# Log-concavity in 1-d Coulomb gas ensembles

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# Introduction

# Log-concave measures, functions and sequences

- A measure $\mu$ on $\mathbb{R}^n$ is log-concave if for any measurable sets $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$$

- A function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if $f = e^{-\phi(x)}$, where $\phi(x)$ is a convex function.
- Log-concave $\mu$ on $\mathbb{R}^d$ has density w.r.t. Lebesgue measure iff it is not supported on affine hyperplane [C. Borell, 1975]
- In the discrete setting, a sequence $\{a_n\}_{n\in\mathbb{Z}}$ is log-concave if

$$\boldsymbol{a_k^2 \geq a_{k-1}a_{k+1}}.$$

- In $\mathbb{Z}^d$, for $d \geq 2$, there are multiple definitions of convexity which are not equivalent.
- A random variable or its probability distribution is log-concave if it has log-concave density function (on $\mathbb{R}^n$) or log-concave mass function (on $\mathbb{Z}$).

## An open problem in combinatorics

- $S_n$ = symmetric group of all permutations of $[n]$.
- $\ell_n(\sigma)$ = length of the longest increasing subsequence of the permutation $\sigma \in S_n$.
- For example, if $\sigma = 42135$, then $\ell_5(\sigma) = 3$ as $2, 3, 5$ is an increasing subsequence of length 3.
- Define

$$L_{n,k} = \{\sigma \in S_n : \ell_n(\sigma) = k\} \text{ and } \ell_{n,k} = |L_{n,k}|.$$

**Chen's conjecture, [2008]**

For any fixed $n$, the sequence $\ell_{n,1}, \ell_{n,2}, ..., \ell_{n,n}$ is log-concave.

Eg: For $n = 5$, we have $1, 41, 61, 16, 1$.

# Connection to probability

- The asymptotics of $\ell_n(\sigma)$ for a uniformly chosen random permutation is very well understood.
- The works of Logan, Shepp [1977] and Vershik, Kerov [1977] show that
  $\frac{\mathbb{E}[\ell_n(\sigma)]}{\sqrt{n}} \to 2$ as $n \to \infty$.
- Baik, Deift, Johansson [1998] proved that $\frac{\ell_n(\sigma)-2\sqrt{n}}{n^{1/6}}$ converges in distribution to a non-degenerate distribution $TW_2$.

$$F_2(x) = \exp\left(-\int_x^\infty (t-x)u^2(t)dt\right),$$

where $u$ satisfies Painlevé-II equation $u''(x) = xu(x) + 2u^3(x)$ with $u(x) \sim Ai(x)$ as $x \to \infty$. $\boldsymbol{TW_2}$ **is the distribution with c.d.f** $\boldsymbol{F_2(x)}$.

# Is $TW_2$ log-concave?

- Chen's conjecture is log-concavity of p.m.f of $\ell_n(\sigma)$.
- Chen's conjecture is related to log-concavity of $TW_2$.

$$\frac{\ell_n(\sigma) - 2\sqrt{n}}{n^{1/6}} \to TW_2$$

- Bóna, Lackner, Sagan [2017] show that $TW_2$ is log-concave on positive reals (proof attributed to P.Deift).

**Questions:** Is $TW_2$ log-concave? Is Chen's conjecture true?

- Yes, $TW_2$ distribution is log-concave.
- We prove Poissonized version of Chen's conjecture. The main conjecture is still open.

# Overview

1. Continuous setting
    - Log-concavity of ordered elements in 1-d Coulomb gas ensembles
    - Log-concavity of $TW_\beta$, $\text{Airy}_2$ process.
2. Discrete setting
    - Log-concavity of 1-d marginals in discrete ensembles.
    - Log-concavity of last passage times
3. Poissonized version of Chen's conjecture

# Continuous setting

# Coulomb gas ensembles

1-d Coulomb gas ensembles are probability measures on $\mathbb{R}^n$ with the density function $f_{n,\beta}$. Let $x = (x_1, x_2, \ldots, x_n)$

$$f_{n,\beta}(x) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |x_j - x_k|^\beta \quad e^{-\sum\limits_{k=1}^{n} V(x_k)}$$

where $\beta > 0$ is a parameter (temperature) and $V$ is a function (potential) satisfying growth conditions.

For $V$ quadratic and $\beta = 1, 2, 4$ the Coulomb gas ensemble is joint law of eigenvalues in Gaussian orthogonal, unitary, symplectic ensemble respectively.

# Log-concavity of ordered elements

Let $x_{(n)}$ denote the maximum of $x_1, \ldots, x_n$.

