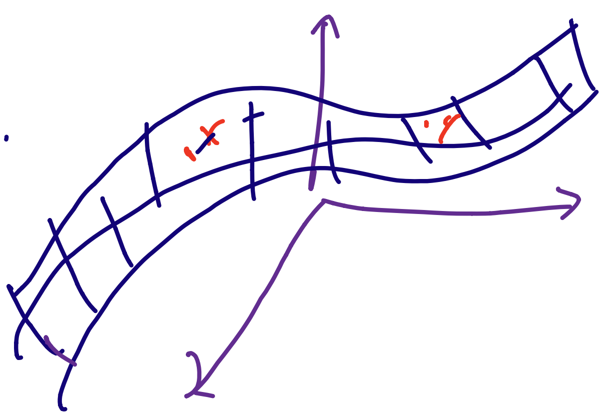
12/5/20

Rigidity of Riemannian embeddings of discrete metric spaces

Joint work with M. Eilat.

- M - complete connected Riemannian manifold
- M as a metric space. $d(x,y)$



- Given a metric space X (finite/countable) we write that

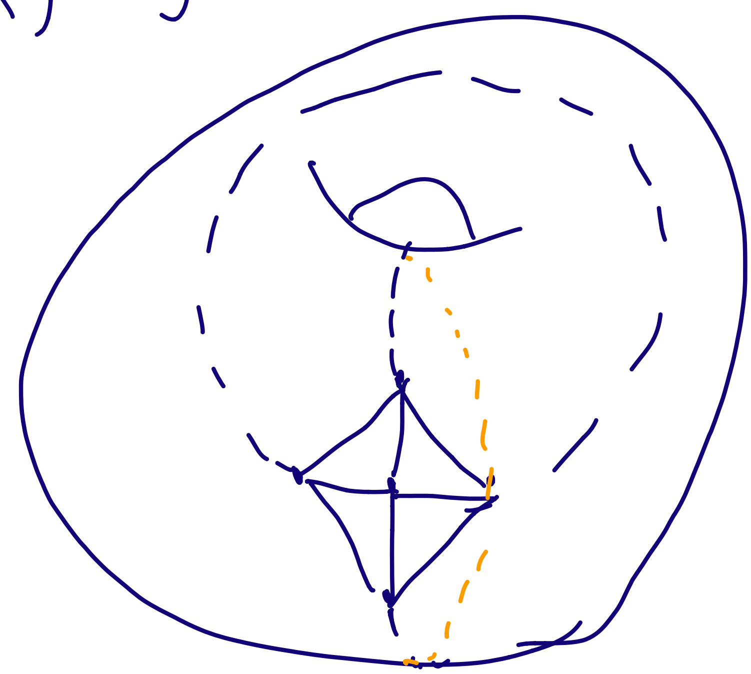
$$X \hookrightarrow M$$

if $\exists f: X \to M$ s.t.

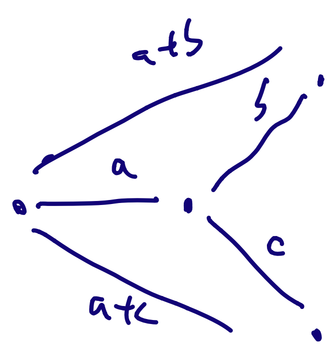
$$\forall a,b \quad d_X(a,b) = d_M(f(a), f(b))$$

Folklore result: X finite metric space, non-branching. Then $\exists M$, $\dim(M)=2$, $X \hookrightarrow M$.

Example: $\#(X)=5$



non-branching metric space:



What about countable metric spaces?

Def: $X \subseteq \mathbb{R}^n$ is a net, if $\exists \delta > 0$

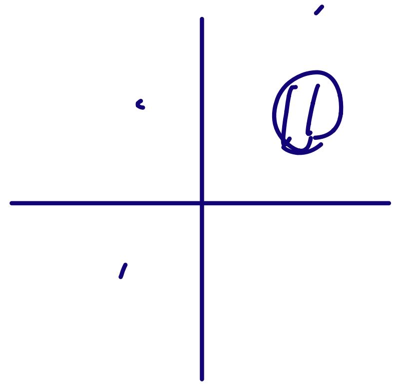
$\forall x \in \mathbb{R}^n \quad \exists y \in X \quad |x-y| \leq \delta.$

Thm 1 Suppose that $X \subseteq \mathbb{R}^2$ is net, and M is a 2D Riemannian mfld, complete connected, such that

$$X \hookrightarrow M$$

Then M is flat and isometric to the Euclidean plane.

Remark: Works for Riemann, but not Finsler



Corollary: $X \subseteq \mathbb{R}^3$ discrete, not contained in any affine plane, but contains a net some 2D plane. Then X does not

embed in any complete manifold of fin. dim.
Riem.

E.g. $X = \mathbb{Z}^2 \times \{0\} \cup \{(0,0,1)\}$.

