

Intersections of dilated convex sets and mixed-degree variance inequalities

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October 11, 2026

Abstract

For every even log-concave density, the measure of an intersection of independently dilated symmetric convex sets is log-concave in the logarithms of the dilation factors. We emphasize this equivalent formulation of log-Brunn–Minkowski and derive it from a functional dilation theorem. For homogeneous potentials of different degrees, the same argument gives a covariance bound by their mean energies and sharp gamma bounds for their joint moment-generating function. The proof uses a moment-coordinate tensor estimate and retains a quantitative matrix remainder. We include examples, scaling identities, and further variance and spectral consequences, with attribution of the analytic input and without claims of historical novelty.

1 Introduction and main results

Suppose several symmetric convex constraints are relaxed or tightened at different rates. The central geometric statement of this note is that the measure of their intersection is log-concave in all logarithmic scale parameters simultaneously.

Throughout, a symmetric set means a set symmetric about the origin, and a convex body is compact with nonempty interior. The convex indicator I_K is zero on K and $+\infty$ outside it. An even log-concave measure is written $d\mu = e^{-V}dx$, with V proper, lower semicontinuous, even and convex. Unless otherwise stated, $\text{dom } V$ has nonempty interior. The measure need not be a probability measure; in particular, $V = 0$ gives Lebesgue measure.

1.1 The intersection inequality

Theorem A (Independent dilations of convex constraints). *Let $d\mu = e^{-V}dx$ be as above, and let $K_1, \dots, K_m$ be closed symmetric convex sets with $0 < \mu(\bigcap_i K_i) < \infty$. Define*

$$K(t) = \bigcap_{i=1}^m e^{t_i} K_i, \quad t \in \mathbb{R}^m.$$

Then $\log \mu(K(t))$ is finite and concave on $\mathbb{R}^m$. Equivalently, for $s, t \in \mathbb{R}^m$ and $0 \leq \lambda \leq 1$,

$$\mu\left(\bigcap_i e^{(1-\lambda)s_i + \lambda t_i} K_i\right) \geq \mu\left(\bigcap_i e^{s_i} K_i\right)^{1-\lambda} \mu\left(\bigcap_i e^{t_i} K_i\right)^{\lambda}. \quad (1.1)$$

The sets K_i and their intersection may be unbounded.

The integrability condition is automatic if $\bigcap_i K_i$ is a convex body. In particular, symmetric slabs are allowed when their common intersection is bounded. Under a finite measure, unbounded intersections of positive measure are included as well.

The parameters in (1.1) are independent. Some constraints may expand while others contract. For one set this is the scalar B-inequality. For slabs, it says that the measure of

$$P(s) = \{x : |\langle v_j, x \rangle| \leq e^{s_j}, j = 1, \dots, N\} \quad (1.2)$$

is log-concave in the logarithms of its prescribed supporting offsets, provided the vectors v_j span $\mathbb{R}^n$. No independence of the constraints or orthogonality of their normals is required.

The slab formulation gives the log-Brunn–Minkowski inequality by approximation in the supporting directions. If K, L are symmetric convex bodies, let

$$K +_{0,\lambda} L = \{x : |\langle v, x \rangle| \leq h_K(v)^{1-\lambda} h_L(v)^\lambda \text{ for every } v \in \mathbb{S}^{n-1}\}.$$

Then Theorem A implies

$$\mu(K +_{0,\lambda} L) \geq \mu(K)^{1-\lambda} \mu(L)^\lambda. \quad (1.3)$$

Conversely, (1.3) implies the intersection inequality: the logarithmic sum of the two endpoint intersections is contained in the intersection at the interpolated parameters, with truncation if the endpoint intersections are unbounded. We prove both passages in Section 4. Thus the intersection formulation expresses the geometric inequality directly in terms of independently varying convex constraints.

1.2 A functional theorem and its variance consequence

The proof uses the following functional statement, in which no homogeneity is assumed.

Theorem B (Joint dilation concavity). *Let $U, W_1, \dots, W_m : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be proper, lower semicontinuous, even convex functions, normalized to vanish at the origin. Assume*

$$0 < \int_{\mathbb{R}^n} e^{-U(x) - \sum_i W_i(x)} dx < \infty.$$

Then

$$\Phi(t) = \log \int_{\mathbb{R}^n} \exp \left[-U(x) - \sum_i W_i(e^{t_i} x) \right] dx \quad (1.4)$$

is finite and jointly concave on $\mathbb{R}^m$.

Taking $U = V - V(0)$ and $W_i = I_{K_i}$, and replacing t by $-t$, gives Theorem A. The potentials W_i may instead describe soft constraints, for example norms or convex energies. A fixed symmetric convex restriction can be incorporated into U .

For homogeneous potentials, dilation is multiplication by a positive coefficient. This yields a matrix inequality that allows different degrees.

Theorem C (Mixed-degree covariance and gamma bounds). *Let U be proper, lower semicontinuous, even and convex, with $U(0) = 0$ and full-dimensional domain. Let $\psi_i : \mathbb{R}^n \rightarrow [0, \infty)$ be finite even convex functions such that*

$$\psi_i(rx) = r^{q_i} \psi_i(x) \quad (r > 0), \quad q_i \geq 1.$$

Assume $0 < \int e^{-U - \sum_i \psi_i} < \infty$. For $s \in \mathbb{R}^m$ put

$$\begin{aligned} Y_i(s, x) &= e^{s_i \psi_i(x)}, & Z(s) &= \int e^{-U - \sum_i Y_i(s, x)} dx, \\ d\nu_s &= Z(s)^{-1} e^{-U - \sum_i Y_i(s, x)} dx, & m_i(s) &= \mathbb{E}_s Y_i. \end{aligned} \quad (1.5)$$

Then $\Psi(s) = \log Z(s)$ is smooth and concave, and

$$\text{Cov}_s(Y_1, \dots, Y_m) \preceq \text{diag}(m_1(s), \dots, m_m(s)). \quad (1.6)$$

Equivalently, for every $b \in \mathbb{R}^m$,

$$\text{Var}_s \left(\sum_i b_i Y_i \right) \leq \sum_i b_i^2 m_i(s). \quad (1.7)$$

Moreover, for all real $\theta_i < 1$,

$$\mathbb{E}_s \exp \left(\sum_i \theta_i Y_i \right) \leq \prod_i (1 - \theta_i)^{-m_i(s)}. \quad (1.8)$$

The background U may include the indicator of a symmetric convex body.

