Affirmative Resolution of Bourgain's Slicing Problem using Guan's Bound

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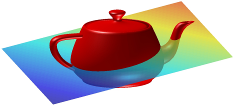
Joint work with Joseph Lehec



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Bourgain's Slicing Problem

Question (Bourgain '86)

Consider a convex body $K \subseteq \mathbb{R}^n$ of volume one. Does there exist a hyperplane $H \subseteq \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) \geq c,$$

for a universal constant $c > 0$?

- The context of Bourgain's seemingly innocent question: the maximal function operator M_K associated with a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$. Bourgain proved

$$\|M_K\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C.$$

- Much mathematics was developed around this question, culminating in an **affirmative answer** in K.-Lehec, 2024.
- We are still waiting for a short and sweet proof...

Bourgain's Slicing Problem

Theorem (K.-Lehec '24, building upon Guan '24)

Consider a convex body $K \subseteq \mathbb{R}^n$ of volume one. Then there exists a hyperplane $H \subseteq \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) \geq c,$$

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The Busemann-Petty problem from the 1950s

It was explained by Milman and Pajor '89 (in the centrally-symmetric case) that the Slicing Theorem implies:

Theorem (corrected Busemann-Petty conjecture)

Let $K, T \subseteq \mathbb{R}^n$ be centered convex bodies such that

$$Vol_{n-1}(K \cap \theta^{\perp}) \leq Vol_{n-1}(T \cap \theta^{\perp}) \qquad \text{for all } \theta \in S^{n-1}.$$

Then $Vol_n(K) \leq C \cdot Vol_n(T)$, for some universal constant $C > 0$.

- Busemann and Petty proved the above with $C = 1$ if $T = -T$ and the convex body K is, say, a Euclidean ball.
- In the 1950s, they conjectured that $C = 1$ for all $K = -K$. This turns out to be true if $n \leq 4$ and false if $n \geq 5$. (Lutwak '88, Zhang, Gardner - Koldobsky - Schlumprecht '90s)
- **Fails already for the cube and Euclidean ball in high dimensions.** We lose a factor of $\approx \sqrt{e/2}$ (Ball '86).

$SL_n(\mathbb{R})$-invariant ways to measure "size"

Given a convex body $K \subseteq \mathbb{R}^n$ we may consider:

1. Its volume.
2. The determinant of its covariance matrix $\mathrm{Cov}(K) \in \mathbb{R}^{n\times n}$,

$$\mathrm{Cov}_{ij}(K) = \int_K x_i x_j \frac{dx}{Vol_n(K)} - \int_K x_i \frac{dx}{Vol_n(K)} \int_K x_j \frac{dx}{Vol_n(K)}.$$

Determined by the volume of the **Legendre ellipsoid of inertia** of K, which has the same 2^{nd} moments as K.

Definition (Isotropic constant)

For a convex body $K \subseteq \mathbb{R}^n$ we define

$$L_K = \left(\frac{\det \mathrm{Cov}(K)}{Vol_n(K)^2} \right)^{1/(2n)}.$$

- We have $L_K = L_{T(K)}$ for any invertible, affine map T.

Isotropic position and volume of slices

- $K \subseteq \mathbb{R}^n$ is an **isotropic convex body** if it's centered and its Legendre ellipsoid is a ball, i.e., $\mathrm{Cov}(K) \in \mathbb{R}^{n \times n}$ is scalar.

Theorem (Volumes of slices – Hensley '80, Fradelizi '99)

Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body. Then for any two hyperplanes $H_1, H_2 \subseteq \mathbb{R}^n$ through the origin,

$$\frac{Vol_{n-1}(K \cap H_2)}{Vol_{n-1}(K \cap H_1)} \leq \sqrt{6}.$$

In fact, $Vol_{n-1}(K \cap H_i) \sim L_K^{-1} \cdot Vol_n(K)^{(n-1)/n}$.

- Proven using the **Brunn-Minkowski inequality**.
- For any convex $K \subseteq \mathbb{R}^n$, we have $L_K \geq L_{B^n} \geq c$.

