MATH-8803: Topics Course in Concentration of Measure Phenomena and Convexity, Fall 2023, Georgia Tech

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Abstract

These lecture notes are under construction. This proofreading review preserves the organization of the Fall 2023 course. A small number of statements and exercises are explicitly marked for author clarification; those proposed changes have not been finalized. The original figures and bibliography are included in this review project.

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Notation from linear algebra and the geometry of $\mathbb{R}^n$

- $\mathbb{R}^n$ is n-dimensional Euclidean space; $\langle x, y\rangle$ denotes its scalar product.
- $|x| = \|x\|_2$ is Euclidean length. When convenient, x^2 means $|x|^2 = \langle x, x\rangle$.
- For a measurable set, $|A|$ or $|A|_n$ is its n-dimensional Lebesgue volume. On a k-dimensional affine subspace, $|A|_k$ is the induced k-dimensional volume. Surface area is understood as Hausdorff measure.
- For $1 \leq p < \infty$, $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, while $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$.

- $B_p^n = \{x : \|x\|_p \leq 1\}$; B_2^n is the Euclidean unit ball and $\mathbb{S}^{n-1} = \{x : |x| = 1\}$ is the unit sphere.
- $A + B = \{a + b : a \in A,\ b \in B\}$ is the Minkowski sum.
- For nonzero θ, $\theta^\perp = \{x : \langle x, \theta\rangle = 0\}$. When θ is a unit vector, $\theta^\perp + t\theta = \{x : \langle x, \theta\rangle = t\}$. For general nonzero θ, the latter level set is $\theta^\perp + (t/|\theta|^2)\theta$.
- A half-space has the form $\{x : \langle x, \theta\rangle \leq t\}$, where $\theta \neq 0$.
- A strip has the form $\{x : |\langle x - y, \theta\rangle| \leq t\}$, where $t \geq 0$ and $\theta \neq 0$.
- $[s] = \lfloor s \rfloor$ denotes the greatest integer no larger than s.

Conventions for presentation. Unless otherwise specified, symmetry means symmetry about the origin. For probability measures, isotropic means mean zero and covariance Id; a volume-one convex body in isotropic position instead has covariance L_K^2 Id. We use $C_{\text{Poin}}(\mu)$ for the coefficient in $\text{Var}_\mu(f) \leq C_{\text{Poin}}(\mu) \int |\nabla f|^2 d\mu$. Some sources use its square root, and constants from those sources are converted accordingly.

In differential calculations we first work with smooth functions, strictly positive Hessians when inverses are used, and enough decay to justify differentiation and integration by parts. Cutoffs, mollification, addition of a small quadratic term, and approximation of convex sets give the usual extensions. We mention extra hypotheses where they change a mathematical statement, while keeping routine approximation details brief. Statements about the state of research are presented in the historical context of the course.

1 Volumes in high dimensions, hands-on computations, and pretty pictures

1.1 Introduction

The first lecture is all about some pretty pictures (including the one from the course website) and getting our hands on some basic volume computations in high dimensions. We will generally work in $\mathbb{R}^n$ and be interested in what happens as the dimension tends to infinity. High-dimensional phenomena appear when complex systems that depend on many parameters actually behave in a simple way. Concentration of measure is one example. Roughly speaking, it tells us that Lipschitz functions on certain high-dimensional spaces behave almost like constant functions. In high dimensions, if you walk a bit far, you quickly end up nowhere.

Where do concentration phenomena stem from?

Example 1.1 (Central Limit Theorem)**.** *Let $X_1, \ldots, X_n$ be independent random variables, each uniformly distributed on $[-1, 1]$. Since $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1/3$, as n tends to infinity*

the random variable

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

converges in distribution to the Gaussian random variable with mean 0*, variance* $1/3$*, and density*

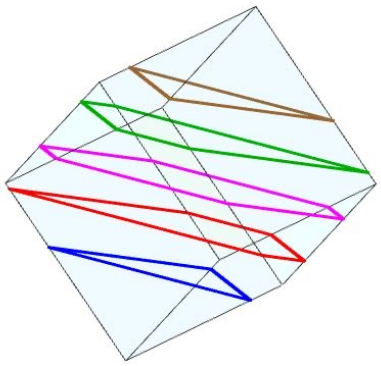
$$\sqrt{\frac{3}{2\pi}} e^{-3t^2/2}.$$

To get a geometric interpretation, note that the vector $X = (X_1, \ldots, X_n)$ is distributed uniformly over the cube $B_\infty^n = [-1, 1]^n$. For $\theta = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$, the example says that $\langle X, \theta \rangle$ is approximately Gaussian when n is large.