The right side of (1.8) is the joint moment-generating function of independent gamma variables with shapes $m_i(s)$ and rate one. The actual energies need not be independent. Both (1.7) and (1.8) are sharp for potentials supported on independent coordinate blocks. When $U = 0$, the degrees determine an exact common-scaling direction:

$$\Psi(s + tq) = \Psi(s) - nt, \quad \sum_i q_i m_i(s) = n, \quad [\text{diag}(m(s)) - \text{Cov}_s(Y)]q = 0, \quad (1.9)$$

where $q = (q_1, \dots, q_m)^T$. Thus the degrees cancel from the covariance bound but remain visible in its equality structure.

1.3 Analytic content, provenance, and organization

The analytic input is the moment-coordinate tensor estimate attributed to [9, Lemma 6.1]. A useful consequence is an estimate that splits an observable into two parts. For a smooth centered probability measure $d\nu = Z^{-1} e^{-Q} dx$ with $H = D^2 Q \succeq \varepsilon I$, let τ be its moment-map Stein kernel, defined in Section 2, and let $g^\perp$ be the orthogonal projection of g off the affine functions. Then, for smooth $g \in L^2(\nu)$ and smooth F ,

$$\|g^\perp\|_2^2 \leq \mathbb{E} \text{tr}(\tau D^2 F H^{-1} D^2 F) + \mathbb{E} \langle H^{-1} \nabla(g - F), \nabla(g - F) \rangle. \quad (1.10)$$

For even ν and g , the left side is $\text{Var}_\nu g$. Neither test function is required to be convex. This freedom is essential for joint concavity, since a parameter direction may have coefficients of either sign.

In the smooth setting, the functional theorem has a quantitative form. Writing $G_i = D^2[W_i(e^{t_i \cdot})]$, $H = D^2 U + \sum_i G_i$ and $S_a = \sum_i a_i G_i$, we prove

$$-a^T D^2 \Phi(t) a \geq \mathbb{E}_t \text{tr} \left[(\tau_t + X X^T) \left(\sum_i a_i^2 G_i - S_a H^{-1} S_a \right) \right] \geq 0. \quad (1.11)$$

The remainder is exact for quadratic potentials, including noncommuting quadratic forms. Optimizing (1.10) also gives $\|g^\perp\|_2^2 \leq E_1(g) E_2(g) / (E_1(g) + E_2(g))$, where E_1 and E_2 are the gradient and Hessian

energies defined in Section 6. For the moment operator $Af = x \cdot \nabla f - \tau : D^2 f$, the condition $H \succeq \alpha \tau^{-1}$ implies that every L^2 eigenvalue above 1 is at least $1 + \alpha$. The exact tensor deficit is recorded separately; no geometric stability conclusion is inferred from it.

Moment measures originate in [1, 2]; their differential and spectral structure is developed in [7]. We use the Stein-kernel identification of [6, Theorem 2.3] and the transport-Hessian identity of [8, Proposition 4.8]. The tensor estimate, its square completion, and the range-density argument are attributed to [9, Sections 4–6]; their calculations are included in the appendices. The common-degree application to powers of linear forms is treated in [9, Proposition 2.1 and Section 7]. We adapt that calculation to individual degrees.

The Gaussian B-theorem [3], the variance methods of [4, 5], and Saroglou’s relations between B-properties, logarithmic sums, and even log-concave measures [11] provide the geometric context. The log-Brunn–Minkowski conclusion is also obtained in [9]. Our purpose is to explain its intersection formulation and the functional and probabilistic consequences in one argument, without claiming historical novelty. The proof below derives the intersection inequality from the analytic input, without taking log-Brunn–Minkowski as an additional premise.

Section 2 proves (1.10) and its Euler form. Section 3 proves Theorem B and (1.11). Section 4 proves Theorem A and develops its geometric interpretations. Section 5 proves Theorem C, the scaling identities, and concentration estimates. Sections 6–7 give the further analytic consequences. The appendices supply the moment construction, tensor calculation, density argument, and convex approximation.

2 Moment measures and the analytic bridge

Let $d\nu = Z^{-1}e^{-Q}dx$ be centered, with Q smooth and $H = D^2Q \succeq \varepsilon I$. The moment construction supplies a smooth convex ϕ such that

$$(\nabla\phi)_\#(e^{-\phi(z)}dz) = \nu, \quad x = \nabla\phi(z), \quad \tau(x) = D_z^2\phi(z), \quad M = \tau^{-1}. \quad (2.1)$$

The source has mass one, the gradient is a diffeomorphism, and $0 < \tau \preceq \Lambda I$. These facts follow from the compact moment construction [1, 2], Klartag’s bound [7, Proposition 3.1 and Remark 3.5], and the exhaustion recalled in Appendix A. The field τ is the Hessian evaluated at $z = (\nabla\phi)^{-1}(x)$, not the Hessian of a function of x .

The Stein identity is

$$\mathbb{E}[Xa(X)] = \mathbb{E}[\tau(X)\nabla a(X)], \quad X \sim \nu. \quad (2.2)$$

It follows by integrating $\phi_i a(\nabla\phi)e^{-\phi}$ by parts in z . This identifies τ as the moment-map Stein kernel [6]. For matrices use $B : C = \text{tr}(B^T C)$. Write

$$\delta v = \nabla Q \cdot v - \text{div } v, \quad Af = \delta(\tau \nabla f) = x \cdot \nabla f - \tau : D^2 f.$$

On matrices, δ acts row by row. Unless otherwise indicated, derivatives are in the target variable x .

Lemma 2.1 (The analytic input). *For $f \in C_c^\infty(\mathbb{R}^n)$ put*

$$B = \tau D^2 f \tau, \quad w = \delta B, \quad u = \delta w.$$

Then

$$w = \tau \nabla(A - 1)f, \quad u = A(A - 1)f, \quad (2.3)$$

$$\mathbb{E}uF = \mathbb{E}w \cdot \nabla F = \mathbb{E}B : D^2 F, \quad (2.4)$$

$$\mathbb{E}(Af)^2 - \mathbb{E}fAf = \mathbb{E}\|\tau^{1/2}D^2 f \tau^{1/2}\|_{\text{HS}}^2, \quad (2.5)$$

$$\mathbb{E}u^2 \geq \mathbb{E}w^T H w + \mathbb{E} \text{tr}(MBHB). \quad (2.6)$$

Moreover, the range of $A(A-1)$ on C_c^∞ is dense in the orthogonal complement of the affine functions in $L^2(\nu)$.

The identities and estimate are attributed to [9, Sections 4–6]; (2.5) is the moment specialization of [8, Proposition 4.8]. For the density assertion we use the cutoff argument of [9, Lemma 5.1], retaining the affine kernel before imposing parity. Appendix B gives the calculations and proof.