Theorem (K.-Lehec '24, building upon Guan '24)

For any convex $K \subseteq \mathbb{R}^n$ we have $c \leq L_K \leq C$.

Isotropic position and volume of slices

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Theorem (K.-Lehec '24, building upon Guan '24)

For any convex $K \subseteq \mathbb{R}^n$ we have $c \leq L_K \leq C$.

What are the values of these universal constants?

- The numerical values that our proof yield are very large.
- Perhaps the extremal cases (with/without central symmetry) are:

$$L_{[0,1]^n} = \frac{1}{\sqrt{12}}, \qquad L_{\Delta^n} = \frac{(n!)^{1/n}}{(n+1)^{(n+1)/(2n)}\sqrt{n+2}} \approx \frac{1}{e}.$$

Relations to classical conjectures

1. If L_K is maximized for the simplex Δ^n, then the Mahler volume-product conjecture follows (the non-symmetric case, proven in 2D by Mahler, 1939). See K. '18.
2. If among centrally-symmetric bodies, L_K is maximized for the cube, then the Minkowski lattice conjecture follows (proven in 2D by Minkowski, 1901). See Magazinov '18.

More consequences of the Slicing Theorem

From Milman-Pajor '89, K.-Milman '05 and the Slicing Theorem:

Theorem (Sylvester problem)

Let $K \subseteq \mathbb{R}^n$ be a convex body. Select $n+2$ random points, i.i.d uniformly in K. Write $p(K)$ for the probability for a **convex position**. Then,

$$c/\sqrt{n} \leq (1-p(K))^{1/n} \leq C/\sqrt{n}.$$

Theorem (Steiner symmetrization of most of a convex body)

For any convex body $K \subseteq \mathbb{R}^n$ there exists a convex $T \subseteq K$ with

$$Vol_n(T) \geq 0.9 \cdot Vol_n(K)$$

such that $\forall \varepsilon > 0$, after $\lfloor \varepsilon n \rfloor$ Steiner symmetrizations of T, we reach $\tilde{T}$ with Banach-Mazur distance to a Euclidean ball:

$$d_{BM}(\tilde{T}, B^n) \leq C(\varepsilon).$$

Relation to Milman's ellipsoids

Theorem (V. Milman, '80s)

Let $K \subset \mathbb{R}^n$ be a centered convex body. Then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$, with $Vol_n(\mathcal{E}) = Vol_n(K)$, such that

$$Vol_n(K \cap C\mathcal{E}) / Vol_n(K) \geq c^n.$$

- Quite a few consequences: reverse Brunn-Minkowski, Bourgain-Milman inequality, Quotient of Subspace and also

$$\max\{N(K,\mathcal{E}), N(\mathcal{E},K)\} \leq e^{Cn}.$$

Corollary (Slicing thm + Paouris large deviation estimate '05)

Let $K \subset \mathbb{R}^n$ be a centered convex body. Then its normalized **Legendre ellipsoid** $\mathcal{E} \subset \mathbb{R}^n$, with $Vol_n(\mathcal{E}) = Vol_n(K)$, satisfies

$$Vol_n(K \cap C\mathcal{E}) / Vol_n(K) \geq 1 - e^{-\sqrt{n}}.$$

Previous bounds for the isotropic constant

Write $L_n = \sup_{K \subseteq \mathbb{R}^n} L_K$. Trivial bound $L_n \le C\sqrt{n}$. Better bounds:

- $n^{1/4} \cdot \log n$ using sub-Gaussian processes and K-convexity, Bourgain '91.
- $n^{1/4}$ using covariance of exponential tilts $e^{x \cdot y} 1_K(x)$, Paouris LDP and Bourgain-Milman, K. '05.
- $n^{1/4}$ via thin shell and **Eldan's stochastic localization**, covariance of $e^{x \cdot y - t|x|^2/2} 1_K(x)$, Lee-Vempala '16.
- $e^{C\sqrt{\log n \cdot \log \log n}}$ using growth regularity of the covariance process, **Chen '20**.
- $\log^4 n$ and $\log^{2.223\ldots} n$ by combining the above with spectral analysis, K.-Lehec '22, Jambulapati-L.V. '22.
- $\sqrt{\log n}$ by an improved Lichnerowicz inequality, using Bochner's formula, K. '23.
- $\log \log n$ using self-controlled growth estimates for the covariance process, analysis of 3-tensors, Guan '24.

