Sharp stability of the Brunn Minkowski inequality

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Minkowski sum

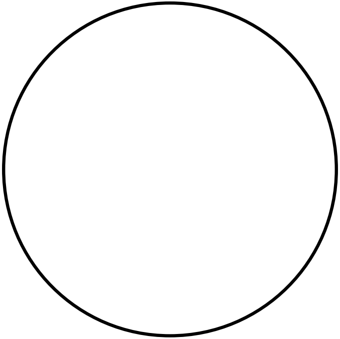
Minkowski sum

$$A + B := \{a + b : a \in A, b \in B\}$$

$$cA := \{ca : a \in A\}$$

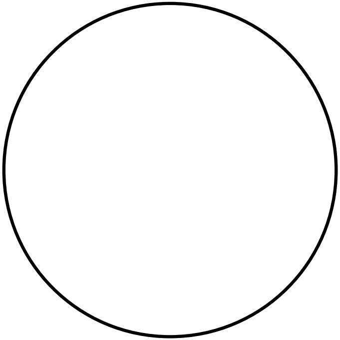
$$\text{e.g. } \frac{A + A}{2} \supset A$$

Examples of semisums



A

Examples of semisums



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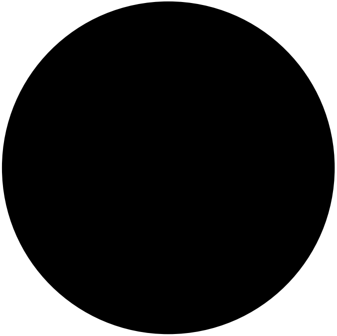
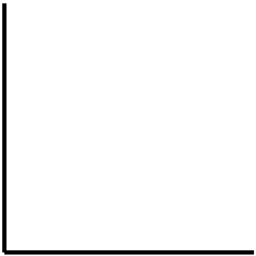


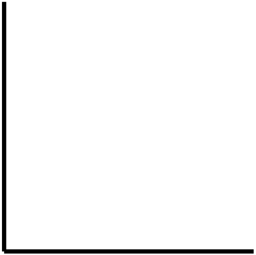
$\frac{A+A}{2}$

Examples



A

Examples



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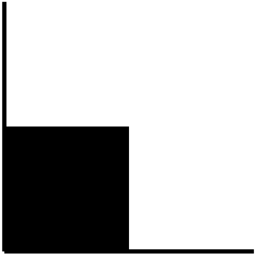

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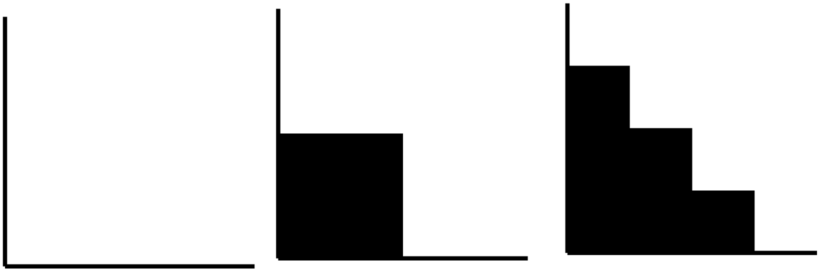
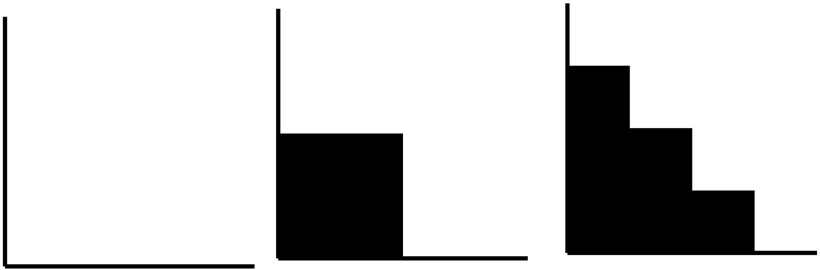
Examples



A $\qquad \frac{A+A}{2}$ $\qquad \frac{\frac{A+A}{2}+\frac{A+A}{2}}{2} = \frac{A+A+A+A}{4}$

Examples



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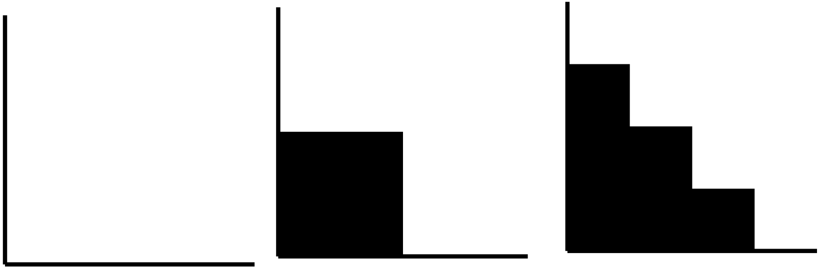
Observations

$$\frac{\overbrace{A+\ldots+A}^{k \text{ times}}}{k} \subset \mathrm{co}(A)$$

Examples



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Observations

Suspicion:

$$\left| \mathrm{co}(A) \setminus \frac{\overbrace{A + \dots + A}^{k \text{ times}}}{k} \right| \to 0 \text{ as } k \to \infty$$

Brunn-Minkowski inequality

Brunn-Minkowski inequality (1887,1896)

For $A, B \subset \mathbb{R}^n$ measurable with $|A|, |B| > 0$,

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$$

with equality iff A, B are homothetic convex bodies.

