# Expected face numbers of random beta polytopes

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# Angle sums in a triangle

- In a triangle, the sum of angles is constant.
- In a tetrahedron (more generally, in a simplex), it is not constant.
- What are the maximal and minimal values? Answered by Höhn (1953) and Perles and Shephard (1967).
- What are the average values? (Will be discussed below).

# Angles at vertices

## Definition of angles at vertices

- Consider a simplex $S = [X_0, X_1, \ldots, X_d]$ in $\mathbb{R}^d$.
- Ball: $B(X_0, \varepsilon) = \{y \in \mathbb{R}^d : \|y - X_0\| \le \varepsilon\}$.
- Internal angle at vertex $X_0$:

$$\beta(X_0, S) = \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(B(X_0, \varepsilon) \cap S)}{\mathrm{Vol}_d(B(X_0, \varepsilon))}.$$

- Normalization: Angle of the full space is 1, angle of the half-space is $1/2$.
- Angle sum at vertices:

$$\sigma_0(S) = \sum_{i=0}^{d} \beta(X_i, S).$$

# Maximal angle sum at vertices

**Theorem (Höhn, Perles-Shephard)**

For every $d$-dimensional simplex we have

$$0 < \sigma_0(S) \leq \frac{1}{2} \quad \text{(strict for } d \geq 3\text{)}.$$

**Idea of proof**

- Assume $X_0 = 0$ is the origin.
- Take a random direction $U$ (uniformly distributed on $\mathbb{S}^{d-1}$).
- Project the simplex onto the hyperplane $U^\perp$.
- Probability that $X_0 = 0$ is **inside** the projection is $2\beta(X_0, S)$.
- $2\sigma_0(S)$ is the probability that the projection is a $(d-1)$-dimensional simplex.
- $\sigma_0(S) \leq 1/2$.

# Angles at faces

## Definition

- Let $S = [X_0, X_1, \ldots, X_d]$ be a simplex in $\mathbb{R}^d$.
- Take some $k$-dimensional face, for example $F = [X_0, X_1, \ldots, X_k]$.
- Take some point in the relative interior of $F$, for example $m := \frac{1}{k+1}(X_0 + \ldots + X_k)$.
- Tangent cone at $F$:

$$T(F,S) = \{y \in \mathbb{R}^d : m + \varepsilon y \in S, \text{ if } \varepsilon > 0 \text{ is small}\}.$$

- Angle of $S$ at $F$:

$$\beta(F,S) = \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Vol}_d(B(m,\varepsilon) \cap T(F,S))}{\mathrm{Vol}_d B(m,\varepsilon)}.$$

# Angle sums at faces: relations

## Definition

- Angle sum at $k$-dimensional faces:

$$\sigma_k(S) = \sum_{F \in \mathcal{F}_k(S)} \beta(F, S).$$

- $\sigma_0(S)$ is the sum of angles at vertices.
- $\sigma_1(S)$ is the sum of “dihedral” angles at edges.
- $\sigma_{d-1}(S) = (d+1)/2$.

## Theorem (Gram-Euler relation)

$$\sigma_0(S) - \sigma_1(S) + \ldots \pm \sigma_{d-1}(S) = (-1)^{d-1}.$$

## Remarks

There exist more general Poincaré relations for simplicial polytopes (related to Dehn-Sommerville relations).

# Maximal and minimal values

**Theorem (Höhn, Perles-Shephard)**

- Maximal values: For $j = 0, \ldots, d-2$ we have
$$\max_{S \in \text{simplices}} \sigma_j(S) = \frac{1}{2}\binom{d}{j}.$$
- Minimal values: For $j = 0, \ldots, [(d-3)/2]$ we have
$$\min_{S \in \text{simplices}} \sigma_j(S) = 0.$$
- Minimal values: For $j = [(d-1)/2], \ldots, d-2$ we have
$$\min_{S \in \text{simplices}} \sigma_j(S) = \frac{1}{2}\binom{[(d+1)/2]}{d-j} + \frac{1}{2}\binom{[(d+2)/2]}{d-j}.$$

# Average values: angles at vertices

## Theorem [K]

Let $X_0, \dots, X_d$ be i.i.d. and uniform on the sphere $\mathbb{S}^{d-1}$. Then,

$$\mathbb{E}\sigma_0(S) = \begin{cases} \frac{1}{8}, & \text{for } d = 3; \\ \frac{539}{288\pi^2} - \frac{1}{6}, & \text{for } d = 4; \\ \frac{25411}{7340032}, & \text{for } d = 5; \\ \frac{1}{6} + \frac{113537407}{48384000\pi^4} - \frac{2144238917}{1141620480\pi^2}, & \text{for } d = 6; \\ \dots \end{cases}$$

Rational for odd $d$. Polynomial of $\pi^{-2}$ with rational coefficients for even $d$.