Let $\Sigma = \mathbb{E}XX^T$ and define

$$g^\perp = g - \mathbb{E}g - \langle \Sigma^{-1} \mathbb{E}[Xg], x \rangle.$$

This removes the affine component; Σ is positive definite.

Proposition 2.2 (Splitting an observable). *For smooth $g \in L^2(\nu)$ and smooth F ,*

$$\|g^\perp\|_2^2 \leq \mathbb{E} \operatorname{tr}(\tau D^2 F H^{-1} D^2 F) + \mathbb{E} \langle H^{-1} \nabla(g - F), \nabla(g - F) \rangle. \quad (2.7)$$

An infinite right side is allowed. Neither g nor F needs to be convex. If ν and g are even, the left side is $\operatorname{Var}_\nu g$.

Proof. The adjunctions give $\mathbb{E}ug = \mathbb{E}B : D^2F + \mathbb{E}w \cdot \nabla(g - F)$. Cauchy–Schwarz in the direct sum of matrix and vector L^2 spaces bounds its square by the right side of (2.7) times $\mathbb{E} \operatorname{tr}(MBHB) + \mathbb{E}w^T H w \leq \mathbb{E}u^2$. For the matrix pairing use $H^{1/2} B M^{1/2}$ and $H^{-1/2} D^2 F \tau^{1/2}$. Density permits $u \rightarrow g^\perp$ in L^2 , proving the claim. $\square$

Define the Euler derivative $\mathcal{D}F = x \cdot \nabla F$; the notation $\mathcal{D}^2 F = \mathcal{D}(\mathcal{D}F)$ is distinct from the Euclidean Hessian $D^2 F$.

Corollary 2.3 (Euler–Hessian estimate). *If ν, F are even, F is smooth, and $\mathcal{D}F \in L^2(\nu)$, then*

$$\operatorname{Var}_\nu(\mathcal{D}F) \leq \mathbb{E} \operatorname{tr}[(\tau + xx^T) D^2 F H^{-1} D^2 F]. \quad (2.8)$$

Proof. Use $g = \mathcal{D}F$ in (2.7) and $\nabla(\mathcal{D}F - F) = D^2 F x$. $\square$

The coefficient one is sharp. For $F = Q$, when all quantities and integrations by parts are finite, both sides equal $n + \mathbb{E}x^T H x$:

$$\mathbb{E} \mathcal{D}Q = n, \quad \operatorname{Var}(\mathcal{D}Q) = n + \mathbb{E}x^T H x, \quad \mathbb{E} \operatorname{tr}(\tau H) = n.$$

Bounded $D^2 Q$ is one sufficient condition. The variance identity follows by integrating $\operatorname{div}(x \mathcal{D}Q e^{-Q})$.

3 Joint dilation concavity and a matrix remainder

First let $U, W_1, \dots, W_m$ be smooth even convex functions, with $D^2 U \succeq \varepsilon I$ and first two derivatives of at most polynomial growth. At $t \in \mathbb{R}^m$, set

$$W_{i,t}(x) = W_i(e^{t_i} x), \quad Q_t = U + \sum_i W_{i,t}, \quad \Phi(t) = \log \int e^{-Q_t} dx.$$

Expectations are for $\nu_t = e^{-Q_t} dx / \int e^{-Q_t}$, with moment Hessian τ_t . Put $G_i = D^2 W_{i,t}$, $J = D^2 U$, $H = J + \sum_i G_i$, and $K_t(x) = \tau_t(x) + xx^T$.

Theorem 3.1 (Quantitative joint concavity). *For $a \in \mathbb{R}^m$ define*

$$S_a = \sum_i a_i G_i, \quad R_a = \sum_i a_i^2 G_i - S_a H^{-1} S_a.$$

Then

$$-a^T D_t^2 \Phi(t) a \geq \mathbb{E}_t \operatorname{tr}(K_t R_a) \geq 0. \quad (3.1)$$

In particular, Φ is jointly concave.

Proof. For $F_a = \sum_i a_i W_{i,t}$, differentiation gives

$$a^T D_t^2 \Phi a = \operatorname{Var}_t(\mathcal{D}F_a) - \sum_i a_i^2 \mathbb{E}_t \mathcal{D}^2 W_{i,t}.$$

Corollary 2.3 bounds the variance by $\mathbb{E}_t \operatorname{tr}(K_t S_a H^{-1} S_a)$. The Stein identity gives

$$\mathbb{E}_t \mathcal{D}^2 W_{i,t} = \mathbb{E}_t (x \cdot \nabla W_{i,t} + x^T G_i x) = \mathbb{E}_t \operatorname{tr}(K_t G_i).$$

Subtracting proves the first inequality. For every $v \in \mathbb{R}^n$,

$$v^T R_a v = \min_{z \in \mathbb{R}^n} \left\{ z^T J z + \sum_i (a_i v - z)^T G_i (a_i v - z) \right\} \geq 0. \quad (3.2)$$

The minimizer is $z = H^{-1} S_a v$. Since $K_t \succ 0$, $\operatorname{tr}(K_t R_a) \geq 0$. $\square$

The coefficients a_i may have either sign, which explains why the Euler estimate is needed for tests that are not necessarily convex. Formula (3.2) measures the discrepancy between the prescribed dilation rates and one common vector z , using the Hessians as quadratic costs.

For one potential the remainder matrix is

$$G - G(G + J)^{-1} G = (G^{-1} + J^{-1})^{-1} \quad (G, J \succ 0), \quad (3.3)$$

the matrix parallel sum. The left expression also handles singular G . With $J \succ 0$, it vanishes precisely on $\ker G$. Consequently Φ is strictly concave for a nonconstant W under the smooth assumptions: evenness excludes a nonconstant affine W , so $D^2 W$ is nonzero on an open set.

Example 3.2 (Exact quadratic remainder). Take $U(x) = \frac{1}{2} x^T A_0 x$ and $W_i(x) = \frac{1}{2} x^T A_i x$, where $A_0 \succ 0$ and $A_i \succeq 0$. Then

$$H = A_0 + \sum_i e^{2t_i} A_i, \quad \Phi(t) = \frac{n}{2} \log(2\pi) - \frac{1}{2} \log \det H, \quad \tau_t = H^{-1}.$$

As $\mathbb{E}_t X X^T = H^{-1}$, direct differentiation gives $-a^T D_t^2 \Phi a = 2 \operatorname{tr}(H^{-1} R_a) = \mathbb{E}_t \operatorname{tr}(K_t R_a)$. Thus the remainder is exact, even for noncommuting matrices.

