# On the Log-Brunn-Minkowski conjecture

(based on various joint works with Andrea Colesanti, John Hosle, Alexander Kolesnikov, Arnaud Marsiglietti, Piotr Nayar, Artem Zvavitch.)

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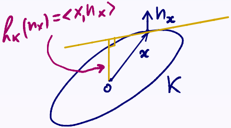
# Notation




- Convex bodies in $\mathbb{R}^n$ denote $K, L, M$;
- Lebesgue volume in $\mathbb{R}^n$ denote $|\cdot|$ or $|\cdot|_n$;
- Recall Minkowski sum of sets $A, B \subset \mathbb{R}^n$ :

$$A + B = \{x + y : x \in A, y \in B\}.$$

- Support function of a convex set $K$ is

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle = \|y\|_{K^\circ};$$

- $h_{K+L} = h_K + h_L$;
- Unit normal to $\partial K$ at $x \in \partial K$ denote $n_x$;
- $h_K(n_x) = \langle x, n_x \rangle$;
- Second fundamental form of $\partial K$ denote $\mathrm{II}$, mean curvature $H_x = tr(\mathrm{II})$.

# The Brunn-Minkowski inequality

## Log-concavity of the Lebesgue measure

$$|\lambda K+(1-\lambda)L|\geq |K|^{\lambda}|L|^{1-\lambda}$$

## $\frac{1}{n}$-concavity of the Lebesgue measure

$$|\lambda K+(1-\lambda)L|^{\frac{1}{n}}\geq \lambda|K|^{\frac{1}{n}}+(1-\lambda)|L|^{\frac{1}{n}}$$

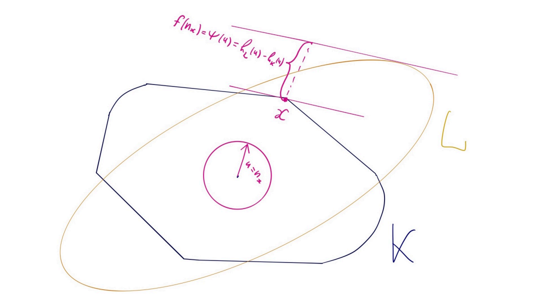
## The isoperimetric inequality

For all Borel-measurable sets $K$ with $|K|=|B_2^n|$, one has $|\partial K|\geq |\partial B_2^n|$.

## Proof

$$|\partial K|=\liminf_{\epsilon\to 0}\frac{|K+\epsilon B_2^n|-|K|}{\epsilon}\geq \liminf_{\epsilon\to 0}\frac{\left(|K|^{\frac{1}{n}}+\epsilon|B_2^n|^{\frac{1}{n}}\right)^n-|K|}{\epsilon}=n|K|^{\frac{n-1}{n}}|B_2^n|^{\frac{1}{n}}.$$

# The local version of the Brunn-Minkowski inequality



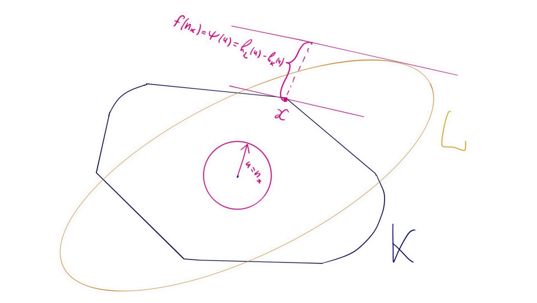

- Fix convex sets $K$ and $L$ with support functions $h_K$ and $h_L$;
- Let $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $\psi(u) = h_L(u) - h_K(u)$;
- For $t \in [0, 1]$, the body $K_t = (1-t)K + tL$ has support function $h_t = h_K + t\psi$ on $\mathbb{S}^{n-1}$;
- The Brunn-Minkowski inequality

$$|\lambda K + (1-\lambda)L| \geq |K|^{\lambda} |L|^{1-\lambda}$$

implies that $\log |K_t|$ is concave;
- Let $F(t) = |K_t|$. We deduce $(\log F)''_{t=0} \leq 0$, or

$$F''(0)F(0) - F'(0)^2 \leq 0.$$

# The local version of the Brunn-Minkowski inequality



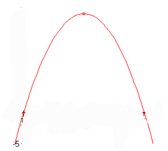

- $F(t) = |K_t|$, $h_t = h_K + t\psi$, BM implies $F''(0)F(0) - F'(0)^2 \leq 0$.
- Let $f : \partial K \to \mathbb{R}$ be given by $f(x) = \psi(n_x) = h_L(n_x) - h_K(n_x)$;
- $F(0) = |K|$;
- $F'(0) = \int_{\partial K} f$;
- $F''(0) = \int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle$;
- Brunn-Minkowski inequality implies, and follows from

