Valuations on Convex Functions

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Valuations on Function Spaces

- $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ space of real valued functions on X
- $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$
- $\langle \mathbb{A}, + \rangle$ abelian semigroup

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- $\mathrm{Z} : \mathcal{F}(X) \to \mathbb{A}$ is a **valuation** $\iff$

$$\mathrm{Z}(f) + \mathrm{Z}(g) = \mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g)$$

for all $f, g \in \mathcal{F}(X)$ such that $f \vee g, f \wedge g \in \mathcal{F}(X)$.

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Examples

- L^1 norm: $\mathrm{Z} : \begin{cases} L^1(\mathbb{R}^n) \to \mathbb{R} \\ f \mapsto \|f\|_1 = \int_{\mathbb{R}^n} |f(x)|\, dx \end{cases}$
- Dirichlet energy: $\mathrm{Z} : \begin{cases} W^{1,2}(\mathbb{R}^n) \to \mathbb{R} \\ f \mapsto \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx \end{cases}$

Valuations on Function Spaces

- Valuations on Sobolev and BV functions:
  L.: AIM 2011, AJM 2012; Wang: IUMJ 2014; Ma: SCM 2016
- Valuations on L_p and Orlicz functions:
  Tsang: IMRN 2010, TAMS 2012; L.: AAM 2013;
  Ober: JMAA 2014; Kone: AAM 2014; Li & Ma: JFA 2017
- **Valuations on convex functions:**
  Cavallina & Colesanti: AGMS 2015; Colesanti, L. & Mussnig: CVPDE 2017, IMRN 2019, IUMJ 2020, JFA 2020, 2020+; Alesker: AG 2019; Mussnig: AiM 2019; CJM 2020; Knoerr 2020+
- Valuations on quasi-concave functions:
  Colesanti & Lombardi: 2017; Colesanti, Lombardi & Parapatits: 2017
- Valuations on Banach lattices and on Lipschitz functions:
  Tradacete, Villanueva: IMRN 2020; Colesanti, Pagnini, Tradacete, Villanueva: AiM 2020
- Valuations on approximable and on definable functions:
  Groemer: Math. Ann. 1972; Baryshnikov, Ghrist & Wright: AIM 2013

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Question:

- Classification of interesting valuations on $\mathcal{F}(X)$

Felix Klein's Erlangen Program 1872

Geometry is the study of invariants of transformation groups.

Groups G acting on $\mathbb{R}^n$

- Group of rigid motions $\mathrm{O}(n) \ltimes \mathbb{R}^n$: $x \mapsto Ux + b$
  where U is an orthogonal $n \times n$ matrix and $b \in \mathbb{R}^n$
- Special linear group $\mathrm{SL}(n)$: $x \mapsto Ax$
  where A is an $n \times n$ matrix of determinant 1
- Special affine group $\mathrm{SL}(n) \ltimes \mathbb{R}^n$: $x \mapsto Ax + b$
  where A is an $n \times n$ matrix of determinant 1 and $b \in \mathbb{R}^n$

Invariance: $\mathrm{Z}(f \circ \phi^{-1}) = \mathrm{Z}(f)$ for all $f \in \mathcal{F}(\mathbb{R}^n)$ and $\phi \in G$

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
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$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

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- Hilbert's Third Problem:
  Dehn 1902, Sydler 1965, Jessen & Thorup 1978, ...

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- Hilbert's Third Problem:
  Dehn 1902, Sydler 1965, Jessen & Thorup 1978, ...
- Classification of valuations:

Blaschke 1937, **Hadwiger** 1949, Schneider 1971, Groemer 1972, McMullen 1977, Betke & Kneser 1985, Klain 1995, Ludwig 1999, Reitzner 1999, Alesker 1999, Bernig 2006, Fu 2006, Hug 2005, Haberl 2006, Schuster 2006, Tsang 2010, Wannerer 2010, Abardia 2011, Parapatits 2011, Faifman 2013, Solanes 2014, Böröczky 2015, Li 2016, Ma 2016, Mussnig 2017, Zeng 2018, ...