What is the geometric meaning of the density $f(t)$ of $\langle X, \theta \rangle$? By Fubini's theorem,

$$f(t) = \frac{|B_\infty^n \cap \{x : \langle x, \theta \rangle = t\}|_{n-1}}{|B_\infty^n|} = 2^{-n} |B_\infty^n \cap \{x : \langle x, \theta \rangle = t\}|_{n-1}.$$

Thus $f(t)$ is the normalized $(n-1)$-dimensional volume of the section of the cube perpendicular to θ at signed distance t from the origin. Although sections of cubes are hard to compute exactly, their distributions can approach a very simple object: a Gaussian.



This phenomenon appears to stem from *independence*. However, a related fact holds far beyond cubes! For a high-dimensional convex body in isotropic position, most one-dimensional marginals are approximately Gaussian, in an appropriate quantitative sense. This is the content of the Central Limit Theorem for convex bodies proved by Bo'az Klartag in 2007.

In fact, much concentration stems from independence, while other concentration phenomena stem from *convexity and isoperimetry*. Concentration via independence is often simpler and better understood, but the geometric approach applies to many dependent systems. This semester, we will discuss concentration stemming from convexity and isoperimetry; concentration via independence will be discussed next semester in the HDP course.

1.2 Computing the volumes of the cube, the cross-polytope and the Euclidean ball

For $1 \le p < \infty$, we define the ℓ_p-ball by

$$B_p^n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \le 1 \right\}.$$

The set B_2^n is the Euclidean ball, while B_1^n is the cross-polytope, or diamond. The set

$$B_\infty^n = \{x \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i| \le 1\}$$

is the unit cube. Note that $B_p^n \subseteq B_q^n$ if $p \le q$.

For a Borel-measurable set $A \subseteq \mathbb{R}^n$, denote its Lebesgue volume by $|A|$ or $|A|_n$. Unless otherwise stated, our sets are Borel-measurable. As a small homework exercise, check that

$$|B_\infty^n| = 2^n, \qquad |B_1^n| = \frac{2^n}{n!}.$$

Computing $|B_2^n|$ is a bit harder. We denote the unit sphere by

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

When applied to a hypersurface, $|\cdot|_{n-1}$ denotes surface area, that is, $(n-1)$-dimensional Hausdorff measure.

Lemma 1.1. $|\mathbb{S}^{n-1}|_{n-1} = n|B_2^n|$.

Proof. Use polar coordinates $x = t\theta$, where $\theta \in \mathbb{S}^{n-1}$ and $t \ge 0$:

$$\begin{aligned} |B_2^n| &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbb{1}_{B_2^n}(t\theta) t^{n-1} \, dt \, d\theta \\ &= \int_{\mathbb{S}^{n-1}} \int_0^1 t^{n-1} \, dt \, d\theta = \frac{1}{n} |\mathbb{S}^{n-1}|_{n-1}. \end{aligned}$$

□

To compute the surface area, we first recall the Gaussian integral.

Lemma 1.2. $\displaystyle\int_{\mathbb{R}} e^{-x^2/2} \, dx = \sqrt{2\pi}$.

Proof. By Fubini's theorem and then polar coordinates,

$$\begin{aligned} \left(\int_{\mathbb{R}} e^{-x^2/2} \, dx \right)^2 &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2} \, dx \, dy \\ &= \int_{\mathbb{S}^1} \int_0^\infty t e^{-t^2/2} \, dt \, d\theta = 2\pi. \end{aligned}$$

The integral is positive, so taking its square root proves the claim. □

Now we return to the sphere.