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$$

(Colesanti 2008; Kolesnikov-Milman 2015-2018)

# Abstract observation

- Take any algebra $A$ which is a vector space over $\mathbb{R}$;
- Let $Q : A \times A \to \mathbb{R}$ be any symmetric bilinear form;
- Suppose for every $a \in A$,
$$Q(a,a) \leq 0. \tag{1}$$
- Fix any element $z \in A$;
- For all $t \in \mathbb{R}$ we have $Q(a+tz, a+tz) \leq 0$, or equivalently
$$Q(a,a) + 2tQ(a,z) + t^2 Q(z,z) \leq 0;$$



- Optimize in $t$, plug optimal $t = -\frac{Q(a,z)}{Q(z,z)}$, get the Schwartz inequality
$$Q(a,a) \leq \frac{Q(a,z)^2}{Q(z,z)} \leq 0 \tag{2}$$
- (2) is sharper than (1);
- (2) is invariant under $a \to a + tz$.

- The local version of the (multiplicative) Brunn-Minkowski inequality:

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$$

- Pick the special function $z(x) = \langle x, n_x \rangle (= h_K(n_x))$;
- Optimize with respect to $f(x) + tz(x)$, using Schwartz inequality get a strengthening

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$$

- When $K = B_2^n$, we get the sharp Poincare inequality on $\mathbb{S}^{n-1}$ :

$$\int_{\mathbb{S}^{n-1}} f^2 - \left(\int_{\mathbb{S}^{n-1}} f\right)^2 \leq \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\nabla_\sigma f|^2,$$

where $\int_{\mathbb{S}^{n-1}}$ is normalized.

- The first eigenvalue of $\Delta$ on $\mathbb{S}^{n-1}$ is $n-1$, and the above is sharp.

# Invariances of the local version of the Brunn-Minkowski inequality

- The Local Brunn-Minkowski inequality

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f\rangle - \frac{n-1}{n}\frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0$$

  is invariant under $f \to f + t\langle x, n_x\rangle$ ("times change");
- It is also invariant under $f \to sf$ (dilating);
- Recall the definition of mixed volumes of convex bodies $K$ and $M$, for $k = 1, ..., n$:

$$V_k(K,M) = \frac{(n-k)!}{n!}|K+tM|^{(k)}_{t=0};$$

- WLOG suppose that $f(x) = h_M(n_x)$ for some convex body $M$ (or else add a large multiple of $h_K(n_x)$). Get Minkowski's second inequality:

$$V_2(K,M) \leq \frac{V_1(K,M)^2}{|K|}.$$

- Upshot: the Minkowski second inequality is equivalent to the Brunn-Minkowski inequality.

- Goal: $\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f\rangle - \frac{n-1}{n}\frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$
- Let $u : K \to \mathbb{R}$ be any function such that $\langle \nabla u, n_x\rangle = f(x)$ for $x \in \partial K$.
- By divergence theorem, $\int_{\partial K} f = \int_K \Delta u$.

**Lemma (Kolesnikov, Milman 2015)**

$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f\rangle \leq \int_K (\Delta u)^2 - \|\nabla^2 u\|^2.$

- Goal follows from finding for every $f : \partial K \to \mathbb{R}$ such $u : K \to \mathbb{R}$ with $\langle \nabla u, n_x\rangle = f(x)$ and

$$\mathbb{E}\|\nabla^2 u\|^2 \geq Var(\Delta u) + \frac{1}{n}(\mathbb{E}\Delta u)^2.$$

**Solvability of the Neumann system**

Let $\Delta u = const$, with the Neumann boundary condition $\langle \nabla u, n_x\rangle = f(x)$.

- For any symmetric matrix $A$, $\|A\|_{HS}^2 \geq \frac{tr(A)^2}{n}$; thus $\|\nabla^2 u\|^2 \geq \frac{1}{n}(\Delta u)^2$.

# The Log-Brunn-Minkowski conjecture

**Logarithmic sum (Definition)**

$$\lambda K +_0 (1-\lambda)L = \bigcap_{u\in\mathbb{S}^{n-1}} \{x\in\mathbb{R}^n : |\langle u,x\rangle| \le h_K(u)^\lambda h_L(u)^{1-\lambda}\}.$$

Note, by AMGM, $\lambda K +_0 (1-\lambda)L \subset \lambda K + (1-\lambda)L$.