Rigid Motion Invariant Valuations

Theorem (Hadwiger 1952)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists \; c_0, c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

- $V_0(K), \dots, V_n(K)$ intrinsic volumes of K
- $V_n(K)$ volume
- $2 \, V_{n-1}(K)$ surface area
- $V_0(K)$ Euler characteristic
- Klain 1995, Klain & Rota 1997

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$$\mathrm{Z}(K) = c_0\, V_0(K) + \dots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Corollary

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, i-homogeneous, rigid motion invariant valuation*

$$\Longleftrightarrow$$

$\exists \;\; c \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c\, V_i(K)$$

for every $K \in \mathcal{K}^n$.

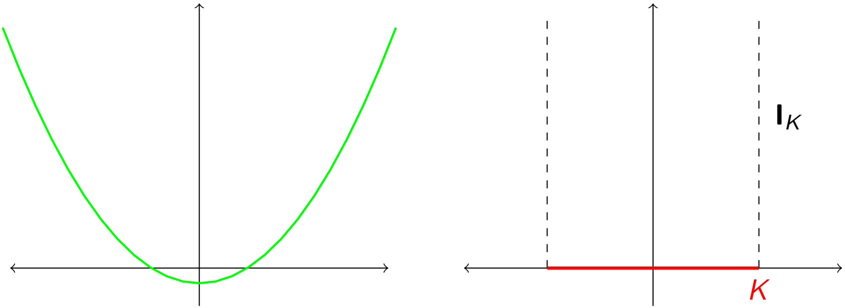
Intrinsic and Mixed Volumes on Function Spaces

- Log-concave functions:
  Klartag, Milman: GD 2005; Rotem: LNM 2012;
  Colesanti, Fragalà: AiM 2013;
- α-concave functions:
  Milman, Rotem: ERAM 2013; Rotem: AiM 2013
- Quasi-concave functions:
  Milman, Rotem: JFA 2013;
  Bobkov, Colesanti, Fragalà: MM 2014

Outline: Valuations on Convex Functions

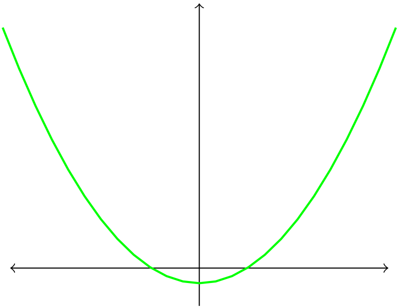
- Real-valued $\mathrm{SL}(n)$ invariant valuations
- Convex-body-valued $\mathrm{SL}(n)$ contravariant valuations
- Translation invariant valuations
- Rigid motion invariant valuations

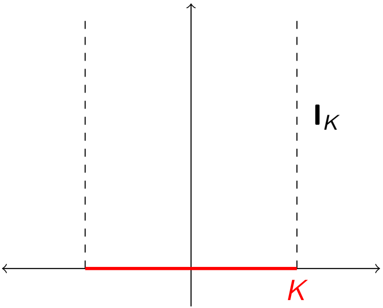
Valuations on Convex Functions



- Convex functions
  $\mathrm{Conv}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to (-\infty, \infty] : u \text{ convex, l.s.c., proper}\}$

Valuations on Convex Functions





- Convex functions
  $\mathrm{Conv}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to (-\infty, \infty] : u \text{ convex, l.s.c., proper}\}$
- u_k is epi-convergent to u in $\mathrm{Conv}(\mathbb{R}^n)$ $\Leftrightarrow$
  - $u(x) \leqslant \liminf_{k\to\infty} u_k(x_k)$ for every (x_k) with $x_k \to x$
  - $\forall x$, $\exists (x_k)$ with $x_k \to x$ such that $u(x) = \lim_{k\to\infty} u_k(x_k)$

SL(n) Invariant Valuations

Theorem (Blaschke 1937)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous,* SL(n) *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists\, c_0, c_n \in \mathbb{R}$*:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

