Singularity of discrete random matrices

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Joint work with Ashwin Sah (MIT) and Mehtaab Sawhney (MIT)

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- Tao-Vu (2006, 2007): $q_n \leq (0.75 + o_n(1))^n$.
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- Notably, $\xi = \text{Ber}(p), p \in (1/2, 1)$ was open.

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$q_n(\xi) = (1 + o_n(1))\,(\mathbb{P}[\text{zero row/column}] + \mathbb{P}[\text{two equal/opposite rows/columns}])$.

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Theorem (J.-Sah-Sawhney, 2020)

For ξ not uniform on its support,

$q_n(\xi) = \mathbb{P}[\text{zero row/column}] + (1 + O(e^{-c_\xi n}))\mathbb{P}[\text{two equal/opposite rows/columns}]$.

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$$q_n(\xi) = 2n(1-p)^n + (n^2 - n)(p^2 + (1-p)^2)^n + \dots$$

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$$q_n(\xi) = \left(|\mathrm{supp}(\xi)|^{-1} + o_n(1)\right)^n .$$

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Theorem (J.-Sah-Sawhney, 2020)

Let ξ be a discrete random variable. There exist $\delta, \rho, \eta > 0$ depending on ξ such that for all $t \leq 1$,

$$\mathbb{P}\left[\inf_{x \in \mathrm{Cons}(\delta,\rho)} \|M_n(\xi)x\|_2 \leq t\right] \leq \mathbb{P}[\text{zero column}] + \mathbb{P}[\text{equal or opposite columns}] + te^{-\eta n} + \dots$$

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Overview of the proof

For simplicity, we will focus on the special case:

Theorem (J.-Sah-Sawhney, 2020)

$$\begin{aligned}\mathbb{P}[M_n(\mathrm{Ber}(p)) \text{ is singular}] &= (1+o_n(1))\mathbb{P}[\text{zero row or column}] \\ &= (2+o_n(1))n(1-p)^n, \quad p \in (0, 1/2).\end{aligned}$$

The anti-concentration phenomenon

Definition (Small ball probability)

$v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is fixed, $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ is a random vector, $r \geq 0$.

$$\mathcal{L}(v_1\xi_1 + \cdots + v_n\xi_n, r) := \sup_{z \in \mathbb{R}} \mathbb{P}\left(|v_1\xi_1 + \cdots + v_n\xi_n - z| \leq r\right).$$

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Examples:

- $v = (1, \ldots, 1)$, $\vec{\xi}$ i.i.d Ber(p), $\mathcal{L}(v_1\xi_1 + \cdots + v_n\xi_n, 1/4) = \Theta_p\left(\frac{1}{\sqrt{n}}\right)$.

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- $v = (10, 100, \ldots, 10^n)$, $\vec{\xi}$ i.i.d Ber(p), $p \in (0, 1/2)$,
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- The bound on $\mathbb{P}[M_n \text{ is singular}]$ is the special case $t = 0$.

Divide and conquer

We analyze $\inf_{x \in \mathbb{S}^{n-1}} \|M_n x\|_2, \inf_{y \in \mathbb{S}^{n-1}} \|y M_n\|_2$ by decomposing $\mathbb{S}^{n-1}$ into a few pieces.

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- For coarse invertibility, do not need to isolate $\mathrm{Elem}(\rho)$.
- For strong invertibility, leading term comes from $\mathrm{Elem}(\rho)$, so this piece requires a precise non-trivial analysis.

Non-elementary almost-constant vectors

Lemma (J.-Sah-Sawhney, 2020)

For all $x \in \mathbb{S}^{n-1} \setminus \mathrm{Elem}(\rho)$,

$$\mathcal{L}(x_1\xi_1 + \cdots + x_n\xi_n, \theta_{p,\rho}) \leq 1 - p - \theta_{p,\rho}.$$

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Idea: Wlog, $|x_1| \geq |x_2| \geq \cdots \geq |x_n|$.

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Lemma (J.-Sah-Sawhney, 2020)

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Proposition (J.-Sah-Sawhney, 2020)

For all $c > 0$,

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Non almost-constant vectors

- So far, we have been able to correctly control

$$\mathbb{P}\left[\inf_{x \in \mathrm{Cons}(\delta,\rho)} \|M_n x\|_2 \le e^{-cn} \text{ OR } \inf_{y \in \mathrm{Cons}(\delta,\rho)} \|y M_n\|_2 \le e^{-cn}\right].$$

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- Thus, it remains to control the infima over $\mathrm{NCons}(\delta,\rho)$ on the event that

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Invertibility via distance - I

- Let $y \in \mathbb{S}^{n-1}$.
- Let the rows of M_n be $R_1, \ldots, R_n$.
- Let the span of rows $R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n$ be H_i.