Convex bodies with large isotropic constant

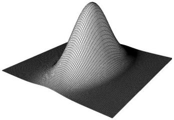
Write λ_K for the uniform probability measure on K. By using **Milman ellipsoid** and covering numbers:

Proposition (essentially from Bourgain, K., Milman '04)

Let $K \subseteq \mathbb{R}^n$ satisfy $L_K \geq L_n/2$. Let $E \subseteq \mathbb{R}^n$ be any subspace of $\dim(E) \geq n/4$. Denote $\mu = (Proj_E)_* \lambda_K$. Then,

$$L_\mu \geq c \cdot L_n.$$

- The measure projection of λ_K is a **log-concave measure:** it has density e^{-H} where $H : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex.
- **Prékopa-Leindler inequality**: Log-concavity is preserved under convolution, push-forward by linear maps, weak limits. Also pointwise products.
- The **isotropic constant** of a log-concave μ is



$$L_\mu = e^{-\mathrm{Ent}(\mu)/n} \cdot \det \mathrm{Cov}(\mu)^{1/(2n)}$$

where $\mathrm{Ent}(\mu) = -\int_{\mathbb{R}^n} f \log f$, with f being the density of μ.

Entropy vs. Covariance? Analyze using heat flow!

- Log-concave measures include uniform measures on convex bodies and Gaussian measures.

Theorem (Slicing thm + Ball '86 (and K. '05 for non-even μ))

For any log-concave probability measure μ in a finite dimensional linear space,

$$1/\sqrt{2\pi e} \leq L_\mu \leq C.$$

(Equality on the left-hand side is classical, attained for Gaussians.)

- How may we analyze the isotropic constant? A suggestion from Ball-Nguyen '13 is to use:

The heat flow in $\mathbb{R}^n$

It preserves log-concavity, it increases the covariance linearly, and it is great with entropy, too!

Entropy production along the heat flow

Definition

A log-concave probability measure μ in $\mathbb{R}^n$ is **isotropic** if it is centered with $\mathrm{Cov}(\mu) = \mathrm{Id}$.

(this is one of two common normalizations in the literature)

- For an isotropic, log-concave μ in $\mathbb{R}^n$ and for $s > 0$ denote

$$\mu_s = \mu * \gamma_s$$

where γ_s is a Gaussian of mean zero and $\mathrm{Cov}(\gamma_s) = s \cdot \mathrm{Id}$.

De Bruijn's identity (from Stam's paper '59)

$$\frac{d}{ds}\mathrm{Ent}(\mu_s) = \frac{1}{2}J(\mu_s).$$

Here, the density of μ_s is $\exp(-\psi_s)$, and the Fisher information is

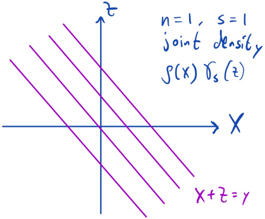
$$J(\mu_s) = \int_{\mathbb{R}^n} |\nabla\psi_s|^2 d\mu_s = \int_{\mathbb{R}^n} \mathrm{Tr}[\nabla^2\psi_s] d\mu_s.$$

By differentiating a Gaussian convolution,

$$\nabla^2 \psi_s(y) = \frac{\mathrm{Id}}{s} - \frac{\mathrm{Cov}(p_{s,y})}{s^2}$$

where, with ρ the log-concave density of μ,

$$p_{s,y}(x) = \frac{\gamma_s(x-y)\rho(x)}{(\rho * \gamma_s)(y)}$$



Observation 1

We have $\mathbf{Cov}(\boldsymbol{p_{s,y}}) \leq \boldsymbol{s} \cdot \boldsymbol{Id}$. By Prékopa, as μ_s is log-concave.