- $V_n(K)$ — n-dimensional volume of K
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- $V_n(K)$ — n-dimensional volume of K
- $V_0(K)$ — Euler characteristic of K
- Versions for valuations on polytopes without translations (Hadwiger 1970) and without continuity (L. & Reitzner 2017)

SL(n) Invariant Valuations on Convex Functions

Theorem (Colesanti, L. & Mussnig: IMRN 2019)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \to [0,\infty)$ *is a continuous,* SL(n) *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists$ *non-negative functions* $\zeta_0 \in C(\mathbb{R})$ *and* $\zeta_n \in A^{n-1}(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n\big(u(x)\big)\, dx$$

for every $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$.

- $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$ coercive, convex functions: $\lim_{|x|\to+\infty} u(x) = +\infty$
- $\mathrm{dom}\, u = \{x \in \mathbb{R}^n : u(x) < \infty\}$
- $A^{n-1}(\mathbb{R}) = \big\{\zeta \in C(\mathbb{R}) : \zeta \geqslant 0, \int_0^\infty t^{n-1}\zeta(t)\, dt < \infty\big\}$.

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for every $K \in \mathcal{K}^n$.

$\mathrm{SL}(n)$ Invariant Valuations on Convex Functions

Theorem (Mussnig: AiM 2019)

$\mathrm{Z} : \mathrm{Conv}_{coe}(\mathbb{R}^n; \mathbb{R}) \to [0, \infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\iff$$

$\exists$ *non-negative functions* $\zeta_0 \in C(\mathbb{R})$, $\zeta_n \in A^{n-1}(\mathbb{R})$, $\zeta_{-n} \in C_b(\mathbb{R})$ *s.t.*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x \in \mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n\big(u(x)\big)\, dx + \int_{\mathbb{R}^n} \zeta_{-n}(\nabla u^*(x) \cdot x - u^*(x))\, dx$$

for every $u \in \mathrm{Conv}_{coe}(\mathbb{R}^n; \mathbb{R})$.

- $\mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n; \mathbb{R})$ finite-valued coercive, convex functions
- $C_b(\mathbb{R})$ continuous functions with support bounded from above
- u^* convex conjugate of u

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for every $u \in \mathrm{Conv}_{coe}(\mathbb{R}^n;\mathbb{R})$.

Theorem (Haberl, Parapatits: JAMS 2014)

$\mathrm{Z} : \mathcal{K}^n_{(0)} \to \mathbb{R}$ *is a continuous and* $\mathrm{SL}(n)$ *invariant valuation*

$$\iff$$

$\exists\ c_0, c_n, c_{-n} \in \mathbb{R}$*:*

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_n\, V_n(K) + c_{-n}\, V_n(K^*)$$

for every $K \in \mathcal{K}^n_{(0)}$.

Minkowski Valuations

Theorem (L.: AiM 2002)

$\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation

$$\iff$$

$\exists\, c \geqslant 0$ such that

$$\mathrm{Z}(K) = c\,\Pi\, K$$

for every $K \in \mathcal{K}^n$.

- Minkowski valuation: $\langle \mathcal{K}^n, + \rangle$ with Minkowski addition
- $\Pi\, K$ projection body of K: $h(\Pi\, K, y) = V_{n-1}(K|y^{\perp})$
- Z is $\mathrm{SL}(n)$ contravariant $\Leftrightarrow \mathrm{Z}(\phi\, K) = \phi^{-t}\,\mathrm{Z}(K)$ for $\phi \in \mathrm{SL}(n)$

Minkowski Valuations on Convex Functions

Theorem (Colesanti, L. & Mussnig: CVPDE 2017)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, monotone,* $\mathrm{SL}(n)$ *contravariant, translation invariant Minkowski valuation*

$$\iff$$

$\exists\, \zeta \in A^{n-2}(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \Pi \langle \zeta \circ u \rangle$$

for every $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$.