**Log-Brunn-Minkowski conjecture (Böröczky, Lutwak, Yang, Zhang 2011)**

For **origin-symmetric convex** sets $K$ and $L$ in $\mathbb{R}^n$,

$$|\lambda K +_0 (1-\lambda)L| \ge |K|^\lambda |L|^{1-\lambda}.$$

- Equivalent to uniqueness of solution of certain Monge-Ampere equations, questions go back to Firey;
- True for $n=2$ (Böröczky, Lutwak, Yang, Zhang 2011), (Stancu for polytopes);
- True for unconditional sets (Saraglou 2013; Cordero-Fradelizi-Maurey; Boroczky, Kalantzopoulos 2020 – more general result);
- True for complex convex bodies (Rotem 2017).

- Fix convex sets $K$ and $L$ with support functions $h_K$ and $h_L$;
- Let $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $\psi(u) = \frac{h_L(u)}{h_K(u)}$;
- Locally, $K_t := tK +_0 (1-t)L$ has support function

$$h_t = h_K \psi^t = h_K + t\varphi + O(t^2),$$

  where $\varphi = h_K \log \frac{h_L}{h_K}$;
- The Log-Brunn-Minkowski inequality implies that $\log |K_t|$ is concave;
- Let $F(t) = |K_t|$. We deduce $(\log F)''_{t=0} \leq 0$, or $F''(0)F(0) - F'(0)^2 \leq 0$.
- Let $f : \partial K \to \mathbb{R}$ be given by $f(x) = \varphi(n_x) = h_K(n_x) \log \frac{h_L(n_x)}{h_K(n_x)}$;
- $F(0) = |K|$;
- $F'(0) = \int_{\partial K} f$;
- $F''(0) = \int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle}$.

**Theorem (Colesanti, L, Marsiglietti 2016)**

The Log-Brunn-Minkowski inequality would imply, for every symmetric convex $K$ and every even function $f : \partial K \to \mathbb{R}$,

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle} \leq \frac{\left(\int_{\partial K} f\right)^2}{|K|}.$$

**Colesanti-L-Marsiglietti**

The local version of the Log-Brunn-Minkowski inequality is true when $K = B_2^n$.

- Indeed, the Local Log-Brunn-Minkowski inequality with $K = B_2^n$ is equivalent to the following Poincare inequality:

$$Var_{\mathbb{S}^{n-1}}(f) \leq \frac{1}{n} \mathbb{E}_{\mathbb{S}^{n-1}} |\nabla_\sigma f|^2,$$

  for all even functions $f$, which is known to be true, moreover, with constant $\frac{1}{2n} < \frac{1}{n}$.

**Kolesnikov-Milman**

The local version of the Log-Brunn-Minkowski inequality is true when $K = B_p^n$, for all $p \in [2, \infty]$.

**Chen-Huang-Li-Liu; Putterman**

The local version of the Log-Brunn-Minkowski inequality implies the global version of the Log-Brunn-Minkowski inequality!

- However, when $K$ is fixed, no global result follows. The global conjecture

$$|\lambda K +_0 (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}$$

with arbitrary symmetric $L$ is not known for any $K$.

- Could one prove the Local Log BM for some nice "speed function" $f$, for all $K$? (one such answer will come after two slides...)

- (Kolesnikov-Milman) The Local Log BM inequality

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle} \leq \frac{\left(\int_{\partial K} f\right)^2}{|K|}$$

  is invariant under $f \to f + t\langle x, n_x \rangle$.

- (Putterman) Therefore, it is equivalent to the strengthening of Minkowski's second inequality

$$n(n-1)V_2(K, M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x \rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K, M)^2}{|K|}.$$

- Furthermore, the Local (and global) Log BM is invariant under linear transformations.
- In the case of Log-Brunn-Minkowski conjecture, the invariance under $f \to f + t\langle x, n_x \rangle$ corresponds to the invariance of the global version under $L \to tL$, while the invariance under $f \to sf$ corresponds to "time change".

**Example: $K = B^n_\infty$; it was previously known to Emanuel Milman**

- The inequality

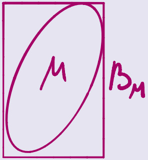
$$n(n-1)V_2(K,M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x\rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K,M)^2}{|K|}$$

becomes (using symmetry!)

$$n(n-1)V_2(B^n_\infty, M) + 2\cdot 2^{n-1}\sum_{i=1}^n h_M^2(e_i) \leq 2^{-4}\cdot 4\cdot 2^{2n-2}\left(\sum_{i=1}^n h_M(e_i)\right)^2.$$