Semisum expression

$$\left|\frac{A+B}{2}\right|^{\frac{1}{n}} \geq \frac{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}{2}$$

Stability parameters

Brunn Minkowski inequality (1887,1896)

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Parameters

$$\delta = \frac{|A + B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}} - 1 \qquad t = \frac{|A|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}$$

Stability parameters intuition

Parameters

$$\delta = \frac{|A+B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}} - 1 \qquad t = \frac{|A|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}$$

Hope

Distance from A, B to homothetic convex sets $\to 0$, as $\delta \to 0$.

Stability parameters intuition

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Hope

Distance from A, B to homothetic convex sets $\to 0$, as $\delta \to 0$.

Caveat

Replacing A by λA with $\lambda \to 0$, gives $\delta \to 0$.

Initial Qualitative Result

Theorem (Christ, 2012)

There exists a function $\epsilon_n(\delta, t) > 0$ such that for every fixed $n, t > 0$, we have $\epsilon_n(\delta, t) \to 0$ as $\delta \to 0$, satifying the following.
For all $A, B \subset \mathbb{R}^n$, there exist homothetic convex sets $K_A \supset A, K_B \supset B$ so that

$$|K_A \setminus A| + |K_B \setminus B| \leq \epsilon_n(\delta, t)(|A| + |B|)$$

Initial Quantitative Result

Theorem (Figalli and Jerison, 2017)

For $n \geq 2$, $\exists b_n, C_n > 0$, $\forall \tau \in (0, \frac{1}{2}]$, $\exists a_n(\tau), d_n(\tau) > 0$ such that the following holds.
For all $A, B \subset \mathbb{R}^n$, with $t \in [\tau, 1-\tau]$ and $\delta \leq d_n(\tau)$, there exist homothetic convex sets $K_A \supset A, K_B \supset B$ so that

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \leq C_n \tau^{-b_n} \delta^{a_n(\tau)}$$

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Central question

What are the optimal exponents a_n and b_n?

Planar Result

Theorem (vH, Spink, Tiba, 2020+)

$\exists C_2 > 0$, $\forall \tau \in (0, \frac{1}{2}]$, $\exists d_2(\tau) > 0$ such that the following holds. For all $A, B \subset \mathbb{R}^2$, with $t \in [\tau, 1-\tau]$ and $\delta \leq d_2(\tau)$, there exist homothetic convex sets $K_A \supset A, K_B \supset B$ so that

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \leq C_n \tau^{-\frac{1}{2}} \delta^{\frac{1}{2}}$$

Equal sets $A = B$

Theorem (Figalli and Jerison, 2015)

For $n \geq 2$, $\exists a_n, C_n, d_n > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, with $\delta \leq d_n$, we have

$$|\text{co}(A) \setminus A| \leq C_n \delta^{a_n} |A|$$

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Theorem (Figalli and Jerison, 2019)

For $n = 1, 2, 3$, $\exists C_n, d_n > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, with $\delta \leq d_n$, we have

$$|\text{co}(A) \setminus A| \leq C_n \delta |A|$$

i.e. $a_1 = a_2 = a_3 = 1$.

Conjecture (Figalli and Jerison, 2019)

$a_n = 1$ for all n.

Main Theorem

Theorem (vH, Spink, Tiba, 2020)

For $n \in \mathbb{N}$, $\exists C_n > 0$, $\forall \tau \in (0, \frac{1}{2}]$, $\exists d_n(\tau) > 0$, such that the following holds. For all $A, B \subset \mathbb{R}^n$ homothetic, with $\delta \leq d_n(\tau)$ and $t \in [\tau, 1-\tau]$, we have

$$|\text{co}(A) \setminus A| \leq C_n \delta \tau^{-1} |A|$$

Corollary (vH, Spink, Tiba, 2020)

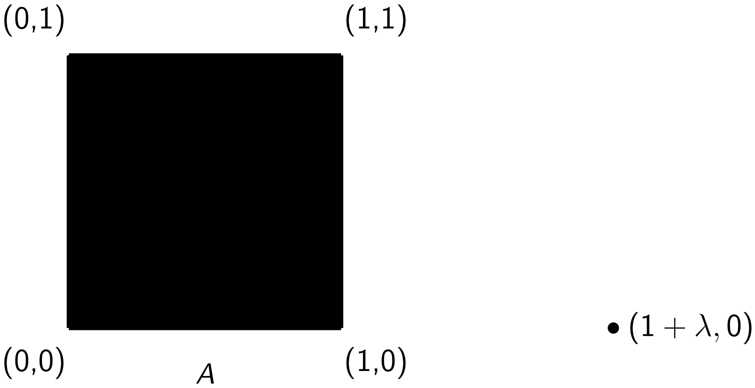
For $n \in \mathbb{N}$, $\exists C_n, d_n > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, with $\delta \leq d_n$, we have

$$|\text{co}(A) \setminus A| \leq C_n \delta |A|$$

i.e.