Lemma 1.3.

$$|\mathbb{S}^{n-1}|_{n-1} = \frac{(2\pi)^{n/2}}{J_{n-1}}, \qquad J_{n-1} := \int_0^\infty t^{n-1} e^{-t^2/2}\, dt.$$

Proof. By Fubini's theorem,

$$\int_{\mathbb{R}^n} e^{-|x|^2/2}\, dx = \prod_{i=1}^n \int_{\mathbb{R}} e^{-x_i^2/2}\, dx_i = (2\pi)^{n/2}.$$

On the other hand, polar coordinates give

$$\int_{\mathbb{R}^n} e^{-|x|^2/2}\, dx = |\mathbb{S}^{n-1}|_{n-1} \int_0^\infty t^{n-1} e^{-t^2/2}\, dt.$$

Comparing these expressions proves the lemma. □

Next, we need to estimate J_{n-1}. One could express it using the Gamma function, but we employ the *Laplace method*, a useful tool for integrals of this type.

Proposition 1.1 (An example of the use of the Laplace method)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that:*

- *F attains its unique absolute maximum at s_0;*
- *there exist $a, b > 0$ such that $F(s) \le F(s_0) - b$ whenever $|s - s_0| > a$;*
- *$\int_{\mathbb{R}} e^{F(s)}\, ds < \infty$;*
- *F is twice continuously differentiable near s_0, and $F''(s_0) < 0$.*

Then, as $m \to \infty$,

$$\int_{\mathbb{R}} e^{mF(s)}\, ds = (1 + o(1)) e^{mF(s_0)} \sqrt{\frac{2\pi}{-mF''(s_0)}}.$$

The same statement holds on an interval containing s_0 in its interior, provided the maximum is uniformly separated from values outside each neighborhood of s_0 and the other assumptions hold there.

Proof. See the homework problem with a series of hints! □

Consider

$$F_n(t) = (n-1)\log t - \frac{t^2}{2}, \qquad t > 0.$$

Solving $F_n'(t) = 0$ gives its maximum at $t_0 = \sqrt{n-1}$, and

$$F_n''(t_0) = -\frac{n-1}{t_0^2} - 1 = -2.$$

To apply Proposition 1.1 with a fixed function, put $m = n-1$ and $t = \sqrt{m}\, s$. Then

$$J_m = m^{(m+1)/2} \int_0^\infty \exp\left(m\left(\log s - \frac{s^2}{2}\right)\right) ds.$$

The function $F(s) = \log s - s^2/2$ has its unique maximum at $s = 1$, where $F(1) = -1/2$ and $F''(1) = -2$. Thus

$$\begin{aligned} J_{n-1} &= \int_0^\infty e^{F_n(t)}\, dt \\ &= (1+o(1)) e^{F_n(t_0)} \sqrt{\frac{2\pi}{-F_n''(t_0)}} \\ &= (1+o(1))\sqrt{\pi}\,(n-1)^{(n-1)/2} e^{-(n-1)/2}. \end{aligned}$$

Combining this with Lemma 1.3, we conclude:

Corollary 1.1.

$$|\mathbb{S}^{n-1}|_{n-1} = (1+o(1)) \frac{(2\pi)^{n/2} e^{(n-1)/2}}{\sqrt{\pi}\,(n-1)^{(n-1)/2}}.$$

Equivalently,

$$|B_2^n| = (1+o(1)) \frac{1}{\sqrt{\pi n}} \left(\frac{2\pi e}{n}\right)^{n/2}.$$

Remark 1.1. *The volume and surface area of the Euclidean unit ball tend to zero as $n \to \infty$, but not monotonically from the start. For example, $|B_2^n|$ increases up to $n = 5$ and then decreases.*