- Mixed volumes are monotone, thus $V_2(B^n_\infty, M) \leq V_2(B^n_\infty, B_M)$, where $B_M$ is the parallelepiped with sides $2h_M(e_1), ..., 2h_M(e_n)$.
- $n(n-1)V_2(B^n_\infty, B_M) = 4\cdot 2^{n-2}\sum_{i\neq j} h_M(e_i)h_M(e_j)$.
- Thus the inequality boils down to an equality

$$\left(\sum_{i=1}^n h_M(e_i)\right)^2 = \left(\sum_{i=1}^n h_M(e_i)\right)^2.$$



# The local Log-Brunn-Minkowski inequality for interval $M$

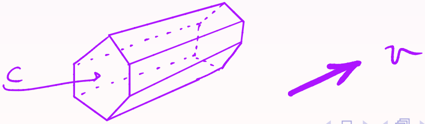
**Theorem (Kolesnikov, L, 2020+)**

For every symmetric convex bounded set $K$ in $\mathbb{R}^n$ with non-empty interior, for $f(x) = t\langle x, n_x\rangle + |\langle v, n_x\rangle|$, for any $t \in \mathbb{R}$ and any $v \in \mathbb{R}^n$, the Local Log-Brunn-Minkowski inequality is true. In other words,

$$\int_{\partial K} H_x \langle n_x, v\rangle^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} |\langle n_x, v\rangle|, \nabla_{\partial K} |\langle n_x, v\rangle|\rangle + \frac{\langle n_x, v\rangle^2}{\langle x, n_x\rangle} \leq$$

$$\frac{1}{|K|} \left( \int_{\partial K} |\langle n_x, v\rangle| \right)^2 .$$

Furthermore, the equality is attained if and only if $K = C + [-v, v]$ for some symmetric convex $C \subset w^\perp$, for some vector $w \in \mathbb{R}^n \setminus v^\perp$.



# The local Log-Brunn-Minkowski inequality for interval $M$

## Proof

- Recall the support function of an interval: $h_{[-v,v]}(u) = |\langle u, v\rangle|$;
- By invariance properties, it is enough to show that

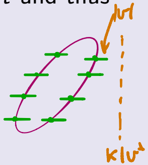
$$n(n-1)V_2(K,M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x\rangle} \le \frac{n^2 V_1(K,M)^2}{|K|}$$

  is true when $M = [-v, v]$, for any vector $v \in \mathbb{R}^n$;
- Cauchy's projection formula:

$$nV_1(K,[-v,v]) = |K + t[-v,v]|'_{t=0} = 2|K|v^{\perp}| = \int_{\partial K} |\langle n_x, v\rangle|;$$

- The function $|K + t[-v,v]| = |K| + 2t|v| \cdot |K|v^{\perp}|$ is linear in $t$ and thus $n(n-1)V_2(K,M) = |K + t[-v,v]|''_{t=0} = 0.$
- Our goal rewrites:

$$\int_{\mathbb{S}^{n-1}} \frac{|\langle u, v\rangle|^2}{h_K(u)} dS_K(u) \le \frac{4|K|v^{\perp}|^2}{|K|}.$$

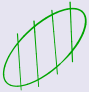


# The local Log-Brunn-Minkowski inequality for interval $M$

**Proof (continued)**

- Goal: $\int_{\mathbb{S}^{n-1}} \left(\frac{|\langle u,v\rangle|}{h_K(u)}\right) |\langle u,v\rangle| dS_K(u) \leq \frac{4|K|v^{\perp}|^2}{|K|}$
- By Fubbini's theorem, for every $u \in \mathbb{S}^{n-1}$, $|K| = \int_{-h_K(u)}^{h_K(u)} |K \cap (u^{\perp} + tu)| dt$, and thus

$$\frac{1}{h_K(u)} \leq \frac{2}{|K|} |K \cap u^{\perp}|.$$



- Since the projection of a subset is smaller than the projection of a set,

$$\frac{|\langle u,v\rangle|}{h_K(u)} \leq \frac{2}{|K|} |K \cap u^{\perp}| \cdot |\langle u,v\rangle| = \frac{2}{|K|} |K \cap u^{\perp}|v^{\perp}| \leq \frac{2}{|K|} |K|v^{\perp}|.$$

- We conclude

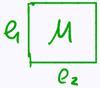
$$\int_{\mathbb{S}^{n-1}} \left(\frac{|\langle u,v\rangle|}{h_K(u)}\right) |\langle u,v\rangle| dS_K(u) \leq \frac{2|K|v^{\perp}|}{|K|} \int_{\mathbb{S}^{n-1}} |\langle u,v\rangle| dS_K = \frac{4|K|v^{\perp}|^2}{|K|}.$$

