Rank of Sparse Bernoulli Matrices

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Singularity of discrete random matrices

Conjecture 1: Let B be the $n \times n$ matrix random sign matrix.

$$\mathbb{P}(s_n(B) = 0) = \left(\frac{1}{2} + o(1)\right)^n.$$

For $n \times n$ matrix A, $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \geq 0$ denote its singular values in non-increasing order.

$$s_k(A) = \max_{k-\dim H} \min_{x \in H \cap S^{n-1}} ||Ax||.$$

What it means? Roughly 2 columns or 2 rows are identical up to the sign difference.

Singularity of discrete random matrices

1967	Komlós:	$o_n(1)$
1995	Kahm, Komlós,and Szemerédi	$exp(-cn)$
2006, 2007	Tao and Vu	$\left(\frac{3}{4}+o(1)\right)^n$
2010	Bourgain, Vu, and Wood	$\left(\frac{1}{\sqrt{2}}+o(1)\right)^n$
2018	Tikhomirov	$\left(\frac{1}{2}+o(1)\right)^n$

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Conjecture 2: Let A be the $n \times n$ random matrix with i.i.d Bernoulli(p) entries, where $\mathrm{p} = \mathrm{p_n} \in (0, \frac{1}{2})$. Then,

$$\mathbb{P}(s_n(A) = 0) = (1 + o(1))\mathbb{P}(either\ a\ row\ or\ column\ of\ A\ equals\ 0)$$

- The case when $p < \log n / n$ is immediate. The probability that A has a 0 row of column is a high probability event.

Singularity of discrete random matrices

Theorem 3 (Litvak-Tikhomirov): Let A be the $n \times n$ random matrix with i.i.d Bernoulli(p) entries, where $p = p_n$ satisfies

$$C \frac{\log(n)}{n} \leq p \leq C^{-1}.$$

Then,

$$\mathbb{P}(s_n(A) \leq t)$$
$$\leq (1 + o_n(1))\mathbb{P}(either\ a\ row\ or\ a\ column\ of\ A\ equals\ 0) + \exp(3\log(n)^2)\, t$$

If $p \geq q$ for some constant $q > 0$.

$$\mathbb{P}(s_n(A) \leq t)$$
$$\leq (1 + o_n(1))\mathbb{P}(either\ a\ row\ or\ a\ column\ of\ A\ equals\ 0) + C_q n^{2.5} t$$

Singularity of discrete random matrices

- Jain, Sah, and Sawhney settle the case when $c < p < \frac{1}{2}$.
- Basak and Rudelson settle the case when $\frac{\log(\mathrm{n})}{n} \leq p \leq (1 + o_n(1))\frac{\log(\mathrm{n})}{n}$.

Rank of discrete random matrices

Conjecture 4 : Let B be the $n \times n$ matrix random sign matrix. For any non-negative integer k,

$$\mathbb{P}(s_{n-k}(B) = 0) = \left(\frac{1}{2} + o(1)\right)^{kn}.$$

Rank of discrete random matrices

Theorem (Costello, Vu)

Let A_{sym} be a n by n symmetric Bernoulli matrix for $\frac{(1+\epsilon)\log n}{2n} \leq p < c$. Then, with high probability, the corank of A_{sym} is the number of zero columns/rows. (c can be greater than $\frac{1}{2}$. Also some extensions from the A_{sym} model)

Corllolary (Basak-Rudelson, Litvak-Tikhomirov, Jain-Sah-Sawhney)

For $p \in \left[\frac{\log n}{n}, \frac{(1+o(1))\log n}{n}\right) \cup (\frac{C\log n}{n}, \frac{1}{2})$, with high probability, the corank of A is the maximum of number of zero columns and number of zero rows.

Rank of sparse Bernoulli matrices

Theorem 5 (H.): Let A be the $n \times n$ random matrix with i.i.d Bernoulli(p) entries, where $p = p_n$ satisfies

$$1 \le \liminf \frac{pn}{\log(n)} \le \limsup \frac{\mathrm{pn}}{\log(n)} < +\infty.$$

Then, for any positive integer k,

$$\mathbb{P}(s_{n-k+1}(A) \le t)$$
$$\le \left(1 + o_n(1)\right)\mathbb{P}(either\ k\ rows\ or\ k\ columns\ of\ A\ equal\ 0) + n^{2k+o(1)}t.$$

- Setting t=0, a similar type of result for Conjecture 4, but for sparse Bernoulli matrices.
- When $k = 1$, settle Conjecture 2 (singularity of Bernoulli matrices) in the regime of p described above.