Proof of Theorem B. Normalize every potential to have minimum zero. For any t , let

$$r = \min(1, e^{t_1}, \dots, e^{t_m}), \quad R = \max(1, e^{t_1}, \dots, e^{t_m}).$$

A convex function P with $P(0) = 0$ is nondecreasing along each ray. Consequently,

$$Q_0(rx) \leq Q_t(x), \quad Q_t(x/R) \leq Q_0(x).$$

Changing variables yields

$$R^{-n} \int e^{-Q_0} \leq \int e^{-Q_t} \leq r^{-n} \int e^{-Q_0}. \quad (3.4)$$

Thus all integrals are finite and nonzero.

Add $\delta|x|^2/2$ to the background, approximate the convex potentials by the smooth functions of Appendix C, and apply Theorem 3.1. Dominated convergence passes the three-point log-concavity inequality to the original potentials with the added quadratic term. Letting $\delta \downarrow 0$ and using monotone convergence proves the assertion. This passage proves concavity; it does not assert convergence of moment Hessians or quantitative remainders. $\square$

4 Intersections and logarithmic sums

4.1 Proof and examples of the intersection inequality

Proof of Theorem A. Since V is even, proper and convex, it attains its minimum at the origin. Take $U = V - V(0)$ and $W_i = I_{K_i}$ in Theorem B. Then

$$\int e^{-U(x) - \sum_i W_i(e^{-t_i} x)} dx = e^{V(0)} \mu\left(\bigcap_i e^{t_i} K_i\right).$$

The integral at $t = 0$ is finite and positive by hypothesis. Theorem B and the linear substitution $t \mapsto -t$ give the result. If the original intersection is a convex body, the integrability condition is automatic: $e^{-V} \leq e^{-V(0)}$, and both the intersection and the domain of V contain a neighborhood of the origin. $\square$

Example 4.1 (Several convex constraints). For symmetric convex bodies K, L , the function

$$(s, t) \mapsto \mu(e^s K \cap e^t L)$$

is jointly log-concave. In particular, one may interpolate between a configuration with a large copy of K and a small copy of L and another with the opposite balance. The same statement holds for any finite number of norm balls or ellipsoids; their axes need not agree.

Example 4.2 (Fixed restrictions). For a fixed symmetric convex body K_0 , replace V by $V + I_{K_0}$. Whenever the intersections have nonempty interior,

$$t \mapsto \mu\left(K_0 \cap \bigcap_i e^{t_i} K_i\right)$$

is log-concave. Equivalently, K_0 can be included as a constraint with its parameter held at zero. This also allows unbounded varying constraints whose intersection becomes bounded only after imposing K_0 .

Example 4.3 (Slabs and convex sublevel constraints). For spanning vectors $v_1, \dots, v_N$, apply the theorem to the slabs $K_j = \{x : |\langle v_j, x \rangle| \leq 1\}$. It gives the asserted concavity of $\mu(P(s))$ in (1.2). More generally, if p_i are finite even convex functions homogeneous of degrees $q_i \geq 1$, then

$$s \mapsto \mu\{x : p_i(x) \leq e^{s_i} \text{ for every } i\}$$

is log-concave whenever the common unit sublevel set is a convex body. Here $\{p_i \leq e^{s_i}\} = e^{s_i/q_i} \{p_i \leq 1\}$.

The theorem dilates each constraint by a scalar, leaving its supporting directions fixed. Applying an arbitrary diagonal linear map to one general body may change its supporting directions and is a different operation. The intersection theorem by itself therefore does not assert the unrestricted strong B-property for an arbitrary measure and body.

4.2 Passage to log-Brunn–Minkowski

For symmetric convex bodies K, L , choose an increasing sequence of finite sets of unit directions, whose union is dense and which include a fixed spanning collection. The polytopes defined by the inequalities

$$|\langle v_j, x \rangle| \leq h_K(v_j), \quad |\langle v_j, x \rangle| \leq h_L(v_j)$$

decrease to K and L , respectively. At the interpolated parameters, the corresponding polytopes have constraints

$$|\langle v_j, x \rangle| \leq h_K(v_j)^{1-\lambda} h_L(v_j)^\lambda$$

and decrease to $K +_{0,\lambda} L$. The spanning collection bounds all these sets in a fixed compact set. The slab case of Theorem A, followed by continuity of measure from above, gives (1.3).

Conversely, suppose (1.3) holds for μ and all symmetric convex bodies. Put $A = \bigcap_i e^{s_i} K_i$ and $B = \bigcap_i e^{t_i} K_i$, and first suppose that A and B are bounded. For each i and each direction where h_{K_i} is finite,

$$h_A(v)^{1-\lambda} h_B(v)^\lambda \leq e^{(1-\lambda)s_i + \lambda t_i} h_{K_i}(v).$$

The support-halfspace description of a closed convex set therefore gives

$$A +_{0,\lambda} B \subseteq \bigcap_i e^{(1-\lambda)s_i + \lambda t_i} K_i.$$

Apply (1.3) and monotonicity of μ to obtain (1.1). If A or B is unbounded, apply the same support bounds and (1.3) to $A \cap RB_2^n$ and $B \cap RB_2^n$, and let $R \rightarrow \infty$. Monotone convergence of the endpoint measures proves the assertion. Directions with infinite support value impose no constraint.

These passages clarify the role of the intersection theorem. For a fixed even log-concave measure in a fixed dimension, its validity for all such families of constraints is equivalent to log-Brunn–Minkowski for that measure. Saroglou’s transfer theorem further relates the Lebesgue and even log-concave versions [11, Theorem 3.1 and Corollary 3.2]. The proof in this note reaches the intersection statement directly through the functional theorem and the analytic input.

5 Mixed degrees: proof, equality, and concentration

5.1 Logarithmic coefficients and the covariance matrix

Use the notation of Theorem C, and write $C(s) = (\text{Cov}_s(Y_i, Y_j))_{i,j}$. Finiteness for every s follows from Theorem B with $t_i = s_i/q_i$. It can also be checked directly: choose $0 < r \leq 1$ with $r^{q_i} \leq e^{s_i}$ for every i . Convexity gives $U(rx) \leq U(x)$, hence $Z(s) \leq r^{-n} Z(0)$.

Differentiation in s is justified locally by slightly decreasing the positive coefficients of the ψ_i ; polynomial factors in the energies are absorbed by the remaining exponential. All orders of differentiation are allowed. In particular,

$$\partial_i \Psi = -m_i, \quad \partial_{ij} \Psi = \text{Cov}_s(Y_i, Y_j) - \delta_{ij} m_i. \quad (5.1)$$

Thus concavity and the covariance bound are exactly equivalent.