## Remark

Similar formulas exist for points uniformly distributed in the ball, for instance $\mathbb{E}\sigma_0(S) = \frac{401}{2560}$ for $d = 3$.

# Beta distributions

## Definition

*d*-dimensional *beta distribution* has Lebesgue density

$$\frac{\Gamma\left(\frac{d}{2}+\beta+1\right)}{\pi^{\frac{d}{2}}\Gamma\left(\beta+1\right)}\left(1-\|x\|^2\right)^{\beta}\mathbb{1}_{\{\|x\|<1\}}. \tag{1}$$

Parameter: $\beta > -1$.

## Examples

- For $\beta = 0$: uniform distribution on the ball.
- For $\beta \downarrow -1$: uniform distribution on the sphere.
- For $\beta \to +\infty$: normal distribution.

## Properties

- Orthogonal projection of beta distribution is again beta.
- Restriction to beta to affine subspace is again beta.

# Beta simplices

## Notation

Consider $n$ points $X_1, \ldots, X_n$ having beta distribution with parameter $\beta \geq -1$ in $\mathbb{R}^{n-1}$.

Expected sum of internal angles of the simplex $[X_1, \ldots, X_n]$ at its $k$-vertex faces:

$$\mathbb{J}_{n,k}(\beta) = \mathbb{E}\sigma_{k-1}([X_1, \ldots, X_n]).$$

## Question

Explicit formula for $\mathbb{J}_{n,k}(\beta)$?

# Beta simplices

## Theorem [K]

Let $n \geq 3$ and $k \in \{1, \ldots, n\}$. For all $\alpha \geq n-3$ we have

$$\mathbb{J}_{n,k}\left(\frac{\alpha - n + 1}{2}\right) = \binom{n}{k} \int_{-\pi/2}^{+\pi/2} c_{\frac{\alpha n}{2}} (\cos x)^{\alpha n + 1} \left(\frac{1}{2} + \sqrt{-1} \int_0^x c_{\frac{\alpha - 1}{2}} (\cos y)^{-\alpha - 1} \mathrm{d}y\right)^{n-k} \mathrm{d}x,$$

where $c_\beta := \Gamma(\beta + \frac{3}{2})/(\sqrt{\pi}\,\Gamma(\beta + 1))$.

## Remark

For integer or half-integer $\beta$: either rational, or a polynomial in $\pi^{-2}$ over $\mathbb{Q}$. The latter sometimes simplifies to a rational multiple of $\pi^{-2m}$.

# Beta prime distribution

## Definition

$d$-dimensional *beta prime distribution* has Lebesgue density

$$\frac{\Gamma(\beta)}{\sqrt{\pi}\,\Gamma\left(\beta - \frac{1}{2}\right)} \left(1 + \|x\|^2\right)^{-\beta}, \qquad x \in \mathbb{R}^d.$$

Parameter: $\beta > d/2$.

## Remarks

- For $\beta = (d+1)/2$: Cauchy distribution.
- Generalization of the Student distribution.

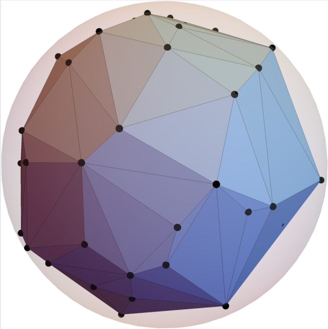
## Result

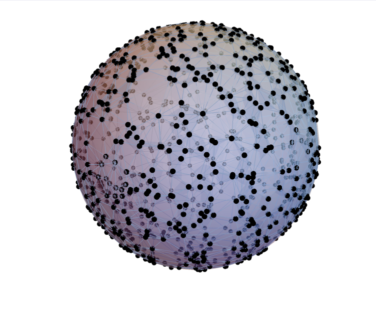
Let $\tilde{\mathbb{J}}_{n,k}(\beta)$ be the expected sum of internal angles of the beta prime simplex. An explicit formula for $\tilde{\mathbb{J}}_{n,k}(\beta)$ exists.