**Theorem 1.** Let $(X_1, X_2, \ldots, X_n) \sim f_{n,\beta}$, where

$$f_{n,\beta}(x_1, \ldots, x_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |x_j - x_k|^\beta \; e^{-\sum_{k=1}^{n} V(x_k)}.$$

If $V$ is convex, then $X_{(k)}$ is log-concave, for all $k \in [n]$.

**Proof:** $\mathcal{W}_n = \{x \in \mathbb{R}^n : x_1 < x_2 < \cdots < x_n\}$.

$$\overrightarrow{f}_{n,\beta}(x_1, \ldots, x_n) = \frac{\mathbb{1}_{x \in \mathcal{W}_n} n!}{Z_{n,\beta}} \prod_{j<k} |x_j - x_k|^\beta \; e^{-\sum_{k=1}^{n} V(x_k)}$$

$\overrightarrow{f}_{n,\beta}$ is log-concave and $X_{(k)}$ is the $k$-th marginal of $\overrightarrow{f}_{n,\beta}$.

By Prékopa-Leindler, $X_{(k)}$ is log-concave.

# A family of distributions

Let $(X_1, \dots X_n) \sim f_{n,\beta}$ where,

$$f_{n,\beta}(x_1, \dots, x_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |x_j - x_k|^\beta \exp\left(-\frac{\beta}{4} \sum_i x_i^2\right).$$

Then $\boldsymbol{n^{1/6}(X_{(n)} - 2\sqrt{n}) \to TW_\beta}$ (Ramírez,Rider,Virág 2006).

- As $X_{(n)}$ is log-concave, $n^{1/6}(X_{(n)} - 2\sqrt{n})$ is log-concave.
- By preservation of log-concavity under affine transformations and weak limits, $TW_\beta$ is log-concave.

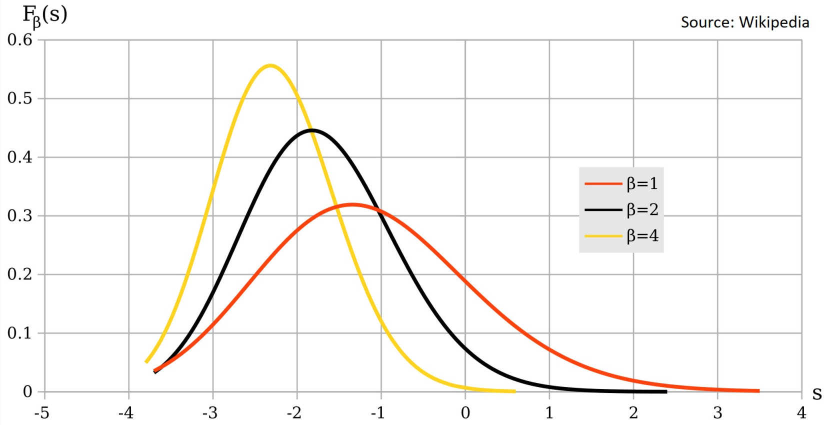
# $TW_\beta$ distributions

- $TW_\beta$ are ubiquitous in random matrix theory, large scale statistics in KPZ equation, current fluctuations in ASEP.
- Despite this very few properties of $TW_\beta$ are known
- $\mathbb{P}(TW_\beta > t) \sim \exp(-\frac{2\beta}{3} t^{3/2})$ as $t \to \infty$.
- $\mathbb{P}(TW_\beta < -t) \sim \exp(-\frac{\beta}{24} t^3)$ as $t \to \infty$ (RRV, 2006).
- For $\beta_2 > \beta_1$ we have $\beta_1^s TW_{\beta_1} \geq \beta_2^s TW_{\beta_2}$ if and only if $s \in [1/3, 2/3]$ (Pedreira, 2022).

# TW laws are log-concave

**Corollary 1.**

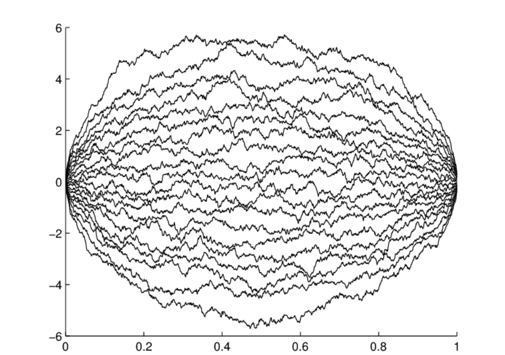
- For every $\beta > 0$, $TW_\beta$ is a log-concave measure.
- Density functions of $TW_\beta$ exist and are log-concave.



