\xi \right)^{-1/n}$$

$$S(K) = |||\nabla \chi_K|||_1$$

$$S^{\mathcal{A}}(K) := \mathrm{vol}(\Pi^\circ K)^{-1/n} = \mathcal{E}_1 \chi_K$$

Sobolev

$$|||\nabla f|||_p \geq C_{n,p} \|f\|_{\frac{np}{n-p}}$$

Zhang

$$\mathcal{E}_p f \geq C_{n,p} \|f\|_{\frac{np}{n-p}}$$

Isoperimetric

$$S(K) \geq c_n \,\mathrm{vol}(K)^{\frac{n-1}{n}}$$

Petty-Projection

$$S^{\mathcal{A}}(K) \geq c_n \,\mathrm{vol}(K)^{\frac{n-1}{n}}$$

$$|||\nabla f^*|||_p \leq |||\nabla f|||_p$$

$$\mathcal{E}_p f^* \leq \mathcal{E}_p f$$

$$\lambda_{p,\Omega} \geq \lambda_{p,\Omega^*}$$

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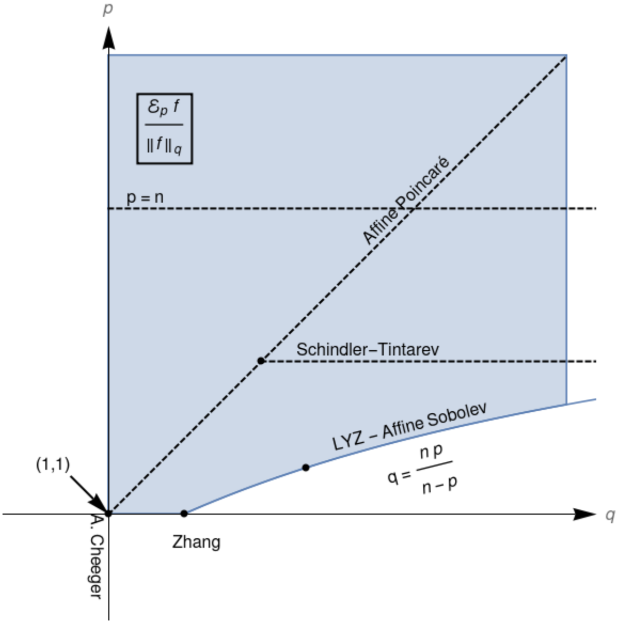
Observation

$W_0^{1,p}(\Omega) = \{f : \Omega \to \mathbb{R} |\ \mathcal{E}_p f < \infty\}$
is the correct space to work with affine Rayleigh quotients

Affine Rayleigh quotients



Results: bounds, compactness, existence and variation

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \geq 1$.

1. $\mathcal{E}_p f \geq C_{n,p}(\Omega) \|f\|_p^{\frac{n-1}{n}} \||\nabla f|\|_p^{1/n}$

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Results: bounds, compactness, existence and variation

Theorem

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Yes! Let's call $\Delta_p^{\mathcal{A}}$ the *p-affine laplacian*

The affine eigenvalue $\lambda_{p,\Omega}^{\mathcal{A}}$

Properties

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1. $\lambda^{\mathcal{A}}_{p,T(\Omega)} = \lambda^{\mathcal{A}}_{p,\Omega}$ for every $T \in \mathrm{SL_n}(\mathbb{R})$.

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The affine PDE

The Equation

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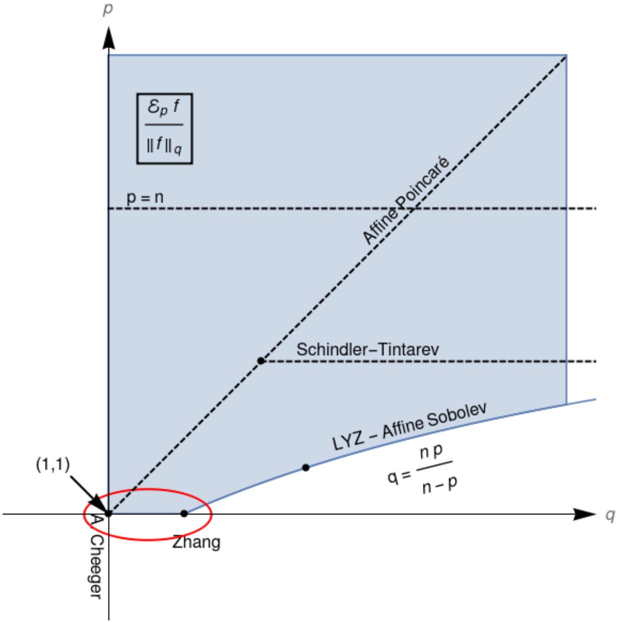
5. We don't know if the eigenvalue is simple.