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- Then,

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- Hence,

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- Rudelson-Vershynin (2008):

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- Why? Recall $\|yM_n\|_2 \ge \max_{i\in[n]}|y_i|\mathrm{dist}(R_i, H_i)$, and

$$y \in \mathrm{NCons}(\delta,\rho) \implies \exists \delta n \text{ indices } i \in [n] \text{ s.t. } |y_i| \ge \rho/\sqrt{n}.$$

Invertibility via distance - III

- Litvak-Tikhomirov (2020): For any $\gamma > 0$, for all n sufficiently large,

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Structure theorem

Upshot: Remains to bound (for $\eta < e^{-2cn}$)

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Let $\varepsilon, \delta, \rho, \gamma > 0$. With probability at least $1 - 4^{-n}$, for any $x(H_1) \in \mathbb{S}^{n-1} \cap H_1^{\perp}$, either:

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Definition (Threshold function)

Fix $p \in (0, 1/2), \gamma \in (0, p)$. For $x \in \mathbb{S}^{n-1}$ and $L \ge 1$, define

$$\mathcal{T}_{p,\gamma}(x, L) = \sup\{t \in (0,1) : \mathcal{L}(x_1\xi_1 + \cdots + x_n\xi_n, t) > Lt, \quad \vec{\xi} \sim \{0,1\}^n_{(p\pm\gamma)n}\}.$$

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- Let H denote the $(n-1) \times n$ matrix formed by the last $n-1$ rows of M_n.

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- Drawback: None of these methods give suitable entropy estimates for the entire range of t.

Overview of the proof of the structure theorem

Theorem (Tikhomirov, 2018)

Fix $p \in (0, 1/2), \varepsilon \in (0, p), \delta \in (0, 1), \rho \in (0, 1)$. There exists $L = L(p, \varepsilon, \delta, \rho)$ such that for any $(1 - p + \varepsilon)^n \leq t \leq 1/\sqrt{n}$,

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Fix $p \in (0, 1/2), \gamma \in (0, p), \delta \in (0, 1), \rho \in (0, 1), \varepsilon \in (0, 1)$. There exists $L = L(p, \gamma, \delta, \rho, \varepsilon)$ such that for any $\binom{n}{pn}^{-1} \exp(\varepsilon n) \le t \le 1/\sqrt{n}$,

$$\mathcal{N}(S_{t,\delta,\rho,p,\gamma}, t\sqrt{n}) \le e^{-\omega(n)} t^{-n},$$

where $S_{t,\delta,\rho,p,\gamma} = \mathrm{NCons}(\delta, \rho) \cap \{\mathcal{T}_{p,\gamma}(x, L) \in [t, 2t]\}$.

Overview of the proof of the structure theorem

Theorem (J.-Sah-Sawhney, 2020)

Fix $p \in (0, 1/2), \varepsilon \in (0, p)$. There exists $L = L(p, \varepsilon) > 0$ such that for any

$$1 \leq N \leq \binom{n}{pn} \exp(-\varepsilon n),$$

$$\#\{x \in \mathbb{Z}^n \cap [N, 2N]^{n/2} \times [-2N, -N]^{n/2} : \mathcal{L}(x_1\xi_1 + \cdots + x_n\xi_n, \sqrt{n}) \geq LN^{-1}\} \leq e^{-\omega(n)} N^n,$$

where

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A similar statement for $\vec{\xi} \sim_p \{0, 1\}^n$ and with $1 \le N \le (1 - p + \varepsilon)^n$ was proved by Tikhomirov (2018).

General distributions

- For general discrete distributions, the key point is to extend our analysis from the slice to a "multislice":

$$\{x \in \{a_1, \dots, a_k\}^n : m_j \text{ indices equal } a_j\},$$

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- Moreover, the class of elementary vectors needs to be enlarged to also cover vectors which are near $e_i \pm e_j$, and the bounds need to depend on $\mathbb{P}[\xi = 0], \mathbb{P}[\xi = \xi']$ (and not on $\sup_{r \in \mathbb{R}} \mathbb{P}[\xi = r]$), which leads to significant additional challenges.

References

- *Singularity of discrete random matrices I*, J.-Sah-Sawhney. arXiv:2010.06553.
- *Singularity of discrete random matrices II*, J.-Sah-Sawhney, arXiv:2010.06554.

Thank you for your attention!