# Beta polytopes

## Definition

- Beta polytope $P_{n,d}^{\beta}$ is a convex hull of $n$ points with beta distribution in $\mathbb{R}^d$.
- Beta prime polytope $\tilde{P}_{n,d}^{\beta}$ is a convex hull of $n$ points with beta prime distribution in $\mathbb{R}^d$.





# Expected face numbers of beta polytopes

**Theorem [K, Thäle, Zaporozhets]**

The expected number of $k$-dimensional faces of a beta polytope $P_{n,d}^{\beta}$ can be expressed through two types of quantities:

- Expected internal angle sums: $\mathbb{J}_{n,k}(\beta)$ and
- Expected external angle sums: $\mathbb{I}_{n,k}(\beta)$:

$$\mathbb{E} f_k(P_{n,d}^{\beta}) = 2 \sum_{s=0}^{\infty} \mathbb{I}_{n,d-2s}(2\beta + d) \mathbb{J}_{d-2s,k+1}\left(\beta + s + \frac{1}{2}\right).$$

**Remark**

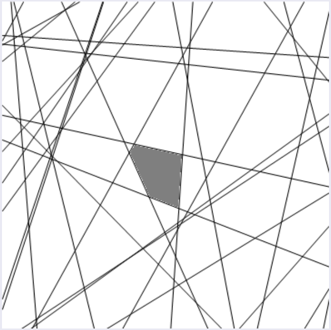
For integer or half-integer $\beta$: Either rational or polynomial in $\pi^{-2}$ with rational coefficients.

Similarly for beta prime polytopes.

# Poisson zero cell

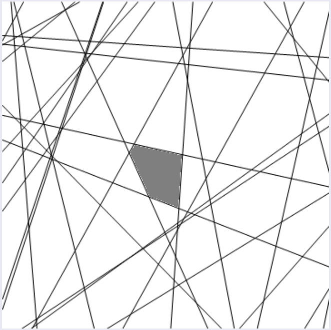
## Poisson hyperplane process

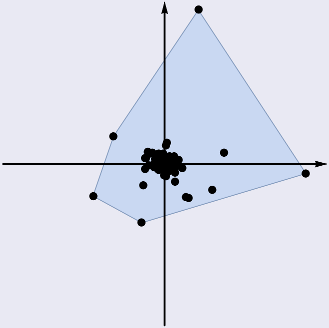
On the space of hyperplanes there is a unique (up to constant) measure invariant under isometries. Consider Poisson point process with this intensity (infinitely many hyperplanes thrown at random in $\mathbb{R}^d$).



# Poisson zero cell and beta prime polytopes

- Consider points dual to these hyperplanes.
- Points form a PPP with intensity $\|x\|^{-d-1}$.
- Limit of $\tilde{P}_{n,d}^{(d+1)/2}$ as $n \to \infty$.





# Poisson zero cell: Expected $f$-vector

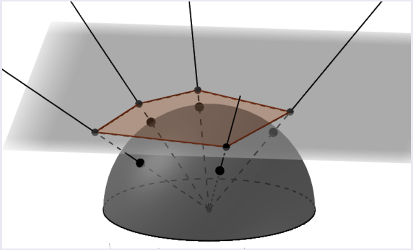
**Theorem [K]**

For all $\ell \in \{1, \dots, d\}$ such that $d - \ell$ is even, we have

$$\mathbb{E} f_\ell(\mathcal{Z}_d) = \frac{\pi^{d-\ell}}{(d-\ell)!}[x^{d-\ell}](1+(d-1)^2x^2)(1+(d-3)^2x^2)\dots = \pi^{d-\ell}\binom{d}{\ell}[x^{d-\ell}]\left(\frac{x}{\sin x}\right)^{d+1}.$$

For odd $d - \ell$: a more complicated formula exists. Even and odd codimensions are related by Dehn-Sommerville relations.

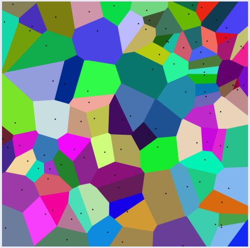
# Angles in half-spaces



- Let $U_1, \dots, U_n$ be uniform on the upper half-sphere in $\mathbb{R}^{d+1}$.
- Their positive hull is a random cone $C_n$.
- Cross-section of $C_n$ is beta prime polytope $\tilde{P}_{n,d}^{(d+1)/2}$.
- Explicit formulas for the expected angle, number of faces, etc. exist.