$$|\text{co}(A) \setminus A| \leq C_n' \left(\left| \frac{A+A}{2} \right| - |A| \right)$$

Optimality of main theorem



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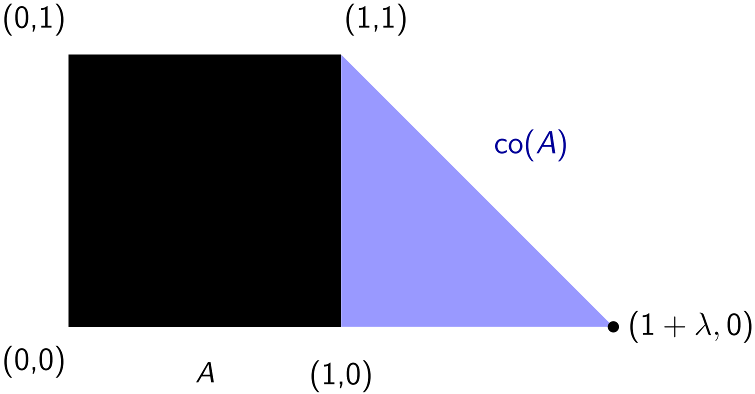

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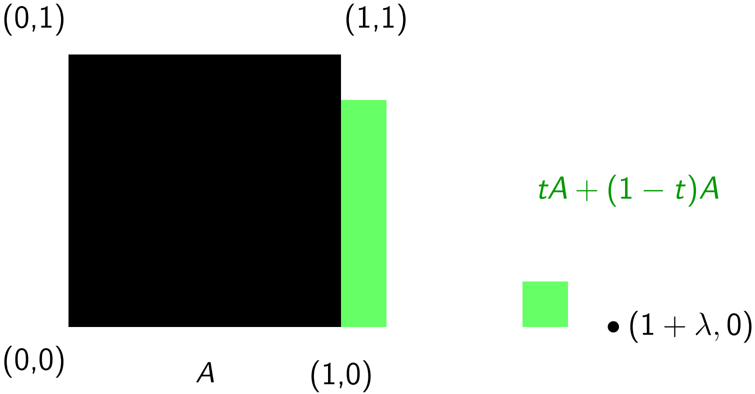
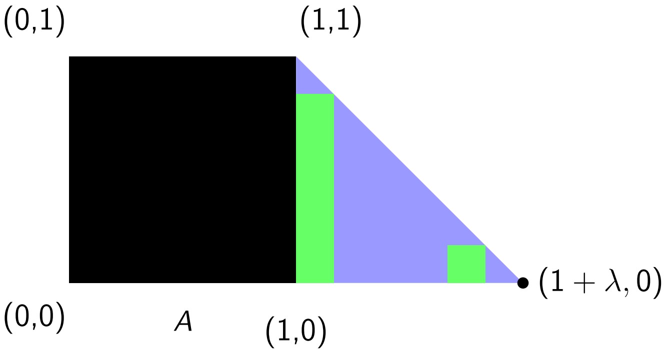


Optimality of main theorem



Optimality of the constants

$$|A| = 1, \qquad |\mathrm{co}(A) \setminus A| = \frac{\lambda}{2}, \qquad |tA + (1-t)A| - |A| \leq t\lambda$$

$$|\mathrm{co}(A) \setminus A| \geq C\delta t^{-1}$$

Other results

Assymetry index

$$\alpha(A,B)=\min_{x\in\mathbb{R}^n}\frac{|(x+A)\Delta sB|}{|A|},\ \text{where } s=|A|^{1/n}|B|^{-1/n}$$

Other results

Assymetry index

$$\alpha(A,B) = \min_{x\in\mathbb{R}^n} \frac{|(x+A)\Delta sB|}{|A|}, \text{ where } s = |A|^{1/n}|B|^{-1/n}$$

Previous results

- (Figalli,Maggi, Pratelli, 2009) For A, B convex, $\alpha(A,B) \leq C_n \delta^{1/2}\tau^{-1/2}$
- (Figalli, Maggi, Mooney, 2018) For A a ball and B arbitrary, $\alpha(A,B) \leq C_n \delta^{1/2}\tau^{-1/2}$
- (Carlen, Maggi, 2018) For A convex and B arbitrary, $\alpha(A,B) \leq C_n \delta^{n+\frac{3}{4}}\tau^{-1/2}$
- (Barchiesi, Julin, 2017) For A convex and B arbitrary, $\alpha(A,B) \leq C_n \delta^{1/2}\tau^{-1/2}$

Connection to Additive Combinatorics

Freiman's Theorem (1964)

$\forall \lambda > 0, \exists d, s > 0$, so that
if $A \subset \mathbb{Z}$ satisfies $|A + A| \leq \lambda |A|$, then A is contained in a *generalised arithmetic progression* of dimenion d of size $s|A|$.