Def: The <u>asymptotic Riemannian dimension</u> of a discrete metric space is the minimal dimension of a Riemannian mfld in which it embeds.

E.g. $X = (\mathbb{Z}^2 \cup \{0\}) \cup \{(0,0,1)\}$ has dim 3.

Theorem 2: Let $X \subseteq \mathbb{R}^n$ be a net, $X \hookrightarrow M$ for some Riemannian M of $\dim(M) = n$ then M is diffeomorphic to $\mathbb{R}^n$.

Remarks: 1) If curvature of M has compact support, then isometric to $\mathbb{R}^n$

2) Works for also for some non-Euclidean spaces.

<u>Def:</u> A discrete subset $X \subseteq M$ is <u>metrically rigid</u> if $\forall \tilde{M}$ $\dim(\tilde{M}) = \dim(M)$

$X \hookrightarrow \tilde{M} \implies \tilde{M}$ is isometric to M

- A net in $\mathbb{R}^2$ is metrically rigid, as well as random instance of a Poisson process, or a quasi-net.

<u>Some history of metric rigidity</u>

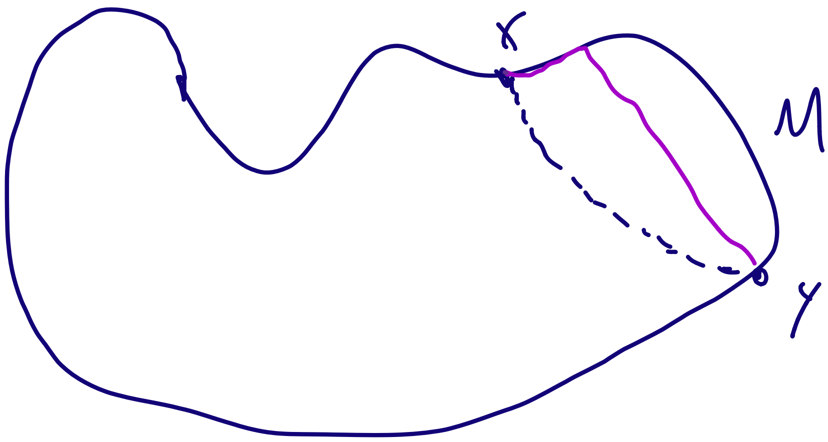
- <u>Hopf '48</u> (extended Morse and Hedlund)

A Riemannian metric on $\mathbb{T}^2$ without conjugate points is flat.

- Burago–Ivanov '94: True in $\mathbb{T}^n$.

Michel's conjecture ('81)

A simple Riemannian mfld with boundary is determined from boundary distances.



- True for strictly convex sets in $\mathbb{R}^n$

Pestov–Uhlmann '05: True in 2D

Bangert–Emmerich '13: M diffeomorphic to $\mathbb{R}^2$, no conjugate points, $x \in M$. Then

$$\liminf_{r \to \infty} \frac{\mathrm{Area}\,(D(x,r))}{r^2} \geq \pi .$$

with equality iff M is flat.

Q1: Metric rigidity of nets in Hadamard manifold.

Q2: How to formulate and prove a finitary version?

Plan of Proof

1) Use $X \curvearrowright M$ to prove that no conjugate points exist.

2) Use large-scale geometry, which is approx. Euclidean.

- Why a point of the net is not conjugate to anything in M?

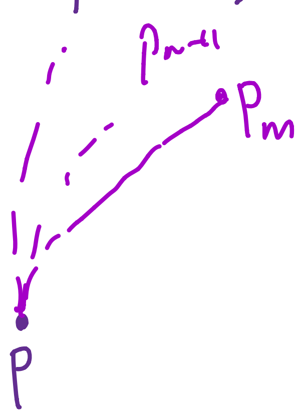
(works in n-dim.)

Write L for the net and of X.

$L \subseteq \mathbb{R}^n$

$L \cap M$

Pick $p \in L$, connect it to $p_m \to \infty$

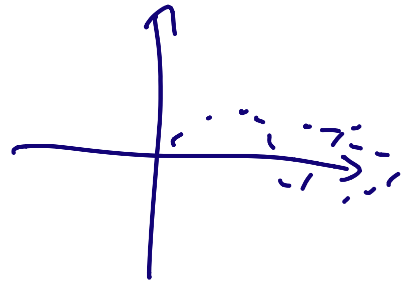


Pass to a subsequence, you get a <u>limiting geodesic ray</u> from p, which is minimizing

<u>Def:</u> $(p_m)_{m \geq 1} \subseteq L$, $v \in S^{n-1}$, we write $p_m \rightsquigarrow v$ if

$$\frac{p_m}{|p_m|} \to v \quad \text{and} \quad p_m \to \infty$$

Our first goal:



1) limiting ray is determined by $v \in S^{n-1}$.

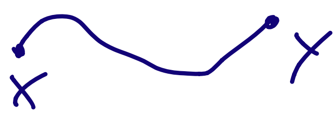
2) varies continuously with $v \in S^{n-1}$

3) Onto

(any ray from p arises this way).