- $A^k(\mathbb{R}) = \{\zeta \in C(\mathbb{R}) \,:\, \zeta \geqslant 0,\, \zeta \text{ is decreasing}, \int_0^\infty t^k \zeta(t)\, dt < \infty\}$.
- $h(\Pi \langle \zeta \circ u \rangle, y) = \int_0^{+\infty} h(\Pi\{\zeta \circ u \geqslant t\}, y)\, dt$
- $\{\zeta \circ u \geqslant t\}$ super-level set of $\zeta \circ u$
- $\Pi \langle \zeta \circ u \rangle$ projection body of the Lutwak-Yang-Zhang operator

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$\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant valuation*

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$\exists\, c \geqslant 0$*:*

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for every $K \in \mathcal{K}^n$.

Translation Invariant Valuations

Theorem (Homogeneous decomposition)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation invariant valuation*

$$\Longrightarrow$$

$\exists\ \mathrm{Z}_0, \dots, \mathrm{Z}_n : \mathcal{K}^n \to \mathbb{R}$ *continuous, translation invariant valuations s.t.* Z_i *is i-homogeneous and*

$$\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n\,.$$

- Hadwiger 1945; McMullen, Meier, Spiegel 1977
- Versions for valuations on polytopes without continuity (McMullen)

Translation Invariant Valuations

Theorem (Polynomiality)

$\mathrm{Z}:\mathcal{K}^n\to\mathbb{R}$ *continuous, m-homogeneous, translation invariant valuation*

$$\Longrightarrow$$

$\exists$ *continuous, symmetric map* $\bar{\mathrm{Z}}:(\mathcal{K}^n)^m\to\mathbb{R}$ *translation invariant and Minkowski additve in each variable s.t.*

$$\mathrm{Z}(\lambda_1 K_1+\cdots+\lambda_k K_k)=\sum_{\substack{i_1,\dots,i_k\in\{0,\dots,m\}\\ i_1+\cdots+i_k=m}}\binom{m}{i_1\,\dots\,i_k}\lambda_1^{i_1}\dots\lambda_k^{i_k}\,\bar{\mathrm{Z}}(K_1\,[i_1],\dots,K_k\,[i_k])$$

for $K_1,\dots,K_k\in\mathcal{K}^n$ *and* $\lambda_1,\dots,\lambda_k\geqslant 0$.

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for $K_1,\ldots,K_k\in\mathcal{K}^n$ *and* $\lambda_1,\ldots,\lambda_k\geqslant 0$.

Corollary

$\mathrm{Z}:\mathcal{K}^n\to\mathbb{R}$ *continuous, 1-homogeneous, translation invariant valuation*

$$\Longrightarrow$$

Z *is Minkowski additive*

Translation Invariant Valuations

Theorem (Hadwiger 1957)

$\mathrm{Z}: \mathcal{K}^n \to \mathbb{R}$ *is a continuous, n-homogeneous, translation invariant valuation*

$$\Longleftrightarrow$$

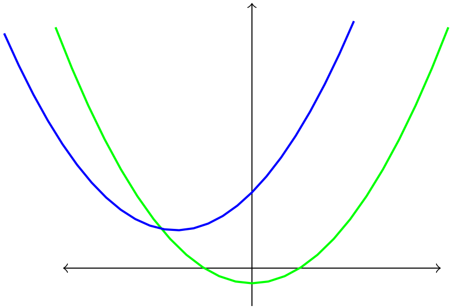
$\exists\, c \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c\, V_n(K)$$

for $K \in \mathcal{K}^n$.

Epi-translation Invariant Valuations

- $\mathcal{F}(\mathbb{R}^n) \subset \mathrm{Conv}(\mathbb{R}^n)$
- $\mathrm{Z} : \mathcal{F}(\mathbb{R}^n) \to \mathbb{R}$ is epi-translation invariant
  $\Leftrightarrow \mathrm{Z}(u \circ \tau^{-1} + c) = \mathrm{Z}(u)$ for all translations $\tau : \mathbb{R}^n \to \mathbb{R}^n$ and $c \in \mathbb{R}$



Epi-translation Invariant Valuations

Theorem (Colesanti, L. & Mussnig: IMRN 2019)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \to [0, \infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation*

$$\Longleftrightarrow$$

$\exists$ *non-negative functions* $\zeta_0 \in C(\mathbb{R})$ *and* $\zeta_n \in A^{n-1}(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x \in \mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n\big(u(x)\big)\, dx$$

for every $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$.