Connection to Additive Combinatorics

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Theorem (Green and Tao, 2006)

$\forall n, \epsilon > 0, \exists w_n > 0$ so that
if $A \subset \mathbb{Z}$, with $|A + A| \leq (2^{n+1} - \epsilon)|A|$, then A is covered by w_n translations of a generalized arithmetic progression of dimension n and size at most $|A|$.

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Theorem (vH, Spink, Tiba, 2020+)

$\forall n, \exists d_n, C_n > 0$ and $\forall \delta \in (0, d_n], \exists m_n(\delta) > 0$ so that
if $A \subset \mathbb{Z}^n$ satisfies $|A + A| \leq (2^n + \delta)|A|$ and A is not contained in $m_n(\delta)$ hyperplanes, then $|\widehat{\text{co}}(A) \setminus A| \leq C_n \delta |A|$

Proof of the Main theorem

Theorem (vH, Spink, Tiba, 2020)

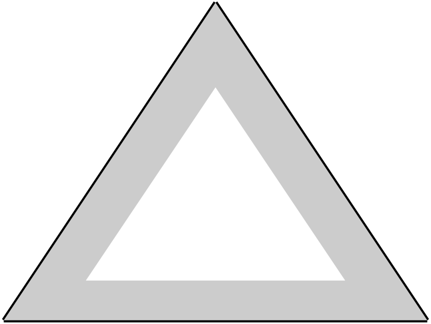
For $n \in \mathbb{N}$, $\exists C_n, d_n > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, with $\delta \leq d_n$, we have

$$|\text{co}(A) \setminus A| \leq C_n \left(\left| \frac{A+A}{2} \right| - |A| \right)$$

Remarks

- $\left| \frac{A+A}{2} \setminus A \right| = \Theta(\delta)$

Proof overview



Overview of the proof

- Reduction to $\mathrm{co}(A) = T$ a unit simplex.
- Guaranteed region; $(1-\epsilon)T \subset \frac{A+A}{2}$.
- Homogeneity of A in T.

Guaranteed region

Lemma (Christ, 2012)

$\forall n, \epsilon > 0, \exists d_n > 0$ such that if $\delta \leq d_n$ then $|\mathrm{co}(A) \setminus A| \leq \epsilon |A|$

Guaranteed region

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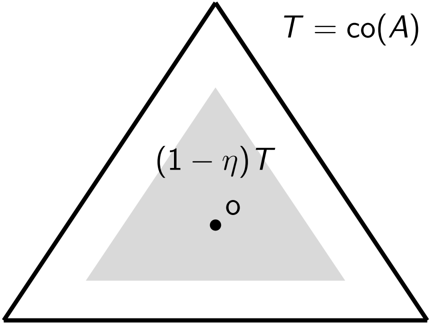
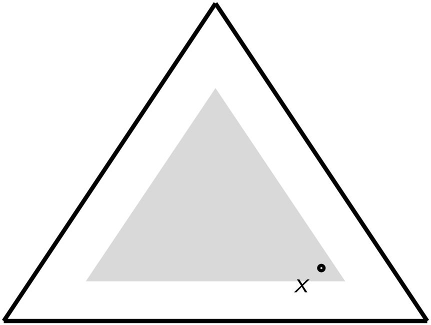
Lemma

$\forall n, \eta > 0, \exists \epsilon > 0$ such that if $|\text{co}(A) \setminus A| \leq \epsilon |A|$, then $(1-\eta)\text{co}(A) \subset \frac{A+A}{2}$



Guaranteed region



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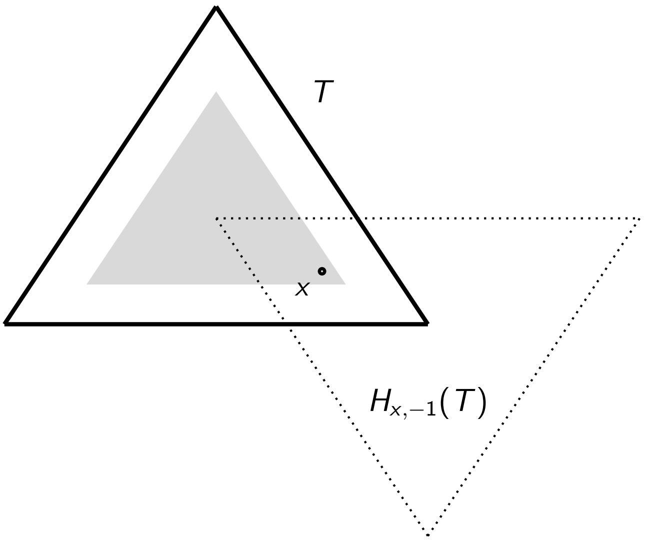