# Airy$_2$ process

- Prähofer, Spohn (2002) introduced Airy$_2$ process $\mathcal{A}_2$ in the study of scaling limit of a random growth model.
- $(B_1(t), \ldots, B_N(t))$ be non intersecting Brownian bridges.

$$2N^{1/6}\left(B_N\left(\frac{1}{2}(1+N^{-1/3}t)\right)-\sqrt{N}\right)\to \mathcal{A}_2(t)-t^2$$




credits: Delvaux and Kuijlaars

# Log-concavity of $\text{Airy}_2$ process

**Theorem 2.** For any $k \geq 1$ and $t_1 < t_2 \cdots < t_k$, the joint distribution $(\mathcal{A}_2(t_1), \mathcal{A}_2(t_2), \ldots, \mathcal{A}_2(t_k))$ is log-concave.

**Proof:**

- $W_1(t), \ldots, W_N(t)$ be independent Brownian bridges.
- Choose $\{(t_{m1}, \ldots, t_{mm})\}_{m \geq 1} \to (0, 1)$ and $t_1, \ldots t_k$ are contained in the mesh for all large enough $m$.
- The joint distribution

$$(W_1(t_{m1}), \ldots, W_N(t_{m1}), \ldots, W_1(t_{mm}), \ldots, W_N(t_{mm}))$$

  is log-concave as it is a Gaussian vector.

# Conditioning trick and Prékopa-Leindler

- Condition on the event $E_m = \{W_1(t_{mi}) < W_2(t_{mi}) \cdots < W_N(t_{mi}), \forall i \in [m]\}$.
- Restricting the Gaussian density to the convex set $\{x \in \mathbb{R}^{mN} : x_{iN+1} < \cdots < x_{iN+N}, \forall i \in \{0, 1, ..., m-1\}\}$.
- By Prékopa-Leindler inequality, conditional on $E_m$, $(W_N(t_1), W_N(t_2), \ldots, W_N(t_k))$ is log-concave.
- Limit of $(W_1(t), \ldots, W_N(t))$ conditioned on $E_m$ converges to $(B_1(t), \ldots, B_N(t))$ as $m \to \infty$.

# Discrete setting

# DOPE

Coulomb gas ensemble on $\mathbb{Z}$

$$\mathbb{P}_{N,w}(h) = \frac{1}{Z_{N,w}} \prod_{1\le i<j\le N} (h_j - h_i)^\beta \prod_{j=1}^{N} w(h_j), h \in \overrightarrow{\mathbb{Z}}^N$$

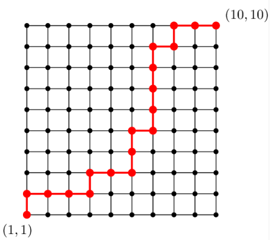
where $\overrightarrow{\mathbb{Z}}^N = \{h \in \mathbb{Z}^N : h_1 < h_2 \cdots < h_N\}$. Referred to as discrete orthogonal polynomial ensembles (DOPE) for $\beta = 2$.

- Analogous to continuous ensemble

$$f_{n,\beta}(x) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |x_j - x_k|^\beta \quad e^{-\sum_{k=1}^{n} V(x_k)}$$

- For $\beta = 2$ and $q \in (0,1)$ and $w(k) = \mathbb{1}_{\{k\ge 0\}} q^k$, Meixner ensemble.

# Last passage times




- Vertex weights $\{\zeta_v\}_{v\in\mathbb{Z}^2}$ are i.i.d Geo$(1-q)$ random variables.
- For each up/right path $\gamma$ from $(1,1)$ to $(n,n)$, compute $\ell(\gamma)=\sum_{v\in\gamma}\zeta_v$. Define last passage time $T_n=\max_{\gamma}\ \ell(\gamma)$.
- $T_N+N-1\overset{d}{=}h_N$ of Meixner ensemble ($\beta=2$ and $w(k)=q^k$). Due to Johannson (2000).

# Log-concavity of DOPE

**Theorem 3.**

If $w$ is a log-concave sequence on $\mathbb{Z}$, then for all $i \in [N]$,

$$\mathbb{P}_{N,w}(h_i = k-1)\mathbb{P}_{N,w}(h_i = k+1) \leq \mathbb{P}_{N,w}(h_i = k)^2.$$

**Corollaries:**

- Each one-dimensional marginal of Meixner ensemble is log-concave.
- Last passage times with Geometric weights are log-concave.