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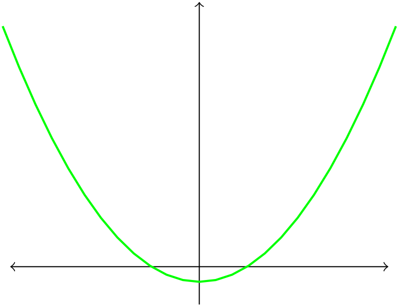
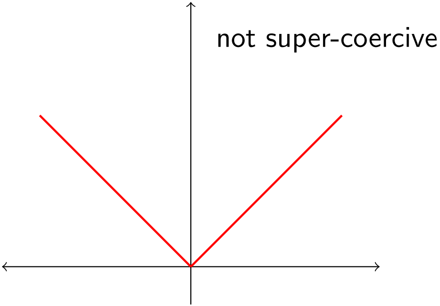
for every $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$.

Theorem (Colesanti, L. & Mussnig: JFA 2020)

Every continuous, epi-translation invariant valuation $\mathrm{Z} : \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n) \to \mathbb{R}$ *is constant.*

Valuations on Super-coercive Convex Functions





- $u \in \mathrm{Conv}(\mathbb{R}^n)$ super-coercive
  $\Leftrightarrow \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty$
- $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) = \big\{u \in \mathrm{Conv}(\mathbb{R}^n) : u \text{ super-coercive}\big\}$

Epi-translation Invariant Valuations

- Epi-multiplication: $\lambda \cdot u(x) = \lambda\, u(\frac{x}{\lambda})$
  for $u \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda > 0$

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- Epi-multiplication: $\lambda \cdot u(x) = \lambda\, u(\frac{x}{\lambda})$
  for $u \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda > 0$
- $\mathrm{Z} : \mathcal{F}(\mathbb{R}^n) \to \mathbb{R}$ is epi-homogeneous of degree i
  $\Leftrightarrow \mathrm{Z}(\lambda \cdot u) = \lambda^i\, \mathrm{Z}(u)$ for all $\lambda > 0$

Epi-translation Invariant Valuations

- Epi-multiplication: $\lambda \cdot u(x) = \lambda\, u(\frac{x}{\lambda})$ for $u \in \mathcal{F}(\mathbb{R}^n)$ and $\lambda > 0$
- $\mathrm{Z} : \mathcal{F}(\mathbb{R}^n) \to \mathbb{R}$ is epi-homogeneous of degree i $\Leftrightarrow \mathrm{Z}(\lambda \cdot u) = \lambda^i\, \mathrm{Z}(u)$ for all $\lambda > 0$

Theorem (Colesanti, L. & Mussnig: JFA 2020)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation*

$$\Longrightarrow$$

$\exists\ \mathrm{Z}_0, \dots, \mathrm{Z}_n : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *continuous, epi-translation invariant valuations s.t.* Z_i *is epi-homogeneous of degree* i *and*

$$\mathrm{Z} = \mathrm{Z}_0 + \cdots + \mathrm{Z}_n\,.$$

Translation Invariant Valuations

Theorem (Colesanti, L. & Mussnig: JFA 2020)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *continuous, epi-translation invariant valuation that is epi-homogeneous of degree m*

$$\Longrightarrow$$

$\exists$ *continuous, symmetric map* $\bar{\mathrm{Z}} : (\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n))^m \to \mathbb{R}$ *epi-translation invariant and epi-additve in each variable s.t.*

$$\mathrm{Z}(\lambda_1 \cdot u_1 \square \cdots \square \lambda_k \cdot u_k) = \sum_{\substack{i_1,\dots,i_k \in \{0,\dots,m\} \\ i_1+\dots+i_k=m}} \binom{m}{i_1 \dots i_k} \lambda_1^{i_1} \dots \lambda_k^{i_k} \, \bar{\mathrm{Z}}(u_1[i_1], \dots, u_k[i_k])$$

for $u_1, \dots, u_k \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ *and* $\lambda_1, \dots, \lambda_k \geqslant 0$.