(the last passage is the Cauchy's projection formula again.) $\square$

# Question: what if $M$ is a 2-dimensional square?

When $M=[-e_1,e_1]\times[-e_2,e_2]$, the Local Log BM inequality

$$n(n-1)V_2(K,M)+\int_{\partial K}\frac{h_M^2}{\langle x,n_x\rangle}dH_{n-1}(x)\le\frac{n^2V_1(K,M)^2}{|K|}$$




becomes

**Conjecture: Local Log BM when $M$ is a square**

$$8|K|span(e_1,e_2)^{\perp}|+\int_{\mathbb{S}^{n-1}}\frac{(|u_1|+|u_2|)^2}{h_K(u)}dS_K(u)\le\frac{4\left(|K|e_1^{\perp}|+|K|e_2^{\perp}|\right)^2}{|K|}.$$

- This does not reduce to the case of one interval:( Example – hexagon on the plane which is close to the square.

**Observation**

If the above inequality is true for all $K$, then the Local Log-Brunn-Minkowski inequality holds whenever

- $K$ is any symmetric convex body and $M$ is a zonoid (limit of a sum of intervals), or
- $K$ is a zonoid and $M$ is any symmetric convex body.

# Another inequality

## Remark

Suppose the Log-Brunn-Minkowski conjecture holds. Then, for all symmetric convex $K$ and $M$,

$$n(n-1)V_2(K,M)+\int_{\partial K}\frac{h_M^2(n_x)}{\langle x,n_x\rangle}\leq\frac{n^2V_1(K,M)^2}{|K|}.$$

Since $V_2(K,M)\geq 0$, this implies

$$\int_{\partial K}\frac{h_M^2(n_x)}{\langle x,n_x\rangle}\leq\frac{n^2V_1(K,M)^2}{|K|}.$$

Equivalently,

$$\int_{\partial K}\frac{\|n_x\|^2}{\langle x,n_x\rangle}\leq\frac{1}{|K|}\left(\int_{\partial K}\|n_x\|\right)^2,$$

where $\|\cdot\|=\|\cdot\|_{M^\circ}$ is a semi-norm.

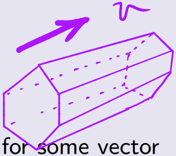
Question: Is the above inequality true?

# Another inequality

## Theorem (Kolesnikov, L, 2020+)

For any symmetric convex bounded set $K$ in $\mathbb{R}^n$ with non-empty interior, and any semi-norm $\|\cdot\|$ on $\mathbb{R}^n$, we have

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x\rangle} \leq \frac{1}{|K|}\left(\int_{\partial K} \|n_x\|\right)^2.$$




Furthermore, the equality occurs if and only if $\|\cdot\| = |\langle \cdot, v\rangle|$, for some vector $v \in \mathbb{R}^n$, and $K = C + [-v, v]$ for some symmetric convex $C \subset w^{\perp}$, for some vector $w \in \mathbb{R}^n \setminus v^{\perp}$.

## Sketch of the proof

Recall that any semi-norm there exists a set $\Omega$ such that

$$\|u\| = \sup_{v\in\Omega} |\langle u, v\rangle|.$$

Similarly to the previous proof, we show

$$\int_{\partial K} \frac{\sup_{v\in\Omega} |\langle n_x, v\rangle|^2}{\langle x, n_x\rangle} \leq \frac{1}{|K|}\left(\int_{\partial K} \sup_{v\in\Omega} |\langle n_x, v\rangle|\right)^2.$$

**Theorem (Kolesnikov, L, 2020+)**

For any symmetric convex bounded set $K$ in $\mathbb{R}^n$ with non-empty interior, and any semi-norm $\|\cdot\|$ on $\mathbb{R}^n$, we have

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x\rangle} \leq \frac{2C_{poin}(K)}{inrad(K)} \cdot \frac{1}{|K|} \left(\int_{\partial K} \|n_x\|\right)^2,$$

where $inrad(K)$ is the radius of the largest ball inside $K$, and $C_{poin}(K) = \inf_{v:1-Lip} \sqrt{Var_K(v)}$.

Note that $\frac{2C_{poin}(K)}{inrad(K)} < 1$ e.g. for $K = B_2^n$, in which case this estimate beats the previous estimate.