# Discrete version of Brunn-Minkowski

**Theorem 4. (Halikias, Klartag, Slomka '21)**

Suppose that for any $\lambda \in [0,1]$, the functions $f, g, h, k : \mathbb{Z}^n \to [0,\infty)$ satisfy the below inequality $\forall x, y \in \mathbb{Z}^n$

$$f(x)g(y) \leq h\left(\lfloor \lambda x + (1-\lambda)y \rfloor\right) k\left(\lceil (1-\lambda)x + \lambda y \rceil\right) \quad \text{then,}$$

$$\left(\sum_{x\in\mathbb{Z}^n} f(x)\right)\left(\sum_{x\in\mathbb{Z}^n} g(x)\right) \leq \left(\sum_{x\in\mathbb{Z}^n} h(x)\right)\left(\sum_{x\in\mathbb{Z}^n} k(x)\right).$$

$$\mathbb{P}_{N,w}(h_N = k) = \sum_{h_1<h_2<\cdots<h_N=k} \prod_{1\leq i<j\leq N} (h_j - h_i) \prod_{j=1}^{N} w(h_j)$$

# Proof of Theorem 4

$$h(x) = k(x) := \prod_{1 \le i < j \le N} (x_j - x_i) \prod_{j=1}^{N} w(x_j)\ \mathbb{1}_{x \in S_k}$$

$$f(x) := \prod_{1 \le i < j \le N} (x_j - x_i) \prod_{j=1}^{N} w(x_j)\ \mathbb{1}_{x \in S_{k-1}}$$

$$g(x) := \prod_{1 \le i < j \le N} (x_j - x_i) \prod_{j=1}^{N} w(x_j)\ \mathbb{1}_{x \in S_{k+1}}$$

$S_k$ is the set $S_k := \{x \in \mathbb{Z}^N : x_1 < x_2 < \cdots < x_N = k\}$.

$$f(x)g(y) \le h\left(\left\lfloor \frac{1}{2}(x+y) \right\rfloor\right) k\left(\left\lceil \frac{1}{2}(x+y) \right\rceil\right) \quad \forall x, y \in \mathbb{Z}^n.$$

# Poissonized version of Chen's conjecture

# Plancherel measure

Fix $N$. For any partition $\lambda \vdash N = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq 1)$, define

$$\mu_N(\lambda) := \frac{d_\lambda^2}{N!}$$

Let $\lambda = (4, 2, 1)$ then $d_\lambda = 7!/(6.4.2.3)$.

| | | | |
|---|---|---|---|
| 6 | 4 | 2 | 1 |
| 3 | 1 | | |
| 1 | | | |

By RSK correspondence there exists a bijection from permutations of $n$ objects to pairs of standard Young tableaux of same shape.

S=

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 6 |
| 3 | 7 | | |
| 5 | | | |

T=

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 5 |
| 4 | 6 | | |
| 7 | | | |

Eg: $(1, 3, 5, 4, 7, 6, 2)$ is mapped to $(S, T)$.

# Permutations to discrete ensembles

By RSK correspondence

- uniformly random permutation corresponds to random partition under Plancherel measure.
- LIS is the length of the first row

Chen's conjecture is equivalent to,

$$\mu_N(\lambda_1 = k-1)\mu_N(\lambda_1 = k+1) \leq \mu_N(\lambda_1 = k)^2$$

Natural bijection from $h = (0 \leq h_1 < h_2 \cdots < h_n)$ to $\lambda$ with $\ell(\lambda) \leq n$. If $n = 7$ and $h = (0, 1, 2, 3, 5, 7, 11)$ it is mapped to $\lambda = (5, 2, 1)$.

Any measure $\mathbb{P}_{N,w}$ on $\overrightarrow{\mathbb{Z}}^N$ induces a measure on partitions with $\ell(\lambda) \leq N$.

# Poissonized Plancherel measure

If $\Lambda$ is the set of all partitions of all non-negative integers, $\mathbb{P}_{n,w}$ with $w(k) = q^k$ induces a certain measure on $\Lambda$, say $\gamma_{n,q}$.

We have that $\gamma_{n,q}(\lambda_1)$ is log-concave.

**Theorem 5. (Kurt Johansson, 2001)** For $q = \alpha/n^2$, as $n \to \infty$

$$\gamma_{n,\alpha/n^2} \to \gamma_\alpha := \text{ Poissonized Plancherel measure with parameter } \alpha.$$

**Poissonized Plancherel measure with parameter $\alpha$:**

Let $N \sim Poi(\alpha)$ and then choose $\lambda \vdash N$ under $\mu_N$.

**Theorem 6.** Poissonized version of Chen's conjecture is true.

$$\gamma_\alpha(\lambda_1 = k-1)\gamma_\alpha(\lambda_1 = k+1) \leq \gamma_\alpha(\lambda_1 = k)^2$$

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Thank
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