Let $X \sim \mu$ and $Z \sim N(0, \mathrm{Id})$ be independent random vectors.

Observation 2

- The measure μ_s is the law of $X + \sqrt{s}Z$.
- $\nabla^2 \psi_s(X + \sqrt{s}Z) = \frac{\mathrm{Id}}{s} - \frac{1}{s^2} \cdot \mathrm{Cov}(X | X + \sqrt{s}Z)$.

Application of the probabilistic interpretation

Lemma (super-additivity)

Let X, Y, Z_1, Z_2 be independent, $Z_1, Z_2 \sim N(0, \mathrm{Id})$. Then,

$$\begin{aligned}
&\mathbb{E}\mathrm{Cov}(X+Y|X+Y+\sqrt{s}(Z_1+Z_2)) \\
&\geq \mathbb{E}\mathrm{Cov}(X+Y|X+\sqrt{s}Z_1, Y+\sqrt{s}Z_2) \\
&= \mathbb{E}\mathrm{Cov}(X|X+\sqrt{s}Z_1) + \mathbb{E}\mathrm{Cov}(Y|Y+\sqrt{s}Z_2).
\end{aligned}$$

- Recall that $J(\mu_s) = n/s - s^{-2} \cdot \mathbb{E}\mathrm{Tr}\left[\mathrm{Cov}(X|X+\sqrt{s}Z)\right]$. Thus, by integrating the derivative of $\mathrm{Ent}(\mu_s)$ we get:

The Shannon-Stam inequality ('48 –'59)

Let X, Y be independent random vectors in $\mathbb{R}^n$ and $0 < \lambda < 1$. Then

$$\mathrm{Ent}\left(\sqrt{\lambda}X + \sqrt{1-\lambda}Y\right) \geq \lambda \cdot \mathrm{Ent}(X) + (1-\lambda)\cdot \mathrm{Ent}(Y).$$

Equality iff Gaussians with proportional covariance matrices.

The Shannon-Stam inequality in the log-concave case

Particularly simple behavior in the i.i.d, log-concave case:

Proposition (Ball-Nguyen '13)

If X and Y are i.i.d and log-concave in $\mathbb{R}^n$, then,

$$\mathrm{Ent}(X) \leq \mathrm{Ent}\left(\frac{X+Y}{\sqrt{2}}\right) \leq \mathrm{Ent}(X) + 2n.$$

- Proof idea: $f(0) \sim e^{-\mathrm{Ent}(X)}$ and $(f * f)(0) = \int f^2 \geq 2^{-n} f(0)$.
- Using a stochastic proof of Shannon-Stam in Lehec '13:

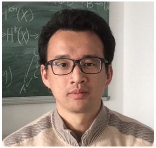
Theorem (reformulation of Eldan and Mikulincer '20)

If X and Y are i.i.d and log-concave in $\mathbb{R}^n$, with law μ, then,

$$2n \geq \mathrm{Ent}\left(\frac{X+Y}{\sqrt{2}}\right) - \mathrm{Ent}(X)$$
$$\geq c \int_0^\infty s \cdot \int_{\mathbb{R}^n} \left| \nabla^2 \psi_s - \int_{\mathbb{R}^n} \nabla^2 \psi_s d\mu_s \right|^2 d\mu_s ds.$$

Guan's Bound

- Last month, **Qingyang Guan** from CAS in Beijing posted a paper on arXiv.
- It solved Bourgain's slicing problem up to $\log \log n$, improving upon $\sqrt{\log n}$, using:



Proposition (Guan '24)

Consider an isotropic, log-concave probability measure μ in $\mathbb{R}^n$. Write $e^{-\psi_s}$ for the density of $\mu_s = \mu * \gamma_s$. Then for $s > 0$,

$$\int_{\mathbb{R}^n} \left| \mathrm{Id} - s \cdot \nabla^2 \psi_s \right|^2 d\mu_s \leq Cn/s^2,$$

where $C > 0$ is a universal constant.