<u>Lipschitz function</u> ("L^1-mass-transport")

. 1-Lip. are "dual" to curves



. upper bound for $d(x,y)$

. 1-Lip

$$|f(x) - f(y)| \leq d(x,y)$$

lower bound.

<u>Example:</u> The Busemann function of a minimizing geodesic ray $\gamma : [0,\infty) \longrightarrow M$,

For $x \in M$,

$$B_\gamma(x) = \lim_{t\to\infty} [t - d(\gamma(t), x)]$$

$$= \lim_{t\to\infty} [d(\gamma(t), \gamma(0)) - d(\gamma(t), x)]$$

- 1-Lip, $B_\gamma : M \to \mathbb{R}$
- If $M = \mathbb{R}^n$, Busemann functions are just

$$f(x) = \langle x, \theta \rangle + c$$

$$\theta \in S^{n-1}, \quad c \in \mathbb{R}.$$

Busemann function along a sequence:

Mark a point $O \in L \subseteq M$, set

$$d_p(x) = d(p, O) - d(p, x)$$

$$p \in L \qquad x \in M$$

Claim: $L \ni p_m \rightsquigarrow v$ then

$$\forall q \in L \qquad d_{p_m}(q) \xrightarrow{m\to\infty} \langle q, v \rangle$$

<u>Def</u> ("ideal boundary")

For $v \in S^{n-1}$,

$$\partial_v M = \left\{ f : M \to \mathbb{R} \,;\, \begin{array}{l} f \text{ 1-Lip.} \\ \forall q \in L \;\; f(q) = \langle q, v \rangle \end{array} \right\}$$

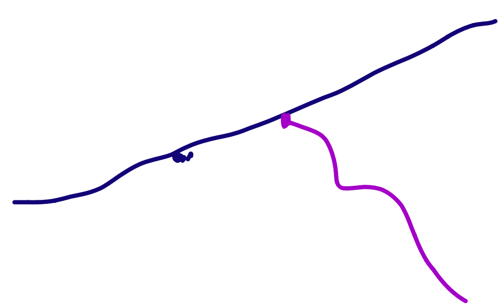
- A unit speed curve $\gamma : [a,b] \to M$ is a <u>transport curve</u> of f, if

$$f(\gamma(t)) - f(\gamma(s)) = t - s \quad \forall s, t \in [a,b]$$

- Evans–Gangbo '99, Feldman–McCann '03:

Two transport curves of f cannot intersect at an interior point, unless they coincide

impossible.



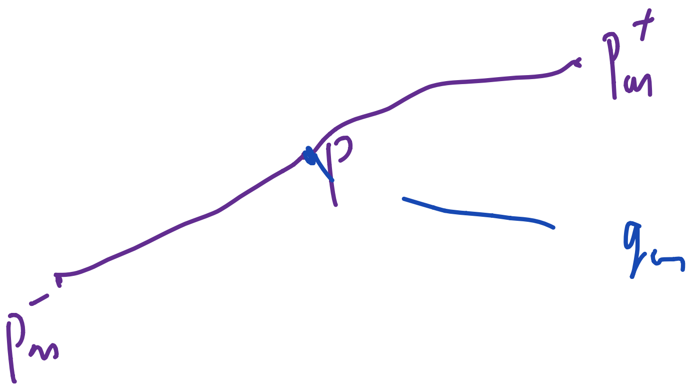
Def: $L \ni P_m \rightsquigarrow v$ narrowly

if $\quad |P_m| - \langle P_m, v \rangle \xrightarrow{m \to \infty} 0$

$L \ni P_m^{\pm} \rightsquigarrow v^{\pm}$ narrowly.

$q_n \rightsquigarrow v$

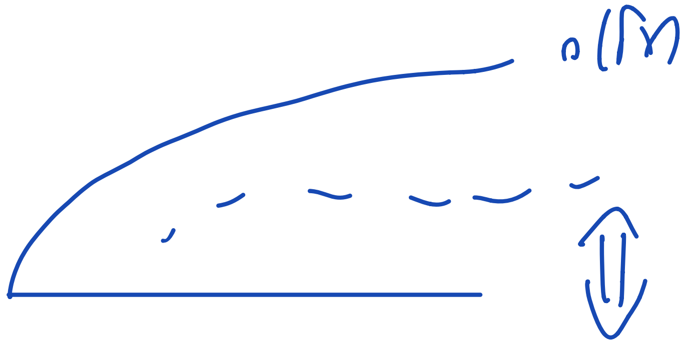
pass $\quad d_{q_m}$ converges to $f \in \partial_r M$



$\rightsquigarrow$ transport line of f

Hence coincide.

$$\frac{|P_m|}{|P_{m-1}|} \to 1$$



$$|p_m| - \langle p_m, e_1 \rangle \longrightarrow 0$$