Remark 1.2. *Note that*

$$\frac{|B_2^{n-1}|_{n-1}}{|B_2^n|_n} = (1+o(1))\sqrt{\frac{n}{2\pi}}.$$

1.3 The first taste of concentration phenomena

Let $0 < \varepsilon \le 1$ and $\theta \in \mathbb{S}^{n-1}$. Consider the volume

$$|B_2^n \cap \{x : |\langle x, \theta\rangle| \le \varepsilon\}|.$$

Fubini's theorem and homogeneity of volume give

$$\begin{aligned} |B_2^n \cap \{x : |\langle x, \theta\rangle| \le \varepsilon\}| &= \int_{-\varepsilon}^{\varepsilon} |\sqrt{1-t^2} B_2^{n-1}|_{n-1}\, dt \\ &= |B_2^{n-1}|_{n-1} \int_{-\varepsilon}^{\varepsilon} (1-t^2)^{(n-1)/2}\, dt. \end{aligned}$$

We could evaluate this asymptotically, but first let us use a coarse bound. The integrand decreases as $|t|$ increases, so Remark 1.2 yields

$$\begin{aligned}\frac{|B_2^n \cap \{x : |\langle x, \theta\rangle| \le \varepsilon\}|}{|B_2^n|} &= \frac{|B_2^{n-1}|_{n-1}}{|B_2^n|} \int_{-\varepsilon}^{\varepsilon} (1-t^2)^{(n-1)/2}\, dt \\ &\ge (1+o(1))\sqrt{\frac{n}{2\pi}}\, 2\varepsilon(1-\varepsilon^2)^{(n-1)/2}.\end{aligned}$$

For $\varepsilon = 1/\sqrt{n}$, this gives the lower bound

$$(1+o(1))\frac{2}{\sqrt{2\pi e}} \approx 0.484.$$

When n is large, the slab of width $2/\sqrt{n}$ is *very thin*, yet it contains at least approximately 48% of the volume of the ball!

Thus, for n = 1,000,000, a constant fraction of the mass of the unit ball lies within distance 0.001 of the equator. Any equator!!!

Remark 1.3. *A more accurate estimate gives a limiting fraction of approximately 68% for this particular slab. More generally, a slab of width $C/\sqrt{n}$ contains at least 99% of the volume for an appropriate absolute constant C and sufficiently large n.*



This begs a question: does most of the mass lie near the center? We will shortly see that the answer is "no", and it could not be more "no"!

1.4 About the thin annulus around the ball

We have just seen that the ball has much of its mass near any given equator, so it "looks like a spinning top in every direction". Since the center belongs to every equatorial slab,

this might suggest that most of the mass is near the center. However, the mass is actually concentrated near the boundary. The point is that concentration near each *fixed* equator does not mean concentration near the intersection of all equatorial slabs.

Define the annulus $A_\varepsilon = B_2^n \setminus (1-\varepsilon)B_2^n$, where $0 \le \varepsilon \le 1$. Then

$$\frac{|A_\varepsilon|}{|B_2^n|} = 1 - (1-\varepsilon)^n.$$

If $n\varepsilon$ is small, this is $n\varepsilon + O(n^2\varepsilon^2)$. More usefully, for fixed $c > 0$ and $\varepsilon = c/n$, the fraction tends to $1 - e^{-c}$ as $n \to \infty$.

Conclusion: by choosing c sufficiently large, we can put as much of the mass of the unit ball as we like within distance c/n of its boundary! Equivalently, most points have distance at least $1 - c/n$ from the origin.

1.5 First hand-wavy glance at the connection between isoperimetry and concentration: most of the mass of a convex body is near the boundary

Definition 1.1. A set $K \subseteq \mathbb{R}^n$ is *convex* if, for every $x, y \in K$ and every $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in K$.

Remark 1.4. *The line segment joining x and y is*

$$[x,y] = \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}.$$

Thus convexity means that the segment joining any two points of K lies in K.

Definition 1.2. A *convex body* is a compact convex set with nonempty interior.

An infinite cylinder is not a convex body because it is not compact; a lower-dimensional disk in $\mathbb{R}^n$ is not a convex body in $\mathbb{R}^n$ because it has empty interior. Both are nevertheless convex sets.

We now want to argue that most of the mass of a convex body is near its boundary. For a convex body K, consider the boundary layer

$$A_\varepsilon = \{x \in K : \operatorname{dist}(x, \partial K) \le \varepsilon\}.$$

We will compare bodies of a fixed volume, since otherwise a dilation of K would change the relevant distance scale.