- Proved with intricate bootstrap using $\int \mathrm{Tr}\, f_k(\nabla^2 \psi_s) d\mu_s$, the improved Lichnerowicz inequality, and 3-tensor analysis.
- This provided the missing link in an approach to Bourgain's slicing problem we discussed with Lehec in 2022.

Focusing on the conditional covariance matrix

Let us change variables $t = 1/s$.
This slightly helps us focus on isotropicity of μ.

Time reversal $s = 1/t$

1. Consider the **covariance matrix**: $A_0 = \mathrm{Id}$ and for $t > 0$,
$$A_t = \mathrm{Cov}(X|X + \sqrt{s}Z) = \mathrm{Cov}(X|tX + \sqrt{t}Z) \leq \mathrm{Id}/t.$$
(helpful to think of $tX + W_t$ where $(W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$)
2. Moreover, $\frac{d}{dt}\mathbb{E}A_t = -\mathbb{E}A_t^2 \leq -(\mathbb{E}A_t)^2 \implies \mathbb{E}A_t \leq \frac{\mathrm{Id}}{1+t}$.
3. The de Brujin identity:
$$\mathrm{Ent}(\mu) = -\frac{1}{2}\int_0^\infty \mathrm{Tr}\left[\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right] dt + \frac{n}{2}\log(2\pi e).$$
4. Guan's bound: $\mathbb{E}|A_t|^2 \leq Cn$.
5. Eldan-Mikulincer: $\int_0^\infty (1+t)\cdot \mathbb{E}|A_t - \mathbb{E}A_t|^2 dt \leq Cn$.

Suppose that $\mathbb{E}A_t \geq c \cdot \mathrm{Id}/(1+t)$ for $t \geq C$

Following Eldan-Mikulincer:

1. Integration by parts using $\frac{d}{dt}\mathbb{E}A_t = -\mathbb{E}A_t^2$ gives
$$\int_0^\infty (1+t)\mathbb{E}\left|\frac{\mathrm{Id}}{1+t} - A_t\right|^2 dt = \int_0^\infty \mathrm{Tr}\left[\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right] dt.$$
In both integrals, the contribution of $t \in [0, C]$ is at most $\tilde{C}n$.
2. From (1) and **Eldan-Mikulincer stability**, Slicing follows from:
$$\int_0^\infty (1+t)\left|\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right|^2 dt \leq \bar{C}n$$
3. **Positive-definite matrices**: from the "Suppose", for $t \geq C$,
$$(1+t)\left|\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right|^2 \leq (1-c)\cdot \mathrm{Tr}\left[\frac{\mathrm{Id}}{1+t} - \mathbb{E}A_t\right].$$
4. From (3) and (1) we obtain (2).

Using Guan's bound and Milman's ellipsoid

We still need to obtain $\mathbb{E}A_t \geq c \cdot \mathrm{Id}/(t+1)$ for $t \geq C$.