Proof of Theorem C. Homogeneity gives $e^{s_i} \psi_i(x) = \psi_i(e^{s_i/q_i} x)$. Theorem B implies that $\Psi(s) = \Phi(s_1/q_1, \dots, s_m/q_m)$ is concave. Equation (5.1) proves (1.6) and (1.7). This argument includes nonsmooth potentials and convex restrictions.

For (1.8), put $\ell_i = \log(1 - \theta_i)$, where $\theta_i < 1$. Then

$$\log \mathbb{E}_s \exp \left(\sum_i \theta_i Y_i \right) = \Psi(s + \ell) - \Psi(s) \leq \nabla \Psi(s) \cdot \ell = - \sum_i m_i(s) \log(1 - \theta_i),$$

by the tangent inequality for a concave function. Exponentiate. $\square$

5.2 A second proof: cancellation of the degrees

The degree cancellation can also be seen directly in the tensor estimate. This adapts the argument of [9, Section 7] to individual degrees. Work in the smooth uniformly convex setting with $q_i > 1$, and take $s = 0$. Let $H_i = D^2 \psi_i$, $m_i = \mathbb{E} \psi_i > 0$, and

$$r_i = \mathbb{E} \operatorname{tr}(\tau H_i) = \mathbb{E} D \psi_i = q_i m_i, \quad d_i = \mathbb{E} u \psi_i = \mathbb{E} B : H_i = \mathbb{E} w \cdot \nabla \psi_i.$$

The Stein identity is justified by cutoffs, since the homogeneous potentials have polynomial growth and the measure has Gaussian tails. Matrix Cauchy–Schwarz gives

$$\mathbb{E} \operatorname{tr}(M B H_i B) \geq \frac{d_i^2}{r_i}.$$

Euler’s identity gives $H_i x = (q_i - 1) \nabla \psi_i$, and hence

$$\mathbb{E} x^T H_i x = (q_i - 1) r_i, \quad \mathbb{E} x^T H_i w = (q_i - 1) d_i.$$

Cauchy–Schwarz for the semidefinite form $\mathbb{E} v^T H_i v$ yields

$$\mathbb{E} w^T H_i w \geq (q_i - 1) \frac{d_i^2}{r_i}.$$

Since $H \succeq \sum_i H_i$, the tensor estimate gives

$$\mathbb{E} u^2 \geq \sum_i (1 + (q_i - 1)) \frac{d_i^2}{r_i} = \sum_i \frac{d_i^2}{m_i}. \quad (5.2)$$

The Hessian term supplies one part, the gradient term supplies $q_i - 1$ parts, and their sum cancels the q_i in $r_i = q_i m_i$. Density extends (5.2) to every centered even $u \in L^2$. For $u = \sum_i b_i (\psi_i - m_i)$,

$$\|u\|_2^2 = \sum_i b_i d_i \leq \left(\sum_i b_i^2 m_i \right)^{1/2} \left(\sum_i d_i^2 / m_i \right)^{1/2} \leq \left(\sum_i b_i^2 m_i \right)^{1/2} \|u\|_2.$$

This proves the covariance bound. Zero summands can be omitted. The first proof covers every degree $q_i \geq 1$ and does not require approximants to remain homogeneous.

5.3 Response and the common-dilation direction

Define the positive semidefinite deficit matrix

$$\mathcal{C}(s) = \operatorname{diag}(m(s)) - C(s) = -D_s^2 \Psi(s). \quad (5.3)$$

By (5.1), $\partial_j m_i = \mathcal{C}_{ij}$. Thus the map from logarithmic coefficients to mean component energies is monotone:

$$(m(s) - m(t)) \cdot (s - t) \geq 0.$$

Increasing a coefficient increases its mean *weighted* energy $e^{s_i} \psi_i$ in the non-strict sense. The unweighted mean $\mathbb{E}_s \psi_i$ is nonincreasing, with derivative $-e^{s_i} \operatorname{Var}_s \psi_i$ in s_i .

Proposition 5.1 (The scaling direction). *If $U = 0$ and $q = (q_1, \dots, q_m)^T$, then*

$$\Psi(s + tq) = \Psi(s) - nt, \quad \sum_i q_i m_i(s) = n, \quad \mathcal{C}(s)q = 0. \quad (5.4)$$

Equivalently, $\mathcal{C}(s)q = \text{diag}(m(s))q$.

Proof. The substitution $y = e^t x$ gives $Z(s + tq) = e^{-nt} Z(s)$. Differentiate first in t and then in s . All differentiations are in the coupling parameters, so no smoothness in x is needed. $\square$

Common spatial scaling contributes an affine function to Ψ . Curvature can occur only in relative scales, and the kernel can be larger than the span of q .

Example 5.2 (Independent blocks and sharpness). Let $\mathbb{R}^n = E_1 \oplus \dots \oplus E_m$ be an orthogonal decomposition, $d_i = \dim E_i$, and $\psi_i(x) = \|x_i\|_{K_i}^{q_i}/q_i$, where K_i is a symmetric convex body in E_i and $U = 0$. Radial integration in each norm shows that $Y_i = e^{s_i} \psi_i(X)$ are independent gamma variables with shape d_i/q_i and rate one. Thus $m_i = d_i/q_i$ and $C = \text{diag}(m)$: equality holds in every direction of (1.7) and throughout (1.8).

Proposition 5.3 (Two components). *Suppose $U = 0$, $m = 2$, and put $c = \text{Cov}(Y_1, Y_2)$. Then $c \geq 0$ and*

$$\sum_{i=1}^2 b_i^2 m_i - \text{Var}(b_1 Y_1 + b_2 Y_2) = \frac{c}{q_1 q_2} (q_2 b_1 - q_1 b_2)^2. \quad (5.5)$$

Proof. The identity $\mathcal{C}q = 0$ forces $\mathcal{C}_{11} = (q_2/q_1)c$ and $\mathcal{C}_{22} = (q_1/q_2)c$, while $\mathcal{C}_{12} = -c$. Positive semidefiniteness gives $c \geq 0$ and (5.5). $\square$

This sign conclusion uses both the absence of a background potential and the presence of only two components. The general result is a matrix inequality and does not prescribe the signs of all off-diagonal entries.