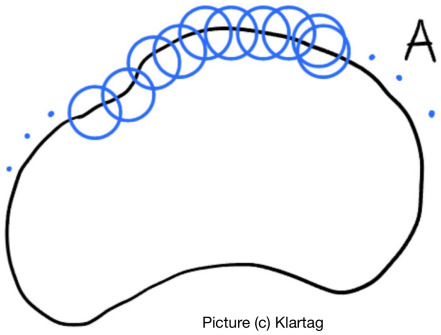
Definition 1.3 (Minkowski sum)**.** For $A, B \subseteq \mathbb{R}^n$, their Minkowski sum is

$$A + B = \{x + y : x \in A,\ y \in B\}.$$

Definition 1.4 (Perimeter)**.** For a Borel set $A \subseteq \mathbb{R}^n$ of finite volume, define its outer Minkowski perimeter by

$$|\partial A| := \liminf_{\varepsilon \downarrow 0} \frac{|(A + \varepsilon B_2^n) \setminus A|}{\varepsilon}.$$

For convex bodies, this agrees with surface area $|\partial A|_{n-1}$.



Picture (c) Klartag

Claim 1.1 (A first-order observation)**.** *For a fixed convex body K, as $\varepsilon \downarrow 0$,*

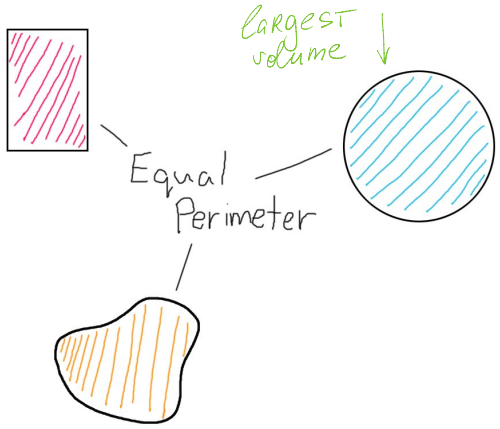
$$|A_\varepsilon| = \varepsilon |\partial K|_{n-1} + o(\varepsilon).$$

A nice way to bound the perimeter from below is the isoperimetric inequality.

Theorem 1.1 (The isoperimetric inequality)**.** *Balls minimize perimeter at fixed volume. More precisely, if $K \subseteq \mathbb{R}^n$ is Borel-measurable, $0 < |K| < \infty$, and $R > 0$ satisfies $|K| = |RB_2^n|$, then*

$$|\partial K| \geq |\partial (RB_2^n)|.$$

A fable has it that Roman soldiers wanted to maximize the amount of land they received as payment for military service:)



The ratio $|\partial K|/|K|^{(n-1)/n}$ is invariant under dilations: volume is n-homogeneous and perimeter is $(n-1)$-homogeneous, so for $t > 0$,

$$\frac{|\partial(tK)|}{|tK|^{(n-1)/n}} = \frac{t^{n-1}|\partial K|}{t^{n-1}|K|^{(n-1)/n}} = \frac{|\partial K|}{|K|^{(n-1)/n}}.$$

Consequently, isoperimetry can be written as

$$\frac{|\partial K|}{|K|^{(n-1)/n}} \geq \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{(n-1)/n}} = n|B_2^n|^{1/n}.$$

For a body of volume one, the right-hand side has order $\sqrt{n}$. Combining this with the first-order observation suggests that a boundary layer of thickness of order $1/\sqrt{n}$ should already contain a substantial fraction of the volume.