# Typical Voronoi cell

- Consider a Poisson point process $P_1, P_2, \ldots$ with intensity 1 in $\mathbb{R}^d$.
- Voronoi cell of $P_i$: the set of points whose distance to $P_i$ is smaller than to $P_j$ for all $j \neq i$.
- Typical Voronoi cell: a cell chosen uniformly at random from all cells in a large window.
- Explicit description: cell of 0 in a Poisson process to which we added 0.



Source: Wikipedia

# Typical Voronoi cell

- Let $\mathcal{V}_d$ be the typical Voronoi cell in $\mathbb{R}^d$.
- Meijering (1953):

$$\mathbb{E}f_0(\mathcal{V}_3) = \frac{96\pi^2}{35}, \ \mathbb{E}f_1(\mathcal{V}_3) = \frac{144\pi^2}{35}, \ \mathbb{E}f_2(\mathcal{V}_3) = 2 + \frac{48\pi^2}{35}.$$

- Miles (1970): Formula for $\mathbb{E}f_0(\mathcal{V}_d)$.

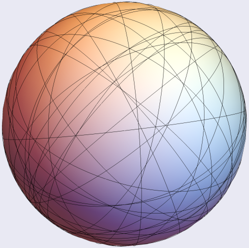
**Theorem [K]**

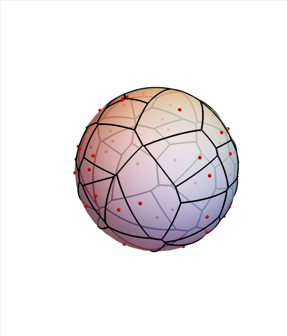
For all $d \in \mathbb{N}$ and $k \in \{1, \ldots, d\}$ such that $dk$ is even, we have

$$\mathbb{E}f_{d-k}(\mathcal{V}_d) = d^d \binom{d}{k} \left( \frac{\sqrt{\pi}\,\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})} \right)^k \operatorname*{Res}_{x=0} \frac{\left(\int_0^x (\sin y)^{d-1} \mathrm{d}y\right)^{d-k}}{(\sin x)^{d^2+1}}.$$

Either rational, or polynomial in $\pi$ over $\mathbb{Q}$, or $q\pi^m$.

# Spherical tessellations





**Theorem [K, Thäle]**

Explicit formulas exist for the expected $f$-vector of the

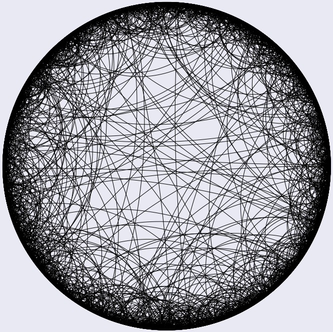
- spherical Poisson zero cell;
- typical cell of the spherical Voronoi tessellation.

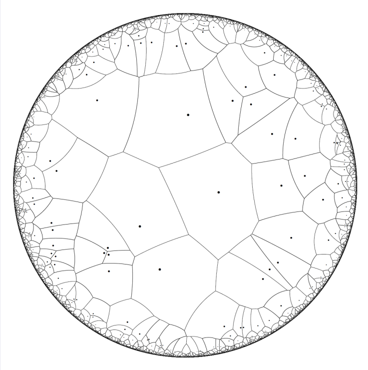
# Hyperbolic tessellations

## Theorem [Godland, K., Thäle]

Explicit formulas exist for the expected $f$-vector of the

- hyperbolic Poisson zero cell;
- typical cell of the hyperbolic Voronoi tessellation.

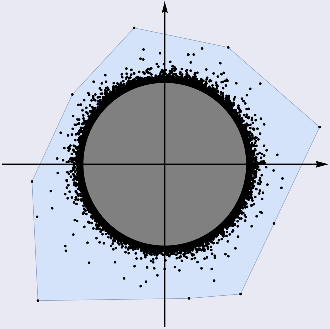




# Beta* polytopes

**Definition**

Beta*-polytope is the convex hull of the Poisson process with intensity $c(\|x\|^2 - 1)^{-\beta}$ on the complement of the unit ball.



**Claim [Godland, K., Thäle]**

Hyperbolic cells reduce (by duality) to beta* polytopes.

# Final slide

Thank you for your attention!