1. Start with "worse possible" $\mu = \lambda_K$. By Milman ellipsoid, for any subspace $E \subseteq \mathbb{R}^n$ with $\dim(E) \geq n/4$,
$$L_n \leq c' \cdot L_{(Proj_E)_* \mu}.$$
2. For $t = t_0 = c$ we have $\mathrm{Tr}\mathbb{E}A_{t_0} \geq n/2$. Indeed, $A_0 = \mathrm{Id}$ and hence from Guan's bound, $\mathbb{E}\mathrm{Tr}[A_t] \geq 1 - Cn \cdot t$ for all t.
3. A third of the eigenvalues of $\mathbb{E}A_{t_0}$ are at least $1/4$. Indeed, this follows from (2) as $\mathbb{E}A_{t_0} \leq \mathrm{Id}$ and all e.v. are in $[0,1]$.
4. Thus there exists a subspace E with $\dim(E) \geq n/3$ and
$$\mathbb{E}A_{t_0,(Proj_E)_*\mu} \geq Proj_E \cdot \mathbb{E}A_{t_0} \cdot Proj_E \geq \mathrm{Id}/4.$$
5. Denote $\tilde{A}_t = A_{t,(Proj_E)_*\mu}$. Then $\mathbb{E}\tilde{A}_t \geq \tilde{c} \cdot \mathrm{Id}/t$ for $t \geq t_0$. Indeed, the matrix $t \cdot \mathbb{E}\tilde{A}_t$ is increasing, since $\frac{d}{dt}\mathbb{E}\tilde{A}_t = -\mathbb{E}\tilde{A}_t^2 \geq -\mathbb{E}\tilde{A}_t/t$. Hence $L_{(Proj_E)_*\mu} < Const$. □

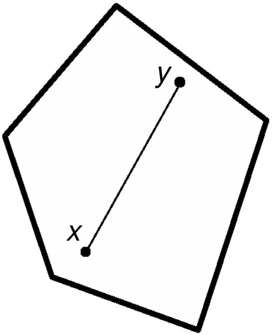
The Poincaré inequality

Theorem (Poincaré, 1890 and 1894)

Let $K \subseteq \mathbb{R}^3$ be convex and open. Let $f : K \to \mathbb{R}$ be C^1-smooth, with $\int_K f = 0$. Then,

$$\int_K f^2 \leq C_P(K) \cdot \int_K |\nabla f|^2$$

where $C_P(K) \leq (9/16) \cdot Diam^2(K)$.



- The smallest possible $C_P(K)$ is the **Poincaré constant** of K or the inverse spectral gap for the Neumann laplacian.
- Proof: Estimate $\int_{K \times K} |f(x) - f(y)|^2 dxdy$ via segments.
- In high dimensions, bounds for the Poincaré constant via the diameter are usually inadequate.

The Kannan-Lovász-Simonovits (KLS) conjecture

- It is conjectured that up to a universal constant, the Poincaré inequality is **saturated by linear functions**.

Conjecture (KLS '95)

Let $X \in \mathbb{R}^n$ be a log-concave random vector. Then,

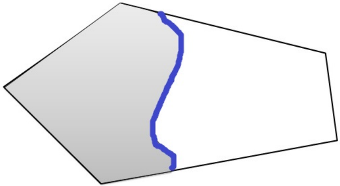
$$\|\mathrm{Cov}(X)\|_{op} \leq C_P(X) \leq C \cdot \|\mathrm{Cov}(X)\|_{op}.$$

- An equivalent formulation of KLS due to Cheeger '70, Buser '82 and Ledoux '04: up to a universal constant, the isoperimetric problem in a convex body $K \subseteq \mathbb{R}^n$ is **saturated by a hyperplane bisection.**

The isoperimetric constant

For an open set $K \subset \mathbb{R}^n$ define

$$h_K = \inf_{A \subset K} \frac{|\partial A \cap K|}{\min\{|A|, |K \setminus A|\}}$$



Thin shell phenomenon

- The KLS conjecture is currently known up to $\log n$ (K. '23).
- A weak form is:

Thin Shell Conjecture (Bobkov and Koldobsky '04, stems from Anttila, Ball and Perissinaki '03)

If X is an isotropic, log-concave random vector in $\mathbb{R}^n$, then

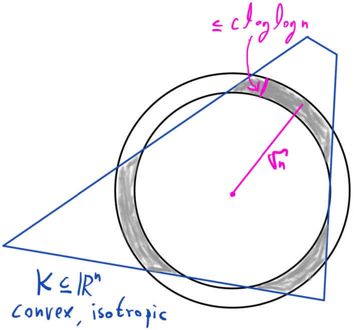
$$\mathbb{E}\left(|X| - \sqrt{n}\right)^2 \leq C.$$

Most of the mass of X is contained in a **thin spherical shell.**

- Using Guan's bound and the approach in K.-Lehec '22 refined:

Theorem (Guan '24)

The thin shell conjecture is true up to $(\log \log n)^2$.