Magnitude of the event

Let $O_{RC}(k)$ *be the event of* A *that either k rows or k columns of A are zero.*

$$\mathbb{P}(O_{RC}) = O(\exp(-(r-1)k\log n)) ,$$

where $r = \frac{pn}{\log n} \geq 1.$

Definition A probability event O of A is called a (r,k) *−high probability event* if
$$\mathbb{P}(O^c) = o\big(\mathbb{P}(O_{RC}(k))\big).$$

Decomposition of $\mathbb{R}^n$ from Litvak-Tikhomirov

For $x \in \mathbb{R}^n$, x^* be the *non-increasing rearrangement* of x.

Let $\sigma_x: [n] \mapsto [n]$ be the permutation so that $\left|x_{\sigma_x(i)}\right| \geq |x_{\sigma_x(j)}|$ whenever $i \geq j$.

$$x_i^* \coloneqq |x_{\sigma_x(i)}|.$$

Decomposition of $\mathbb{R}^n$ from Litvak-Tikhomirov

- $\mathcal{V}$-vectors: collection of vectors x in $\mathbb{R}^n$ so that $\mathrm{i} \mapsto \frac{x^*(i)}{x^*(\lambda n)}$ is not growing too fast as $i \searrow 1$.

Decomposition of $\mathbb{R}^n$ from Litvak-Tikhomirov

- $\mathcal{R}$-vectors: collection of vectors x such that there exists $1 \leq \mathrm{n}_1 \leq n_2 \leq n$ such that $\frac{||x^*_{[n_1,n_2]}||_\infty}{||x^*_{[n_1,n_2]}||_2} \leq \frac{c}{p}$.

Decomposition of $\mathbb{R}^n$ from Litvak-Tikhomirov

- $\mathcal{T}$-vectors: collection of vectors x such that $x^*_{n_1} \gg x^*_{n_2}$ for $(n_1, n_2) \in S$.

Structure of the Proof in Litvak-Tikhomirov

- $\{\forall x \in \mathcal{V}, ||Ax|| \gg 0\}$ is a $(r, 1)$- high probability event.
- $\{\forall x \in \mathcal{R}, ||Ax|| \gg 0\}$ is a $(r, 1)$- high probability event.
- $\{\forall x \in \mathcal{T}, ||Ax||, ||A^{\top}x|| \gg 0\}$ is a $(r, 1)$- high probability event if we condition on $O_{RC}^{c}(1)$.
- $\mathcal{V} \cup \mathcal{R} \cup \mathcal{T} = \mathbb{R}^n$.

Structure of the Proof

With some modifications of $\mathcal{V}$, $\mathcal{R}$, and $\mathcal{T}$,

- $\{\forall x \in \mathcal{V} \cap \mathbb{R}_J, ||A_{I_1,J_1}x|| \gg 0\}$ is a (r, k)- high probability event.

- $\{\forall x \in \mathcal{R}, ||Ax|| \gg 0\}$ is a (r, k)- high probability event.

- $\{\exists I_1, J_1 \subset [n]\ with\ |I| = |J| = n - k + 1\ such\ that$ $\forall x \in \mathcal{T} \cap \mathbb{R}_J, ||A_{I_1,J_1}x|| \gg 0, and\ \forall x \in \mathcal{T} \cap \mathbb{R}_I, ||A^{\top}_{I_1,J_1}x|| \gg 0\}$ is a (r, k)- high probability event if we condition on $O^c_{RC}(k)$.

$\mathcal{T}$-vectors

Condition on typical Column support sizes

- $\mathcal{L}(t)$ be the collection of columns whose support has size at most t.
- $J_0 = \mathcal{L}(t)$ for t depends on r, k.
- $I_0 = \{i \in [n], \exists j \in \mathcal{L}(t)\ s.t. a_{ij} = 1\}$ (union of supports of columns from J)

Matrix Decomposition:

When $(r-1)k < 1 - o(1)$

When $(r-1)k < 1 - o(1)$

When $(r-1)k \geq 1 - o(1)$

When $(r-1)k \geq 1 - o(1)$