HOME WORK VI, HIGH-DIMENSIONAL GEOMETRY AND PROBABILITY, SPRING 2018

Due April 17. All the questions marked with an asterisk(s) are optional. The questions marked with a double asterisk are not only optional, but also have no due date.

Question 1. Show that for every a, b, and for every random variable X on $[a,b]$ with $\mathbb{E}X=0$,

$$\mathbb{E}e^{\lambda X} \leq e^{\frac{\lambda^2(b-a)^2}{8}}.$$

Deduce the general case of Hoeffding's inequality for bounded random variables.

Hint: Modify the proof given in class for symmetric intervals; find a bound from above for the function

$$F(\lambda) = \log\left(\frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}\right).$$

Question 2. Check that $||\cdot||_{\psi_1}$ and $||\cdot||_{\psi_2}$ are indeed norms.

Question 3. Verify all the equivalences in regards to the definition of ψ_1 random variables (use the ideas from the analogous result about ψ_2 proved in class).

Question 4. Check that

a) Normal $N(0,\sigma^2)$ random variable Z is sub-gaussian with $||Z||_{\psi_2} = C\sigma$.

b) Bounded random variable X is sub-gaussian with $||X||_{\psi_2} \leq C\sup_{\omega\in\Omega}|X(\omega)|$.

c) Poisson random variable (with density $f(t)=e^{-t}1_{t\geq 0}$) is NOT sub-gaussian.

Question 5. Show that $||X-\mathbb{E}X||_{\psi_1} \leq C||X||_{\psi_1}$. (Follow the ideas from the analogous statement about sub-gaussian norm proved in class).

Question 6^***.** We proved in class that 1-dimensional marginals of any log-concave random vector are sub-exponential.

a) Is there a log-concave random vector for which all of the marginals are sub-gaussian?

b) Is there a log-concave random vector for which none of the marginals are sub-gaussian?

c) Is there a log-concave random vector for which some of the marginals are sub-gaussian and some are not?

d) Consider a log-concave isotropic random vector X. Estimate

$$inf_{\theta \in \mathbb{S}^{n-1}} ||\langle X, \theta \rangle||_{\psi_1}$$

and

$$sup_{\theta \in \mathbb{S}^{n-1}} ||\langle X, \theta \rangle||_{\psi_1}.$$

$e)^{**}$ Estimate, as well as you can, $|| \, |X| - \mathbb{E}|X| \, ||_{\psi_2}$.