Example 5.4 (Quadratic and quartic contributions). In one dimension take $\psi_2(x) = x^2/2$, $\psi_4(x) = x^4/4$, $U = 0$, and $s = 0$. Denote their means by m_2, m_4 and their covariance by c . Then

$$2m_2 + 4m_4 = 1, \quad \text{Var}(\psi_2) = m_2 - 2c, \quad \text{Var}(\psi_4) = m_4 - \frac{1}{2}c.$$

Here $c > 0$, because both energies are strictly increasing functions of $|X|$. Indeed, for independent copies R, R' ,

$$2 \text{Cov}(f(R), g(R)) = \mathbb{E}[(f(R) - f(R'))(g(R) - g(R'))].$$

Consequently,

$$m_2 b_2^2 + m_4 b_4^2 - \text{Var}(b_2 \psi_2 + b_4 \psi_4) = \frac{c}{8} (4b_2 - 2b_4)^2.$$

The equality direction is $(b_2, b_4) = (2, 4)$, corresponding to spatial dilation. The total energy $\psi_2 + \psi_4$ has deficit $c/2$.

Example 5.5 (Convex restriction). For finite even convex homogeneous potentials ψ_i and any symmetric convex body K , the normalized density $\mathbf{1}_K e^{-\sum_i \psi_i}$ satisfies

$$\text{Var}\left(\sum_i b_i \psi_i\right) \leq \sum_i b_i^2 \mathbb{E} \psi_i.$$

The scaling identities (5.4) generally disappear, because the fixed body introduces another spatial scale.

5.4 Concentration from the gamma bound

Fix s , write $Y_i = Y_i(s, X)$ and $m_i = \mathbb{E}_s Y_i$. A zero mean means that the corresponding energy vanishes almost surely. For $U = 0$, (1.8) is an equality along $\theta_i = 1 - e^{r q_i}$, by (5.4).

For a single component Y of positive mean m , Chernoff's argument with $\theta = r/(m + r)$ gives, for $r > 0$,

$$\mathbb{P}\{Y \geq m + r\} \leq \exp[-r + m \log(1 + r/m)]. \quad (5.6)$$

For $b \in \mathbb{R}^m$, put $T = \sum_i b_i(Y_i - m_i)$, $v = \sum_i b_i^2 m_i$, and $c = \max_i |b_i|$. The inequality $-u - \log(1 - u) \leq u^2/[2(1 - |u|)]$ for $|u| < 1$ yields

$$\log \mathbb{E} e^{\lambda T} \leq \frac{\lambda^2 v}{2(1 - |\lambda|c)} \quad (|\lambda|c < 1).$$

For $v > 0$, use $\lambda = \sqrt{2r/v}/(1 + c\sqrt{2r/v})$ for each tail. Then

$$\mathbb{P}\{|T| \geq \sqrt{2vr} + cr\} \leq 2e^{-r}, \quad r > 0. \quad (5.7)$$

If $v = 0$, then $T = 0$ almost surely. The same mean energies therefore control both the variance and an explicit concentration estimate.

6 Improved variance bounds and spectral separation

Return to the smooth uniformly convex moment setting of Section 2. For smooth $g \in L^2(\nu)$, put

$$E_1(g) = \mathbb{E}\langle H^{-1} \nabla g, \nabla g \rangle, \quad E_2(g) = \mathbb{E} \operatorname{tr}(\tau D^2 g H^{-1} D^2 g).$$

Corollary 6.1 (Optimizing the split). *If these energies are finite and their sum is positive, then*

$$\|g^\perp\|_2^2 \leq \frac{E_1(g)E_2(g)}{E_1(g) + E_2(g)}. \quad (6.1)$$

For even ν, g , this bounds $\operatorname{Var}_\nu g$.

Proof. Choose $F = \theta g$ in (2.7). Minimize $\theta^2 E_2 + (1 - \theta)^2 E_1$ at $\theta = E_1/(E_1 + E_2)$. $\square$

For even observables this refines the classical Brascamp–Lieb bound [10] by a Hessian energy. In particular, $E_1 - \operatorname{Var} g \geq E_1^2/(E_1 + E_2)$. For quadratic observables under the standard Gaussian, $E_1 = E_2$ and the sharp factor is $1/2$. A uniform improvement for all even tests requires additional structure: [8, Theorem 1.13] shows that the factor $1/2$ for all even tests forces a Gaussian among two-homogeneous potentials smooth away from the origin with positive-definite Hessian there. A two-homogeneous potential that is C^2 at the origin is already quadratic.

Proposition 6.2 (Separation of eigenvalues). *Suppose $H \succeq \alpha \tau^{-1}$ for some $\alpha > 0$. Every nonzero $L^2(\nu)$ distributional eigenfunction $Ag = \lambda g$ with $\lambda > 1$ satisfies $\lambda \geq 1 + \alpha$.*

Proof. Local elliptic regularity makes g smooth. Testing against $\eta_r^2 g$ with the cutoffs of Appendix B.3 gives $\mathbb{E}\langle \tau \nabla g, \nabla g \rangle = \lambda \|g\|_2^2$. The cutoff errors vanish by Cauchy–Schwarz. Also $g_r = \eta_r g \rightarrow g$ and $Ag_r \rightarrow \lambda g$ in L^2 . Applying (2.5) to g_r and using Fatou gives

$$\mathbb{E}\|\tau^{1/2} D^2 g \tau^{1/2}\|_{\text{HS}}^2 \leq \lambda(\lambda - 1)\|g\|_2^2.$$

The same cutoffs, tested against 1 and the coordinate functions, show $\mathbb{E}g = 0$ and $(\lambda - 1)\mathbb{E}[Xg] = 0$, hence $g^\perp = g$. Here $X \in L^2$ and τ is bounded, so the additional coordinate-cutoff terms vanish on the annuli. Since $H^{-1} \preceq \alpha^{-1}\tau$,

$$E_1(g) \leq \alpha^{-1}\lambda\|g\|_2^2, \quad E_2(g) \leq \alpha^{-1}\lambda(\lambda - 1)\|g\|_2^2.$$

Use $F = g/\lambda$ in (2.7) to obtain $\|g\|_2^2 \leq \alpha^{-1}(\lambda - 1)\|g\|_2^2$. $\square$

This assertion concerns L^2 eigenfunctions and does not presume a discrete spectrum. For the standard Gaussian, $\tau = H = I$, so it recovers the separation between affine eigenfunctions at 1 and quadratic ones at 2. Spectral estimates and their geometric uses are developed in [7, 8].