The case $p = 1$ and Ω convex



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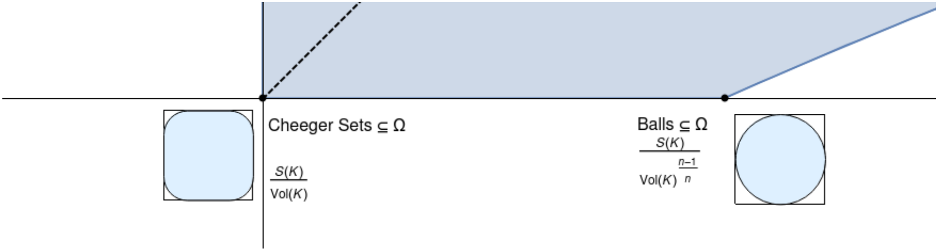
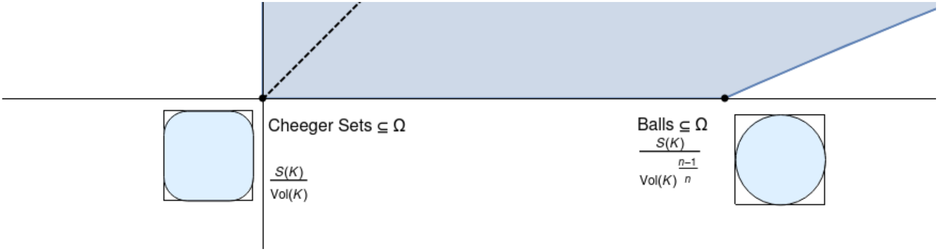
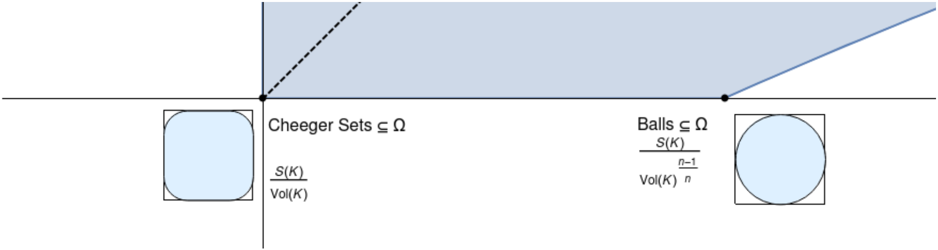
$$\inf_{f \in W_0^{1,p}(\Omega)} \frac{\mathcal{E}_1 f}{\|f\|_q} = \inf_{K \subseteq \Omega} \frac{S^{\mathcal{A}}(K)}{\mathrm{vol}(K)^{1/q}} = \inf_{K \subseteq \Omega} \frac{\mathrm{vol}(\Pi^\circ K)^{-1/n}}{\mathrm{vol}(K)^{1/q}}$$


Cheeger Sets ⊆ Ω
$\frac{S(K)}{\mathrm{Vol}(K)}$
Balls ⊆ Ω
$\frac{S(K)}{\mathrm{Vol}(K)^{\frac{n-1}{n}}}$



$$\text{Cheeger set} \xleftarrow[q\to 1]{} \min_K \left\{ \frac{S(K)}{\mathrm{vol}(K)^{1/q}} \right\} \xrightarrow[q\to \frac{n}{n-1}]{} \text{Largest ball}$$



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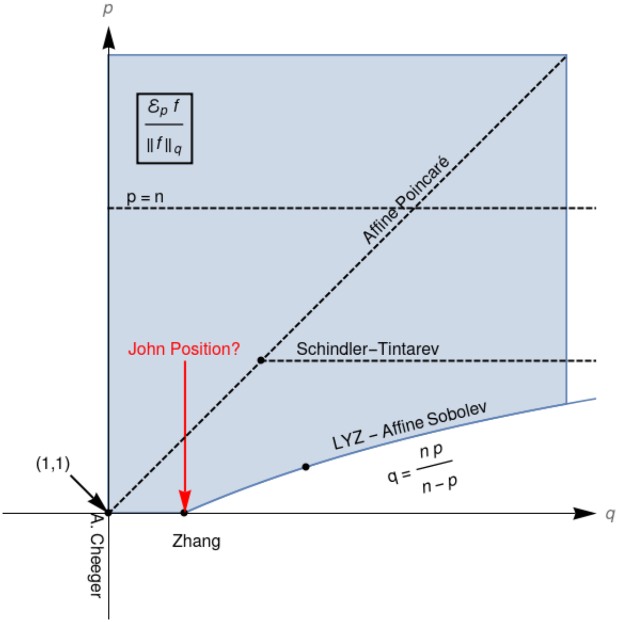
Affine Rayleigh quotients



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Ω is in John position if and only if

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6. Higher eigenvalues? Spectral gap?

The end

Thank you