Some partial results on KLS and thin shell

- Thin shell conjecture $\implies$ Slicing Theorem (Eldan-K. '11).

Sudakov '76, Diaconis-Freedman '84

Thin shell phenomena implies **approx. Gaussian marginals**. Precise estimates (up to $\log$) by Bobkov, Chistyakov, Götze '19.

- Let $t > 0$ and $X \in \mathbb{R}^n$ be a random vector in $\mathbb{R}^n$ with density $e^{-\psi}$ such that $\nabla^2\psi \geq t \cdot \mathrm{Id}$ pointwise.

Theorem (log-concave Lichnerowicz (folklore, see Obata '62))

$$\mathrm{Cov}(X) \leq C_P(X) \leq \frac{1}{t}.$$

- Geometric average of Lichnerowicz and KLS holds:

Theorem (Improved Lichnerowicz, K. '23)

$$C_p(X) \leq \sqrt{\|\mathrm{Cov}(X)\|_{op} \cdot \frac{1}{t}}.$$

How is Guan's bound proved?

- Analyze the covariance matrix A_t. For any function f,

$$\frac{d}{dt}\mathbb{E}\mathrm{Tr} f(A_t) = \mathbb{E}\left(\text{expression involving third moments of } p_t\right)$$

where $p_t = p_{1/t, X+Z/\sqrt{t}}$ is more log-concave than $\gamma_{1/t}$.

- Consider f which is positive, increasing, with $f(t) = t^2$ for $t > 4$ such that

$$|f''(t)| \leq D_0^2 |f(t)|.$$

A self-controlled estimate

Dividing the 3-tensor into a few pieces, and using improved Lichnerowicz yields

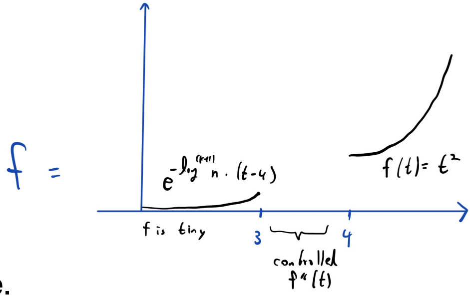
$$\frac{d}{dt}\mathbb{E}\mathrm{Tr} f(A_t) \leq \frac{C}{t} \cdot \mathbb{E}\mathrm{Tr} f(A_t) \qquad (0 < t < D_0^{-4}).$$

How is Guan's bound proved? (continued)

- Divide $[0, c]$ into $\log^* n$ sub-intervals. Set $t_1 = c/\log^2 n$ and for $t \geq 2$,

$$t_k = \frac{c}{|\underbrace{\log \log \ldots \log}_{k \text{ times}} n|^{16}}$$

- For $i \geq 1$, in the interval $[t_i, t_{t+1}]$ use $F_{i,t} = \mathbb{E}\mathrm{Tr} f(A_t)$ with f as in the figure.
- The function $F_{i,t}$ is **much smaller than n**, and **grows regularly** in the interval, thanks to the self-controlled estimate.



At the end of the interval $F_{i,t}$ and $F_{i+1,t}$ are still much smaller than n. May thus proceed to next interval.

Conclusion

- It took a (global) village.
- Is thin shell coming up?
- KLS remains unsolved.

Conclusion

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- Fresh from the oven: **Pierre Bizeul** has simplified our argument, eliminating the usage of information theory!
- Instead of Eldan-Mikulincer, he applies Paouris' small ball estimates from '12.

  Milman ellipsoids are still necessary, as well as Guan's bound and a projection to a subspace of proportional dimension to eliminate small eigenvalues of A_t.

Thank you!

Jean Bourgain by Jan Rauchwerger

Images are courtesy of Qingyang Guan, Vitali Milman and Yoel Shkolnisky.