7 What the exact analytic deficit controls

For a compact test f , define

$$\Delta(f) = \mathbb{E}u^2 - \mathbb{E}w^T H w - \mathbb{E}\text{tr}(MBHB).$$

The square identity in Appendix B.2 reads

$$\Delta(f) = \mathbb{E} \left[\|M^{1/2}RM^{1/2}\|_{\text{HS}}^2 + 2\|S + PJ\|^2 + \frac{1}{2}\|J - PJ\|^2 \right]. \quad (7.1)$$

Here $R = D_z w - B$; it is unrelated to the matrix R_a of Section 3. In constant coordinates normalized by $\tau = I$, put $S_{ijk} = f_{x_i x_j x_k}$, $J_{ijk} = f_{x_i x_a} \phi_{z_a z_j z_k}$, and let P denote full symmetrization. The appendix specifies the coordinate convention and norms. Consequently,

$$\begin{aligned} \mathbb{E}\|J - PJ\|^2 &\leq 2\Delta(f), & \mathbb{E}\|S + PJ\|^2 &\leq \frac{1}{2}\Delta(f), \\ \mathbb{E}\|M^{1/2}RM^{1/2}\|_{\text{HS}}^2 &\leq \Delta(f). \end{aligned}$$

These are quantitative analytic consequences of the tensor identity. A stability theorem for bodies would additionally require control of the losses in duality and approximation, identification of all zero modes, and coercivity modulo those modes. No such geometric stability theorem is asserted here. The exact quadratic and independent-block examples are useful checks on prospective equality or rigidity statements.

A The moment construction

We adapt the whole-space exhaustion of [9, Section 3] to centered, not necessarily even targets. Its inputs are the compact existence and regularity theorem [1, 2] and [7, Proposition 3.1 and Remark 3.5]. Truncate ν to B_R , let its mean be m_R , and shift the target by $-m_R$. The resulting target is centered and has potential $Q(x + m_R)$ with the same Hessian lower bound. Translate the source minimum of its moment potential ϕ_R to zero. Writing $g_R = \phi_R - \phi_R(0)$, we have $0 \leq g_R \leq \Lambda|z|^2/2$ with $\Lambda = 4/\varepsilon$. Compact target support gives exponential source tails and $\mathbb{E}_{e^{-\phi_R} z \cdot \nabla \phi_R} = n$. For $x = \nabla \phi_R(z)$, convexity and the minimum at zero give $x \cdot z \geq 0$ and

$$g_R(y) \geq \max\{0, x \cdot y - x \cdot z\} \geq (x \cdot y)_+ - x \cdot z.$$

Taking source expectation gives

$$g_R(y) \geq \mathbb{E}(x \cdot y)_+ - n = \frac{1}{2}\mathbb{E}|x \cdot y| - n \geq c|y| - n.$$

Centering gives the equality. The final constant is uniform because the shifted target densities are uniformly positive on B_1 for large R . Normalization bounds $\phi_R(0)$.

On any fixed source ball, the moment equation bounds $\det D^2\phi_R$ below by $d > 0$, since $\phi_R, \nabla\phi_R$, and the target normalizations are uniformly controlled there. Thus

$$(d/\Lambda^{n-1})I \preceq D^2\phi_R \preceq \Lambda I.$$

The logarithmic Monge–Ampère equations have uniformly Lipschitz right sides on smaller balls. Interior Evans–Krylov estimates, followed by differentiation and Schauder estimates, give uniform local bounds for all derivatives; see [9, Section 3] and [12]. A subsequence converges locally smoothly. Exponential domination passes normalization and the pushforward identity to the limit.

The gradient image is dense because the target density is positive everywhere. For any x_0 , choose finitely many image points y_i whose convex hull contains a ball around x_0 . Their supporting planes give $\phi(z) - x_0 \cdot z \geq c_0|z| - C$. The minimum of this coercive function realizes $\nabla\phi = x_0$, proving surjectivity.

B Proofs of the analytic input

B.1 Adjunction and the Hessian identity

All fields are as in Lemma 2.1. Compact supports in the target have compact preimages under the moment diffeomorphism.

Proof. Integration by parts in z gives

$$\int \phi_i a(\nabla\phi) e^{-\phi} = \int \partial_i [a(\nabla\phi)] e^{-\phi},$$

hence $\delta\tau = x$ and (2.2). If $C_{ijk} = \phi_{z_i z_j z_k}$, the chain rule gives

$$w_a = x_k \tau_{ai} f_{x_i x_k} - C_{aik} f_{x_i x_k} - \tau_{ai} \tau_{jk} f_{x_i x_k x_j} = \partial_{z_a} (A - 1) f.$$

Repeated indices are summed. Two adjunctions prove (2.4). Finally,

$$\mathbb{E}(Af)(A - 1)f = \mathbb{E}\nabla f \cdot w = \mathbb{E}D^2 f : B,$$

which is (2.5). The Stein identification is [6, Theorem 2.3]; the Hessian identity is the moment-coordinate specialization of [8, Proposition 4.8]. The intertwining and pairing formulas are attributed to [9, Section 4]. $\square$

B.2 The tensor calculation

The following calculation is attributed to [9, Lemma 6.1].

Lemma B.1 (Compensated energy). *For the fields of Lemma 2.1,*

$$\mathbb{E}u^2 \geq \mathbb{E}w^T H w + \mathbb{E} \operatorname{tr}(M B H B). \quad (\text{B.1})$$

More precisely, put $h = D_x^2 f$, $C = D_z^3 \phi$, and $R = D_z w - B$. After integration by parts, the defect in (B.1) is the expectation of

$$\|M^{1/2} R M^{1/2}\|_{\text{HS}}^2 + 2\|S + P J\|^2 + \frac{1}{2}\|J - P J\|^2. \quad (\text{B.2})$$

At a point x_0 , use the constant dual change $z = Tz'$, $x' = T^T x$, with $T = \tau(x_0)^{-1/2}$, so that $\tau' = T^T \tau T = I$ at the point. Transform the functions before computing

$$S_{ijk} = f_{x_i x_j x_k}, \quad J_{ijk} = h_{ia} C_{ajk}, \quad (PJ)_{ijk} = \frac{J_{ijk} + J_{jik} + J_{kij}}{3}.$$

The last two norms are in these normalized coordinates; their sum defines an invariant scalar.