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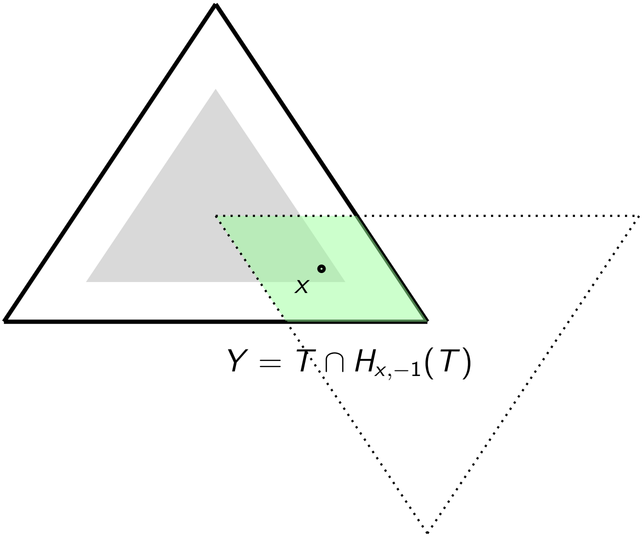
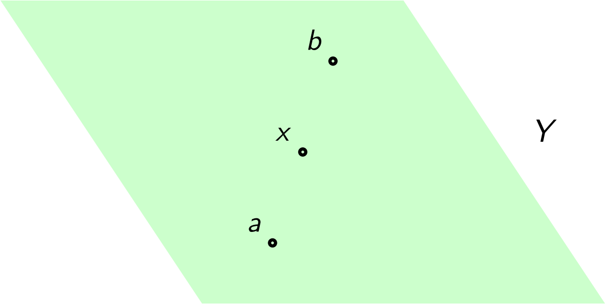


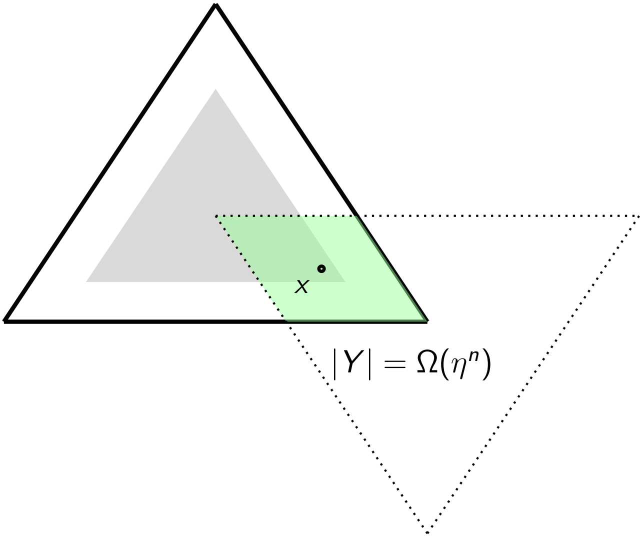
$$Y = T \cap H_{x,-1}(T)$$

Guaranteed region

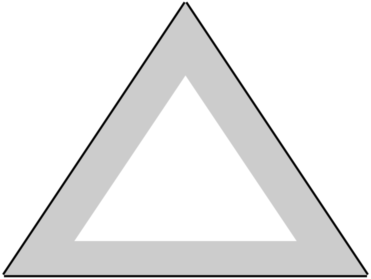


If $|Y| > 2\epsilon|A|$,
then $|A \cap Y| > \frac{1}{2}|Y|$.
Hence, $\exists a, b \in A \cap Y$, with $x = \frac{a+b}{2}$.

Guaranteed region



Homogeneity

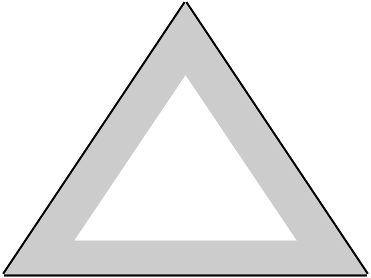


Aim

There exist some $C' > 0$ such that

$$\left|\frac{A+A}{2} \setminus A\right| \geq |(1-\eta)T \setminus A| \geq \frac{1}{2}|T \setminus A| - C' \left|\frac{A+A}{2} \setminus A\right|$$

Homogeneity



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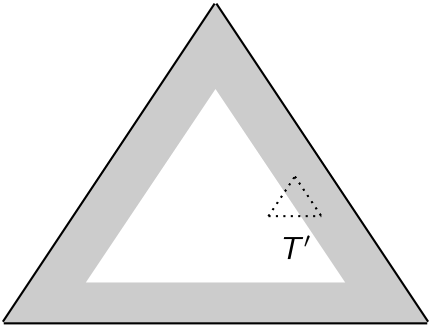
$$|(T \setminus (1-\eta)T) \setminus A| \leq \frac{1}{2}|T \setminus A| + C'\left|\frac{A+A}{2} \setminus A\right|$$

Homogeneity

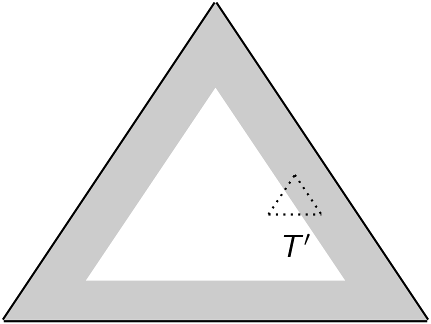


Plan sketch

- For some T's

$$|T' \setminus A| \leq |T'| \cdot \frac{|T \setminus A|}{|T|} + C \left| \frac{A+A}{2} \setminus A \right|$$