## Lemma

Let $K$ be $C^2$-smooth strictly convex body in $\mathbb{R}^n$. Let $\|\cdot\|$ be an arbitrary semi-norm in $\mathbb{R}^n$. Let $u : K \to \mathbb{R}$ be any $C^2$ function such that $\langle \nabla u, n_x \rangle = \|n_x\|$ for all $x \in \partial K$. Then

$$\int_K \|\nabla^2 u\|_{HS}^2 \leq \int_K (\Delta u)^2.$$

## Proof of the Lemma

- Recall (Kolesnikov, Milman): when $\langle \nabla u, n_x \rangle = f : \partial K \to \mathbb{R}$,

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \leq \int_K (\Delta u)^2 - \|\nabla^2 u\|^2.$$

- When $f(x) = \|n_x\| = \|n_x\|_{M^\circ}$, we have

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle = |K + tM|_0'' = n(n-1) V_2(K, M) \geq 0. \square$$

## Proof

- Let $u : K \to \mathbb{R}$ be the solution of the Neumann system
$$\langle \nabla u, n_x \rangle = \|n_x\|, \; x \in \partial K,$$
and
$$\Delta u = \frac{\int_{\partial K} \|n_x\|}{|K|}.$$
- $\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{1}{r} \int_{\partial K} |\nabla u| \langle \nabla u, n_x \rangle$, where $r = inrad(K) = \min \langle x, n_x \rangle$ and we used that $\langle \nabla u, n_x \rangle \geq 0$.
- For any $\alpha, \beta > 0$,
$$div(|\nabla u| \nabla u) = \Delta u |\nabla u| + \langle \nabla^2 u \frac{\nabla u}{|\nabla u|}, \nabla u \rangle \leq$$
$$\frac{\alpha}{2} (\Delta u)^2 + \frac{1}{2\alpha} |\nabla u|^2 + \frac{\beta}{2} \|\nabla^2 u\|_{HS}^2 + \frac{1}{2\beta} |\nabla u|^2.$$

# Proof (continued)

## Proof

- Thus, by divergence theorem, we get

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{1}{r} \int_K \frac{\alpha}{2}(\Delta u)^2 + \frac{1}{2\alpha}|\nabla u|^2 + \frac{\beta}{2}\|\nabla^2 u\|_{HS}^2 + \frac{1}{2\beta}|\nabla u|^2 dx \leq$$

$$\frac{1}{r} \int_K \frac{\alpha}{2}(\Delta u)^2 + \left( \frac{C_{poin}^2}{2\alpha} + \frac{\beta}{2} + \frac{C_{poin}^2}{2\beta} \right) \|\nabla^2 u\|_{HS}^2,$$

  where in the last line we used the Poincare inequality coordinate-wise for $\nabla u$, in view of the fact that $u$ is even and thus $\int_K \nabla u = 0$.

- We let $\alpha = \beta = C_{poin}$, and use the Lemma

$$\int_K \|\nabla^2 u\|_{HS}^2 \leq \int_K (\Delta u)^2,$$

  in order to conclude

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{2C_{poin}}{r} \cdot \int_K (\Delta u)^2.$$

# Proof (continued)

## Proof

- It remains to recall that $\Delta u$ is a constant function, and thus

$$\int_K (\Delta u)^2 dx = \frac{(\int_K \Delta u)^2}{|K|} = \frac{\left(\int_{\partial K} \|n_x\|\right)^2}{|K|},$$

  where in the last passage, the Divergence Theorem was used.

- We conclude that

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{2C_{poin}}{r} \cdot \frac{\left(\int_{\partial K} \|n_x\|\right)^2}{|K|},$$

  and the theorem follows. □

### $L_p$-Minkowski sum (Definition)

$$\lambda K +_p (1-\lambda)L = \bigcap_{u\in\mathbb{S}^{n-1}} \{x\in\mathbb{R}^n : |\langle u,x\rangle|^p \le \lambda h_K(u)^p + (1-\lambda)h_L(u)^p\}.$$

### $L_p$-Brunn-Minkowski conjecture (Böröczky, Lutwak, Yang, Zhang 2011)

For **origin-symmetric convex** sets $K$ and $L$ in $\mathbb{R}^n$, for $p\in[0,1]$

$$|\lambda K +_p (1-\lambda)L| \ge |K|^\lambda |L|^{1-\lambda}.$$

### Equivalently, by homogeneity (and/or the earlier story)

$$|\lambda K +_p (1-\lambda)L|^{\frac{p}{n}} \ge \lambda |K|^{\frac{p}{n}} + (1-\lambda)|L|^{\frac{p}{n}}.$$

- For $p\in[0,1]$,

$$\lambda K +_0 (1-\lambda)L \subset \lambda K +_p (1-\lambda)L \subset \lambda K + (1-\lambda)L.$$

- The conjecture interpolates between the Log-Brunn-Minkowski conjecture ($p=0$) and the Brunn-Minkowski inequality ($p=1$).