Proof. With $(D_x w)_{ij} = \partial_{x_j} w_i$, weighted integration by parts gives

$$\mathbb{E} u^2 = \mathbb{E} w^T H w + \mathbb{E} \operatorname{tr}((D_x w)^2). \quad (\text{B.3})$$

The moment equation is $Q(\nabla \phi) = \phi + \log \det \tau - \log Z$. Write $\partial_i = \partial_{z_i}$, $\partial_i^* = x_i - \partial_i$, $(C_i)_{jk} = C_{ijk}$, $P_0 = \tau h$, and $T = h \tau h$. Differentiating the equation twice yields

$$(\tau H \tau - \tau)_{ij} = (\partial_d - x_d)(M_{dk} C_{ijk}) - \operatorname{tr}(M C_i M C_j). \quad (\text{B.4})$$

Indeed $\tau_{ik} Q_{x_k} = x_i + \operatorname{tr}(M C_i)$ and $\partial_d M_{dk} - x_d M_{dk} = -Q_{x_k}$. The commutator $[\partial_l, \partial_k^*] = \tau_{kl}$ gives $R_{al} = \partial_k^* \partial_l (P_0)_{ak}$; R is symmetric by (2.3). Expanding $D_x w = (B + R)M$ and integrating the cross term, then using (B.4), gives

$$\begin{aligned} \mathbb{E} \operatorname{tr}((D_x w)^2) &= \mathbb{E} \operatorname{tr}(M B M B) + 2 \mathbb{E}(\partial_k h_{al})(\partial_l (P_0)_{ak}) + \mathbb{E} \operatorname{tr}(M R M R), \\ \mathbb{E} \operatorname{tr}(M B H B) &= \mathbb{E} \operatorname{tr}(M B M B) - \mathbb{E}(\partial_d T_{ij}) M_{dk} C_{ijk} - \mathbb{E} T_{ij} \operatorname{tr}(M C_i M C_j). \end{aligned}$$

At $\tau = I$, subtraction leaves, apart from the R term,

$$2\|S\|^2 + 4\langle S, J \rangle + \|J\|^2 + \langle J, J^{(12)} \rangle, \quad J_{ijk}^{(12)} = J_{jik}.$$

Since S is fully symmetric and J is symmetric in its last two indices,

$$\langle S, J \rangle = \langle S, P J \rangle, \quad 3\|P J\|^2 = \|J\|^2 + 2\langle J, J^{(12)} \rangle.$$

Completing the square proves (B.2); (B.3) proves (B.1). Normalization is performed after integration and uses a constant coordinate change at the point. No moving frame is differentiated. $\square$

B.3 Density modulo affine functions

We use the cutoff proof attributed to [9, Lemma 5.1], before its final parity restriction. The eigenvalue-one structure is already present in [7, Section 6].

Lemma B.2 (Range modulo affine functions). *The functions $A(A-1)f$, $f \in C_c^\infty(\mathbb{R}^n)$, are dense in the orthogonal complement of the affine functions in $L^2(\nu)$.*

Proof. Affine functions annihilate the range by (2.4). Conversely, let $g \in L^2$ be orthogonal to it. Elliptic regularity for $A(A-1)g = 0$ makes g smooth. Let η_r equal one on B_r , vanish outside B_{2r} , and have first two derivatives $O(r^{-1})$, $O(r^{-2})$. Put

$$\Gamma(a, b) = \nabla a \cdot \tau \nabla b, \quad X_r = \|\eta_r^2 A g\|_2, \quad Y_r^2 = \mathbb{E} \eta_r^2 \Gamma(g, g), \quad G_0 = \|g\|_2.$$

Boundedness of τ gives $\Gamma(\eta_r, \eta_r) \leq C/r^2$ and $|A \eta_r| \leq C$. The product rule yields, with $\eta = \eta_r$,

$$\begin{aligned} (A-1)(\eta^4 g) &= \eta^4 A g + g \{4\eta^3 A \eta - 12\eta^2 \Gamma(\eta, \eta) - \eta^4\} \\ &\quad - 8\eta^3 \Gamma(\eta, g). \end{aligned}$$

Testing $A(A-1)g = 0$ against $\eta^4 g$ gives the first bound below. The identity $\mathbb{E}\eta^2 g Ag = \mathbb{E}\eta^2 \Gamma(g, g) + 2\mathbb{E}\eta g \Gamma(\eta, g)$ gives the second:

$$X_r \leq CG_0 + (C/r)Y_r, \quad Y_r^2 \leq G_0 X_r + (C/r)G_0 Y_r.$$

Both follow from $A(ab) = aAb + bAa - 2\Gamma(a, b)$ and imply uniform bounds on X_r, Y_r . Hence $v = Ag \in L^2$, $Av = v$, and $A(g - v) = 0$. Testing the last equation with $\eta_r^2(g - v)$ shows that $g - v$ is constant; testing $Av = v$ shows $\mathbb{E}\Gamma(v, v) < \infty$.

For $f_r = \eta_r v$, the product rule then gives $f_r \rightarrow v$ and $Af_r \rightarrow v$ in L^2 : $vA\eta_r$ vanishes by the annular L^2 tail, and $\|\Gamma(\eta_r, v)\|_2 \leq Cr^{-1}(\mathbb{E}\Gamma(v, v))^{1/2}$. Apply (2.5) to f_r . Its right side tends to zero, so $D^2v = 0$ on every fixed ball. Therefore g is affine. $\square$

C Convex approximation

Subtracting a constant from a potential changes the logarithmic partition function only by a constant. Normalize each proper lower semicontinuous even convex potential P by $P(0) = 0$. Its Moreau envelope and smooth approximation are

$$P_j(x) = \inf_y \{P(y) + \frac{j}{2}|x - y|^2\}, \quad \tilde{P}_j = P_j * \rho_{j^{-2}},$$

where ρ_r is an even nonnegative smooth probability kernel supported in B_r . The function P_j is nonnegative, convex and $C^{1,1}$, its gradient has Lipschitz constant j , and $P_j \uparrow P$. Jensen's inequality and the quadratic upper Taylor bound, using the kernel's zero first moment, give

$$0 \leq \tilde{P}_j - P_j \leq \frac{1}{2}j^{-3}.$$

Hence $\tilde{P}_j \rightarrow P$ pointwise in the extended sense. These smooth potentials are even and nonnegative, have Hessians between 0 and jI , and have gradients of at most linear growth.

For the nonsmooth joint dilation theorem, first add $\delta|x|^2/2$ to the background, with $\delta > 0$ fixed. Approximate every potential as above and retain the quadratic term. The smooth theorem applies. The integrands are bounded by $e^{-\delta|x|^2/2}$, so dominated convergence passes the three-point log-concavity inequality to the original potentials with this added term. Then let $\delta \downarrow 0$ and use monotone convergence. The final integrals are finite and positive by (3.4).

This proves the extension used in Theorems B, A, and C. The approximants need not remain homogeneous. For $P = I_K$, the envelope is exactly $P_j = (j/2) \text{dist}(\cdot, K)^2$.

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Acknowledgement and scope. Prepared by ChatGPT for Galyna Livshyts, and revised at her request to present the main results in the introduction with emphasis on the intersection inequality. The analytic input, identities and density argument are attributed above. The additional statements are proved as consequences of those inputs; no claim of historical novelty is made.