Homogeneity



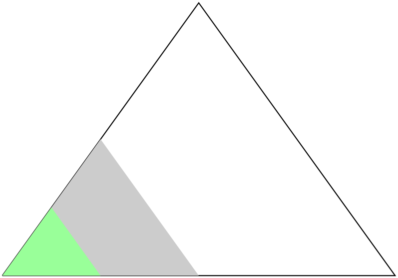
Plan sketch

- For some T's

$$|T' \setminus A| \leq |T'| \cdot \frac{|T \setminus A|}{|T|} + C \left| \frac{A+A}{2} \setminus A \right|$$

- $\exists$ a small family of such T's covering $T \setminus (1-\eta)T$

Fractal structure

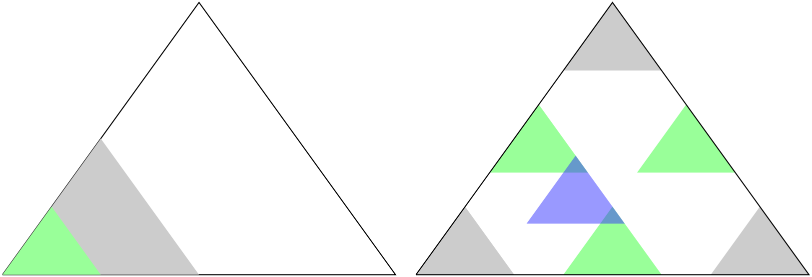


Fractal structure

$$\mathcal{S}_{0,0} := \{T\}$$

$$\mathcal{S}_{i+1,0} := \left\{ \frac{x + T'}{2} : T' \in \mathcal{S}_{i,0}, x \in V(T) \right\}$$

Fractal structure



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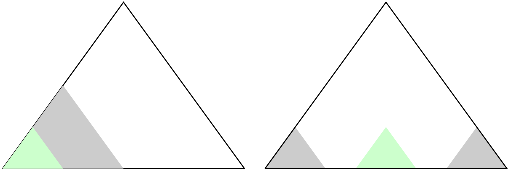
$$\mathcal{S}_{i,j+1} := \left\{ \frac{T' + T''}{2} : T', T'' \in \mathcal{S}_{i,j} \right\}$$

Fractal structure



Fractal structure

$$\mathcal{S}_{0,0} := \{T\}$$

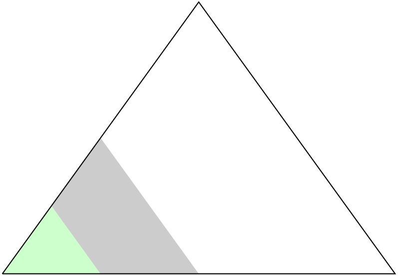
$$\mathcal{S}_{i+1,0} := \left\{ \frac{x_i + T'}{2} : T' \in \mathcal{S}_{i,0} \right\}$$

$$\mathcal{S}_{i,j+1} := \left\{ \frac{T' + T''}{2} : T', T'' \in \mathcal{S}_{i,j} \right\}$$

Remark

$$\bigcup_{j \geq 1} \mathcal{S}_{i,j} \text{ is dense in } T$$

Pulling into a corner



Lemma

$\exists C_{i,0}$ such that $\forall T' \in \mathcal{S}_{i,0}$, we have

$$|T' \setminus A| \leq |T'|\frac{|T \setminus A|}{|T|} + C_{i,0}\left|\frac{A+A}{2} \setminus A\right|$$

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Proof

Induction on i. Let $T'' \in \mathcal{S}_{i-1,0}$ and $x \in V(T)$ such that $T' = \frac{x+T''}{2}$.

Pulling into a corner

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$$\left| T' \cap \frac{A+A}{2} \right| \geq \left| \frac{x + (T'' \cap A)}{2} \right|$$

Pulling into a corner

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$$\left| T' \cap \frac{A+A}{2} \right| \geq \left| \frac{x + (T'' \cap A)}{2} \right| = \frac{|T'|}{|T''|} \cdot |T'' \cap A|$$

Pulling into a corner

Lemma

$\exists C_{i,0}$ such that $\forall T' \in \mathcal{S}_{i,0}$, we have

$$|T' \setminus A| \leq |T'|\frac{|T \setminus A|}{|T|} + C_{i,0}\left|\frac{A+A}{2} \setminus A\right|$$

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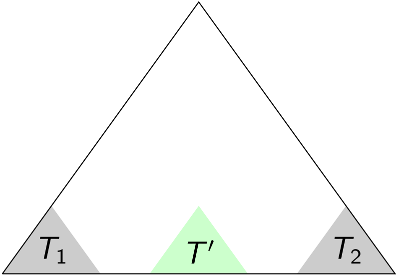
Transporting triangles around