# The $L_p$-Brunn-Minkowski conjecture

- Kolesnikov-Milman developed the local version of the $L_p$-Brunn-Minkowski inequality

$$n(n-1)V_2(K,M) + (1-p)\int_{\partial K} \frac{h_M^2}{\langle x, n_x\rangle} dH_{n-1}(x) \leq \frac{n-p}{n} \frac{n^2 V_1(K,M)^2}{|K|}$$

- Kolesnikov-Milman: true for $p \in [1-cn^{-1.5}, 1]$!
- Chen-Huang-Li-Liu: local implies global (with equality cases)
- Putterman: local implies global (simple and useful proof)
- Conclusion: the $L_p$-Brunn-Minkowski conjecture

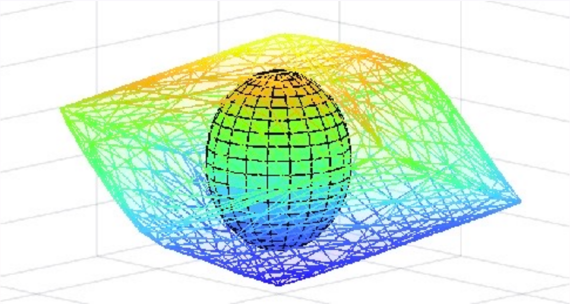
$$|\lambda K +_p (1-\lambda)L|^{\frac{p}{n}} \geq \lambda |K|^{\frac{p}{n}} + (1-\lambda)|L|^{\frac{p}{n}}.$$

is true when $p \in [1-cn^{-1.5}, 1]$!

**Theorem (Hosle, Kolesnikov, L 2020+)**

For origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$ such that $K \subset L$, for $p \in [1 - cn^{-0.75}, 1]$

$$|\lambda K +_p (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}.$$



Remark: note that this is not the dilation-invariant version.

# Log-concave measures: preliminaries

### Log-concave functions

A function is called log-concave if its logarithm is concave, i.e.
$f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$.

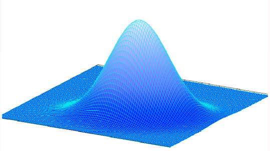
### Log-concave measures

A measure $\mu$ is called log-concave if $\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda}\mu(L)^{1-\lambda}$.

### Borell's theorem (which implies Brunn-Minkowski)

A measure with log-concave density is log-concave.

- Gaussian measure $\gamma$ with density $\frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}}$;
- Lebesgue measure;
- Poisson density...



**Theorem (Saraglou 2014)**

If the $L_p$-Brunn-Minkowski conjecture holds for some $p \in [0,1]$, then for any even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

$$\mu(\lambda K +_p (1-\lambda)L) \geq \mu(K)^{\lambda}\mu(L)^{1-\lambda}.$$

- Considering the case $p = 0$ and $K = aL$, note that the above would imply the B-conjecture of Banazchyk-Latala, posed in the 1990s.
- Cordero-Fradelizi-Maurey 2008: true when $K = aL$ and $\mu$ is Gaussian.
- In the absence of homogeneity, the inequality no longer improves to an additive version...

# The $L_p$-Brunn-Minkowski conjecture for measures

**Theorem (Saraglou 2014)**

If the $L_p$-Brunn-Minkowski conjecture holds for some $p \in [0,1]$, then for any even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

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- Considering the case $p = 0$ and $K = aL$, note that the above would imply the B-conjecture of Banazchyk-Latala, posed in the 1990s.
- Cordero-Fradelizi-Maurey 2008: true when $K = aL$ and $\mu$ is Gaussian.
- In the absence of homogeneity, the inequality no longer improves to an additive version... Except it does!

**L, Marsiglietti, Nayar, Zvavitch 2017**

If the Log-Brunn-Minkowski conjecture holds, then for every even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

$$\mu(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\mu(K)^{\frac{1}{n}} + (1-\lambda)\mu(L)^{\frac{1}{n}}.$$

**Theorem (Hosle, Kolesnikov, L 2020+)**

If the Log-Brunn-Minkowski conjecture holds, then for every even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

$$\mu(\lambda K +_p (1-\lambda)L)^{\frac{p}{n}} \geq \lambda\mu(K)^{\frac{p}{n}} + (1-\lambda)\mu(L)^{\frac{p}{n}}. \qquad (3)$$

Moreover, (3) strengthens when $p$ decreases.

**Conjecture (Gardner, Zvavitch 2007)**

For an **even** log-concave measure $\mu$, and symmetric convex sets $K$ and $L$,

$$\mu(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\mu(K)^{\frac{1}{n}} + (1-\lambda)\mu(L)^{\frac{1}{n}}.$$

- Tkocz-Nayar: the symmetry assumption cannot be replaced by simply origin in the interior, even in the Gaussian case.