- Inf-convolution:
  $u \square v(z) = \inf\{u(x) + v(y) : x, y \in \mathbb{R}^n, x + y = z\}$

Translation Invariant Valuations

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Corollary

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *continuous, epi-translation invariant valuation that is epi-homogeneous of degree 1* $\Longrightarrow$ Z *is epi-additive*

Epi-translation Invariant Valuations

Theorem (Colesanti, L. & Mussnig: JFA 2020)

$\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation that is epi-homogeneous of degree n*

$$\Longleftrightarrow$$

$\exists\, \zeta \in C_c(\mathbb{R}^n)$ *such that*

$$\mathrm{Z}(u) = \int_{\mathrm{dom}(u)} \zeta(\nabla u(x))\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Epi-translation Invariant Valuations

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$\mathrm{Z} : \mathrm{Conv_{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation invariant valuation that is epi-homogeneous of degree n*

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Theorem (Hadwiger 1957)

$\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation invariant valuation that is homogeneous of degree n*

$$\iff$$

$\exists\, c \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Epi-translation Invariant Valuations

Theorem (Colesanti, L. & Mussnig: JFA 2020)

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Theorem (Colesanti, L. & Mussnig: JFA 2020)

$\mathrm{Z}:\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\to\mathbb{R}$ *is a continuous, epi-translation invariant valuation that is epi-homogeneous of degree* 0

$$\Longleftrightarrow$$

$\exists\,c\in\mathbb{R}$ *such that*

$$\mathrm{Z}(u)=c$$

for every $u\in\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Duality

- Legendre transform:

$$u^*(y) = \sup_{x\in\mathbb{R}^n} (\langle x, y\rangle - u(x))$$

- $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) = \{u \in \mathrm{Conv}(\mathbb{R}^n) : u(x) < +\infty$ for all $x \in \mathbb{R}^n\}$
- $^* : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ continuous bijection

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- $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ continuous valuation
  $\Rightarrow \mathrm{Z}^* : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}^*(u) = \mathrm{Z}(u^*),$$

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- $\mathrm{Z}^* : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ epi-translation invariant
  $\Rightarrow \mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ dually epi-translation invariant:

$$\mathrm{Z}(v + \ell + c) = \mathrm{Z}(v)$$

  for all linear functions $\ell : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$

Dually Epi-translation Invariant Valuations

- Alesker 2019: $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$, defined by

$$\mathrm{Z}(v) = \int_{\mathbb{R}^n} \zeta(x) \det(\mathrm{D}^2 v(x)[i], A_1(x), \dots, A_{n-i}(x)) \,\mathrm{d}x$$

for $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C^2(\mathbb{R}^n)$ with $A_1, \dots, A_{n-i}$ continuous, symmetric $n \times n$ matrix-valued functions on $\mathbb{R}^n$ with compact support and $\zeta \in C_c(\mathbb{R}^n)$, extends to a continuous, i-homogeneous, dually epi-translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

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- $\mathrm{Z}^* : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ continuous, epi-translation invariant valuation
  Examples for every degree of epi-homogeneity $i \in \{0, \ldots, n\}$

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  Examples for every degree of epi-homogeneity $i \in \{0, \ldots, n\}$
- For $K \in \mathcal{K}^n$, the support function h_K is in $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and the convex indicator function $\mathbf{I}_K$ is in $\mathrm{Conv_{sc}}(\mathbb{R}^n)$.