Remark 1.5. *The first-order approximation alone does not justify a* 99% *estimate. A rigorous version follows from the Brunn–Minkowski inequality, which we will prove later. If*

$$K_{-\varepsilon} = \{x \in K : x + \varepsilon B_2^n \subseteq K\},$$

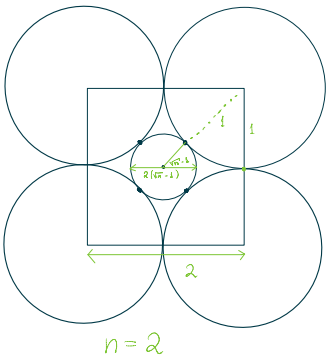
then $K_{-\varepsilon} + \varepsilon B_2^n \subseteq K$. *For* $|K| = 1$, *Brunn–Minkowski gives*

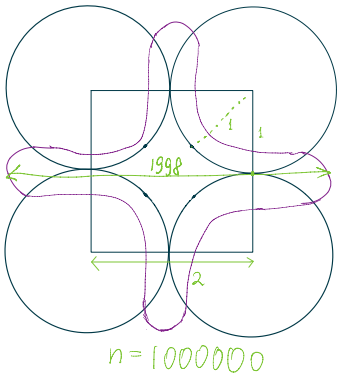
$$\frac{|A_\varepsilon|}{|K|} \geq 1 - \left(1 - \varepsilon|B_2^n|^{1/n}\right)_+^n,$$

where $a_+ = \max\{a, 0\}$. *Since* $|B_2^n|^{1/n}$ *has order* $1/\sqrt{n}$, *taking* $\varepsilon = c/\sqrt{n}$ *with a sufficiently large absolute constant* c *makes this fraction at least* 99%. *For a general volume, the corresponding thickness is* $c|K|^{1/n}/\sqrt{n}$. *The point of this discussion is our first illustration of concentration arising from isoperimetry!*

1.6 Some more fun regarding metric estimates in high dimension

Consider the cube B_∞^n and place a copy of B_2^n at each vertex (see the figure below). The distance from the origin to each vertex is $\sqrt{n}$. Thus the largest ball centered at the origin whose interior avoids all these vertex balls has radius $\sqrt{n} - 1$. This might be surprising: in many directions, that central ball extends much farther from the origin than the cube does when n is large.





1.7 Homework

Question 1.1 (1 point)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that:*

- *F attains its unique absolute maximum at s_0;*
- *there exist $a, b > 0$ such that $F(s) \leq F(s_0) - b$ whenever $|s - s_0| > a$;*
- *$\int_{\mathbb{R}} e^{F(s)}\, ds < \infty$;*
- *F is twice continuously differentiable near s_0, with $F''(s_0) < 0$.*

Prove that, as $m \to \infty$,

$$\int_{\mathbb{R}} e^{mF(s)}\, ds = (1 + o(1))e^{mF(s_0)}\sqrt{\frac{2\pi}{-mF''(s_0)}}.$$

Hint 1: By translating the variable and subtracting a constant, reduce to $s_0 = F(s_0) = 0$.
Hint 2: For every $\epsilon > 0$, choose $\delta > 0$ so that

$$\left| F(s) - \frac{F''(0)s^2}{2} \right| \leq \epsilon s^2 \qquad (|s| < \delta).$$

Hint 3: Use these quadratic bounds to estimate $\int_{-\delta}^{\delta} e^{mF(s)}\, ds$.
Hint 4: Use continuity, uniqueness of the maximum, and the assumption away from s_0 to show that, for every $\delta > 0$, there exists $\eta(\delta) > 0$ such that $F(s) \leq -\eta(\delta)$ for $|s| \geq \delta$.
Hint 5: Estimate the integral outside $(-\delta, \delta)$ using

$$e^{mF(s)} = e^{(m-1)F(s)}e^{F(s)}$$

and the integrability of e^F.
Hint 6: First let $m \to \infty$, and then let $\epsilon \downarrow 0$.

Question 1.2 (1 point)**.** *All the questions below ask for an answer up to a multiplicative factor* $1 + o(1)$ *as* $n \to \infty$*, wherever such an asymptotic is applicable.*

a) Find $|B_2^n|_n / |B_2^{n-1}|_{n-1}$*.*

Hint: Use the formula from Question 1 and Fubini's theorem. This gives an alternative to our computation of $|B_2^n|$*.*