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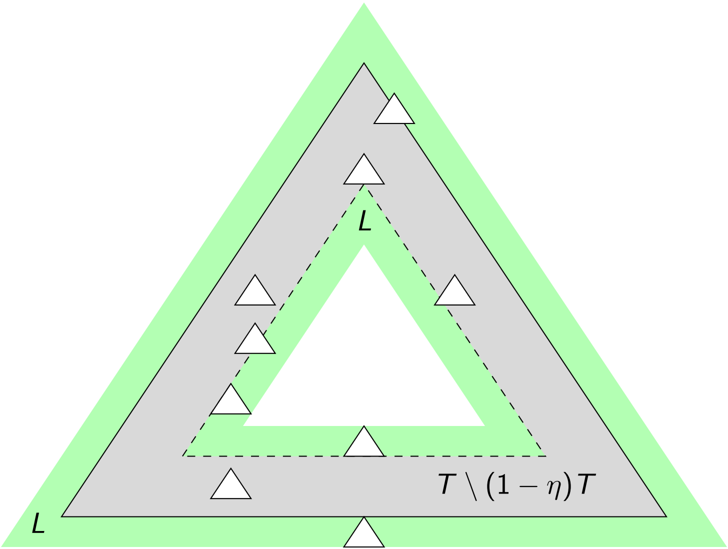
Covering $T \setminus (1-\eta)T$



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Parameters under our control

- Size of the uncovered border; η
- Size of the simplices; λ

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- $\sum_{T' \in \mathcal{B}} |T'| = O(|L|) = O(|T \setminus (1-\eta)T|)$

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- May assume all $T' \in \mathcal{B}$ satisfy $T' \subset T$.
- Let i, so that $2^{-i} > \lambda$.
- $\forall T' \in \mathcal{B}, \exists T'' \subset \mathcal{S}_{i,j}; T' \subset T''$. Let $\mathcal{C} \subset \mathcal{S}_{i,j}$ correspondingly.

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- $\forall T' \in \mathcal{B}, \exists T'' \subset \mathcal{S}_{i,j}; T' \subset T''$. Let $\mathcal{C} \subset \mathcal{S}_{i,j}$ correspondingly.
- Then: $\sum_{T' \in \mathcal{C}} |T'| = O(|T \setminus (1-\eta)T|) = O(\eta)|T|$

Conclusion

Covering result

$\exists \mathcal{C} \subset \mathcal{S}_{i,j}$, with $\sum_{T' \in \mathcal{C}} |T'| = O(\eta)|T|$ and $T \setminus (1 - 2^{-i})T \subset \bigcup_{T' \in \mathcal{C}} T'$

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$$\begin{aligned} |(T \setminus (1-2^{-i})T) \setminus A| &\leq \sum_{T' \in \mathcal{C}} |T' \setminus A| \\ &\leq \sum_{T' \in \mathcal{C}} \frac{|T'|}{|T|} |T \setminus A| + C_{i,j} \left| \frac{A+A}{2} \setminus A \right| \end{aligned}$$

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Choose η so that

$$|(T \setminus (1-2^{-i})T) \setminus A| \leq \frac{1}{2}|T \setminus A| + C'_\eta \left| \frac{A+A}{2} \setminus A \right|$$

Concluding remarks

Theorem (vH, Spink, Tiba, 2020)

For $n \in \mathbb{N}$, $\exists C_n > 0$, $\forall \tau \in (0, \frac{1}{2}]$, $\exists d_n(\tau) > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, and $B = \frac{1-t}{t}A$, with $\delta \leq d_n(\tau)$ and $t \in [\tau, 1-\tau]$, we have

$$|\mathrm{co}(A) \setminus A| \leq C_n \delta \tau^{-1} |A|$$

Constants

- $\exp(\Omega(n)) = C_n = \exp(O(n \log(n)))$
- $d_n(\frac{1}{2}) = \exp(-\Omega(n))$

Thank you!

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Covering $T \setminus (1-\eta)T$

Parameters under our control

- Size of the uncovered border; η
- Size of the simplices; λ

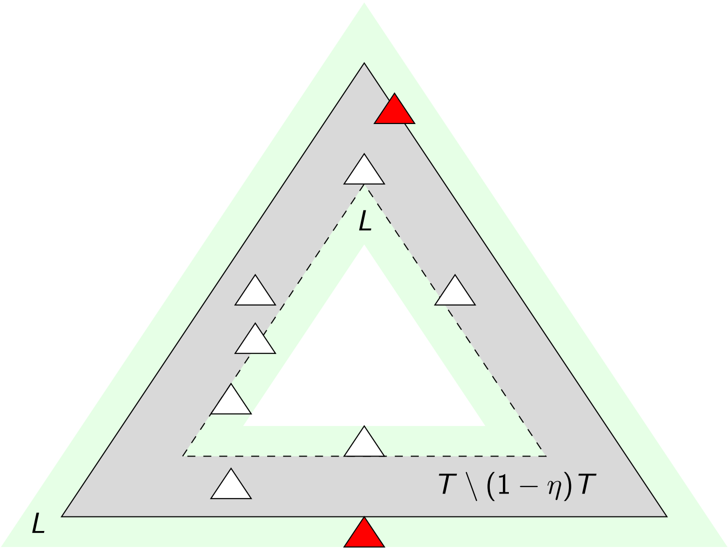
Overview

- Simplices periodically cover $\mathbb{R}^n$ with a certain efficiency α_n.
- Choose $\lambda = \eta$ so that $|L| \leq O_n(|T \setminus (1-\eta)T|) = O_n(\eta)$.
- $\exists$ a subset $\mathcal{B}_\eta \subset \mathbb{R}^n$ of the cover so that;

$$T \setminus (1-\eta)T \subset \mathcal{B}_\eta + \lambda T \subset L$$

and $|\mathcal{B}_\eta| \leq \alpha_n \frac{|L|}{|\lambda T|} = O_n(\eta^{-(n-1)})$