**Theorem (Kolesnikov, L 2018)**

For the Gaussian measure $\mu$, and convex sets $K$ and $L$ containing the origin,

$$\mu(\lambda K + (1-\lambda)L)^{\frac{1}{2n}} \geq \lambda\mu(K)^{\frac{1}{2n}} + (1-\lambda)\mu(L)^{\frac{1}{2n}}.$$

- Eskenazis-Moschidis: for the Gaussian measure $\gamma$ and for symmetric convex sets $K$ and $L$,

$$\gamma(\lambda K+(1-\lambda)L)^{\frac{1}{n}} \geq \lambda\gamma(K)^{\frac{1}{n}}+(1-\lambda)\gamma(L)^{\frac{1}{n}}.$$

**Theorem (Kolesnikov, L 2020+)**

Fix $a \in [0,1]$. For the Gaussian measure $\gamma$ and for symmetric convex sets $K$ and $L$ with $\gamma(K) \geq a$, $\gamma(L) \geq a$,

$$\gamma(\lambda K+(1-\lambda)L)^{\frac{1}{n-F(a)}} \geq \lambda\gamma(K)^{\frac{1}{n-F(a)}}+(1-\lambda)\gamma(L)^{\frac{1}{n-F(a)}},$$

where $\frac{1}{n-F(a)} \to_{a\to 1} \infty$.

**Remarks**

- The power in this inequality tends to infinity even in the non-symmetric case, as is implied by Ehrhard's inequality (but the above is not related to Ehrhard's inequality).
- The function $F(a) := \frac{1}{a} J_{n+1} \circ J_{n-1}^{-1}(a)$, where $J_p(R) := \frac{\int_0^R t^p e^{-\frac{t^2}{2}}\,dt}{\int_0^\infty t^p e^{-\frac{t^2}{2}}\,dt}$, and the rate in the previous theorem is optimal up to a dimensional constant.

Moreover,

**Theorem (Kolesnikov, L 2020+)**

Fix $a \in [0,1]$. Let $\mu$ be a log-concave measure with twice-differentiable density $e^{-V}$, and suppose $\nabla^2 V$ is uniformly strictly non-singular everywhere. Then for symmetric convex sets $K$ and $L$ with $\mu(K) \geq a$, $\mu(L) \geq a$,

$$\mu(\lambda K + (1-\lambda)L)^{p(a)} \geq \lambda \mu(K)^{p(a)} + (1-\lambda)\mu(L)^{p(a)},$$

where $p(a) \to_{a\to 1} \infty$.

Additionally,

**Theorem (Kolesnikov, L 2020+)**

Let $\mu$ be the measure with density $C_n e^{-\|x\|_1}$. Then for symmetric convex sets $K$ and $L$,

$$\mu(\lambda K + (1-\lambda)L)^{\frac{c}{n\log n}} \geq \lambda \mu(K)^{\frac{c}{n\log n}} + (1-\lambda)\mu(L)^{\frac{c}{n\log n}}.$$

The previous result utilizes a deep result of Barthe, Klartag.

**Theorem (Hosle, Kolesnikov, L 2020+)**

Let $\gamma$ be the Gaussian measure, and let $K$ and $L$ be symmetric convex sets containing the ball $rB_2^n$. Then for any $\lambda > 0$,

1. $\gamma(\lambda K +_p (1-\lambda)L) \geq \gamma(K)^{\lambda}\gamma(L)^{1-\lambda}$, whenever $p \geq 0$ and
$$p \geq 1 - \frac{2r^2}{n+1}.$$
2. In particular, the Gaussian Log-Brunn-Minkowski inequality holds for all convex sets $K$ and $L$ containing $\sqrt{0.5(n+1)}B_2^n$.
3. More generally, $\gamma(\lambda K +_p (1-\lambda)L)^{\frac{q}{n}} \geq \lambda\gamma(K)^{\frac{q}{n}} + (1-\lambda)\gamma(L)^{\frac{q}{n}}$, provided that
$$4q + \frac{n+1}{r^2}(1-p) \leq 2.$$
4. Assuming further that $K \subset L$, we show that $\gamma(\lambda K +_p (1-\lambda)L) \geq \gamma(K)^{\lambda}\gamma(L)^{1-\lambda}$, whenever $p \geq 0$ and $p \geq 1 - \frac{r}{\sqrt{n}+0.25}$.

- In one of the steps of the proof, we deduced the "local to global" result for general measures, following the approach of Putterman.

Thanks for your attention!