Hessian Valuations

Theorem (Colesanti, L. & Mussnig: IUMJ 2020)

Let $\zeta \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ have compact support w.r.t. 2nd and 3rd variable. For $i \in \{0, 1, \dots, n\}$, the functional $\mathrm{Z}_{\zeta,i} \colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(u(x), x, y)\, d\Theta_i(u, (x, y)),$$

is a continuous valuation on $\mathrm{Conv}(\mathbb{R}^n)$. If $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n} \zeta(u(x), x, \nabla u(x))\, [\mathrm{D}^2 u(x)]_{n-i}\, dx.$$

- $\Theta_i(u, \cdot)$ Hessian measure of u
- $\mathrm{D}^2 u$ Hessian matrix of u
- $[\mathrm{D}^2 u]_j$ jth elementary symmetric function of the eigenvalues of $\mathrm{D}^2 u$

Hessian Measures

- On smooth functions: $[\mathrm{D}^2 u(x)]_j\, dx$
  Caffarelli, Nirenberg, Spruck: Acta 1985;
  Trudinger, Wang: Annals 1999; ...
- On (semi-)convex, finite-valued functions:
  Colesanti, Hug: TAMS 2000
- Extension to $\mathrm{Conv}(\mathbb{R}^n)$ using Lipschitz regularization:
  Colesanti, L. & Mussnig: IUMJ 2020

Hessian Valuations

Theorem (Colesanti, L. & Mussnig: IUMJ 2020)

Let $\zeta \in C([0,+\infty))$ have compact support. For every $i \in \{0,\dots,n\}$, the functional $\mathrm{Z}_{\zeta,i}\colon \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, defined by

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \zeta(|y|)\, d\Theta_i(u,(x,y)),$$

is a continuous, rotation and epi-translation invariant valuation. If $u \in \mathrm{Conv}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$, then

$$\mathrm{Z}_{\zeta,i}(u) = \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)\, [\mathrm{D}^2 u(x)]_{n-i}\, dx.$$

Rotation and Epi-translation Invariant Valuations

Theorem (Colesanti, L. & Mussnig 2020+)

$\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, rotation and epi-translation invariant valuation that is epi-homogeneous of degree 1*

$$\iff$$

$\exists\, \zeta \in D_1^n$ *such that*

$$\mathrm{Z}(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(|y|)\, \mathrm{d}\Theta_1(u,(x,y))$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

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- D_1^n set of functions $\zeta \in C((0,\infty))$ with bounded support s.t.
  - $\lim_{r\to 0+} r^{n-1}\zeta(r) = 0$
  - $\lim_{r\to 0+} \int_r^{+\infty} s^{n-2}\zeta(s)\,\mathrm{d}s$ exists and is finite

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- $$\int_{\mathbb{R}^n \times \mathbb{R}^n} \zeta(|y|)\,\mathrm{d}\Theta_1(u,(x,y)) = \lim_{r\to 0^+} \int_{\mathbb{R}^n \times \{|y|>r\}} \zeta(|y|)\,\mathrm{d}\Theta_1(u,(x,y))$$

Rotation and Epi-translation Invariant Valuations

Theorem (Colesanti, L. & Mussnig 2020+)

$\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^2) \to \mathbb{R}$ *is a continuous, rotation and epi-translation invariant valuation*

$$\Longleftrightarrow$$

$\exists\, \zeta_0 \in \mathbb{R}, \zeta_1 \in D_1^2$ *and* $\zeta_2 \in C_c([0,\infty))$ *such that*

$$\mathrm{Z}(u) = \zeta_0 + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \zeta_1(|y|)\,\mathrm{d}\Theta_1(u,(x,y)) + \int_{\mathrm{dom}\, u} \zeta_2(|\nabla u(x)|)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^2)$.

Rotation and Epi-translation Invariant Valuations

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for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^2)$.

Theorem (Hadwiger)

$\mathrm{Z} : \mathcal{K}^2 \to \mathbb{R}$ *is a continuous, rigid motion invariant valuation*

$$\iff$$

$\exists\ c_0, c_1, c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0\, V_0(K) + c_1\, V_1(K) + c_2\, V_2(K)$$

for every $K \in \mathcal{K}^2$.

Thank you!