Finding translates inside T



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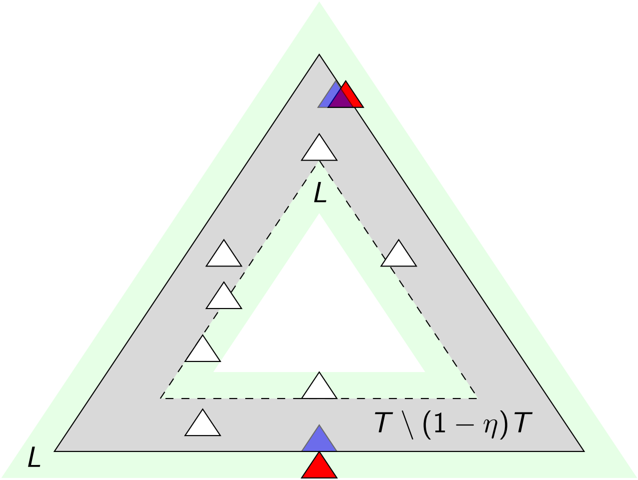
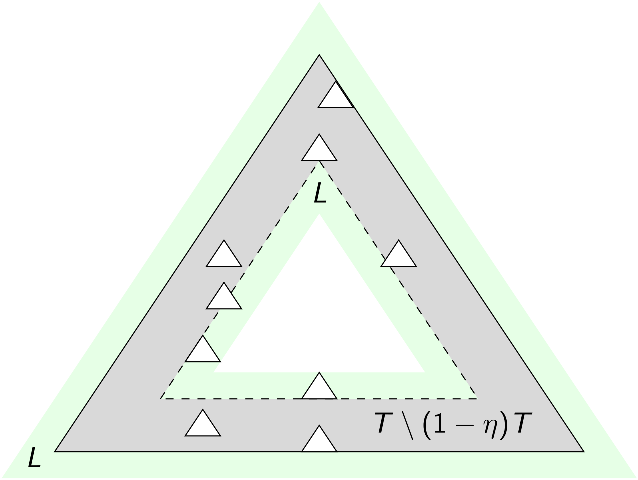


Replace T' by $T'' \subset T$ so that $T' \cap T \subset T''$.

Finding translates inside T



Assume $\mathcal{B}_\eta + \eta T \subset T$.

Finding translates in $\mathcal{S}_{i,j}$

Finding

- Let $\eta = 2^{-i}$
- Recall $\bigcup_{j\geq 0} \mathcal{S}_{i-1,j}$ is dense in T
- $\forall b \in \mathcal{B}_\eta$, $\exists T' \in \bigcup_{j\geq 0} \mathcal{S}_{i-1,j}$, such that $b + 2^{-i}T \subset T'$
- Let $\mathcal{C} \subset \bigcup_{j\geq 0} \mathcal{S}_{i-1,j}$ be the set of these T'
- $|\mathcal{C}| = |\mathcal{B}_\eta| = O(\eta^{-(n-1)}) = O(2^{i(n-1)})$ and $T \setminus (1-\eta)T \subset \bigcup_{T'\in\mathcal{C}} T'$
- $\mathcal{C} \subset \mathcal{S}_{i-1,j}$ for some j depending only on n and i.

Conclusion

Finding

$\mathcal{C} \subset \mathcal{S}_{i-1,j}$, with $|\mathcal{C}| = O(2^{i(n-1)})$ and $T \setminus (1-2^{-i})T \subset \bigcup_{T' \in \mathcal{C}} T'$

$$\begin{aligned}
|(T \setminus (1-2^{-i})T) \setminus A| &\leq \sum_{T' \in \mathcal{C}} |T' \setminus A| \\
&\leq \sum_{T' \in \mathcal{C}} \frac{|T'|}{|T|} |T \setminus A| + C_{i,j} \left| \frac{A+A}{2} \setminus A \right| \\
&\leq O(2^{i(n-1)}) \cdot \left(2^{-ni} |T \setminus A| + C_{i,j} \left| \frac{A+A}{2} \setminus A \right| \right) \\
&\leq O(2^{-i}) |T \setminus A| + C_i' \left| \frac{A+A}{2} \setminus A \right|
\end{aligned}$$

Fix i and thus η so that

$$|(T \setminus (1-2^{-i})T) \setminus A| \leq \frac{1}{2} |T \setminus A| + C_i' \left| \frac{A+A}{2} \setminus A \right|$$