b) Find the volume of

$$\{x \in \mathbb{R}^n : |x| \leq 2,\ x_1 \in [a, b]\},$$

where b1) $a = 0$*,* $b = 0.1$*; b2)* $a = -1/(\sqrt{n} \log n)$*,* $b = 1/n$*.*

Hint: Use the expression for $|B_2^k|$ *derived in class.*

c) Using any method you like, find the volume of

$$\operatorname{conv}\left(\{x \in \mathbb{R}^n : |x| < 3,\ x_2 = 0\} \cup \{x \in \mathbb{R}^n : |x - e_2| < 1,\ x_2 = 1\}\right).$$

d) Let γ *be the standard Gaussian measure on* $\mathbb{R}^n$*, with density* $(2\pi)^{-n/2} e^{-|x|^2/2}$*. Estimate* $\gamma(\{x : |x| > t\})$ *in the different ranges of* $t > 0$*, which may depend on* n*. Find the best approximations you can.*

e) Let μ *be the* ***probability*** *measure with density* $C(n) e^{-|x|^3}$*. Find* $C(n)$*.*

f) For this measure, find the radius R *satisfying* $\mu(RB_2^n) = 1/2$*.*

Question 1.3 (1 point)**.** *Let* $A \subseteq e_1^\perp$ *be a convex set. Find the* n*-dimensional volume of* $\operatorname{conv}(A \cup \{Re_1\})$ *in terms of* $|A|_{n-1}$*,* R*, and* n*.*

Question 1.4 (1 point)**.** *a) Using Laplace's method, prove that at least* 99% *of the volume of the* n*-dimensional Euclidean unit ball is contained in a strip of width* $100/\sqrt{n}$ *around any equator, for sufficiently large* n*.*

b) Prove the same statement for surface area on the sphere. Hint: use spherical coordinates, and take care with the surface-area Jacobian.

Question 1.5 (2 points)**.** *Find a function* $F : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, for every symmetric convex body* $K \subseteq \mathbb{R}^n$ *with* $|K| = 1$*, there exists* $u \in \mathbb{S}^{n-1}$*, possibly depending on* K*, for which*

$$|K \cap u^\perp|_{n-1} \geq F(n).$$

How large a lower bound can you prove? For example, you might investigate the candidate functions

$$20t^{-t}, \quad 5^{-t}, \quad 3t^{-2}, \quad \frac{1}{t}, \quad \frac{10}{\sqrt{t}}, \quad 100t^{-1/4},$$
$$\frac{1}{\log t}, \quad 0.00001, \quad \sqrt{2},$$

or other rates. These are targets to investigate, not assertions that every displayed bound is valid; some constants or proposed rates may be impossible. Expressions involving $\log t$ *are considered for* $t > 1$*. Any valid positive bound is a starting point!*

2 Preliminaries

2.1 Preliminaries from linear algebra

The operator norm of a matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\|A\|_{\mathrm{op}} = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \sup_{x \in \mathbb{S}^{n-1}} |Ax|. \tag{1}$$

Definition 2.1 (Hilbert–Schmidt norm)**.** For an $n \times n$ matrix $A = (a_{ij})$, define

$$\|A\|_{\mathrm{HS}} = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2},$$

where $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ are its singular values.

Recall also that

$$\sigma_n(A) = \inf_{x \in \mathbb{S}^{n-1}} |Ax|.$$

Remark 2.1. *Since $\|A\|_{\mathrm{op}} = \sigma_1$, we have*

$$\|A\|_{\mathrm{op}} \leq \|A\|_{\mathrm{HS}} \leq \sqrt{n}\, \|A\|_{\mathrm{op}}.$$

Indeed, $\sigma_1^2 \leq \sum_i \sigma_i^2 \leq n\sigma_1^2$.

Theorem 2.1 (Spectral decomposition)**.** *Let A be a real symmetric $n \times n$ matrix. There are an orthonormal basis $u_1, \ldots, u_n$ and real eigenvalues $\lambda_1, \ldots, \lambda_n$ such that*

$$A = \sum_{i=1}^{n} \lambda_i u_i \otimes u_i.$$

Here $u \otimes u = uu^\top$, so, writing $u = (u_1, \ldots, u_n)$,

$$u \otimes u = \begin{bmatrix} u_1^2 & u_1 u_2 & \cdots & u_1 u_n \\ u_1 u_2 & u_2^2 & \cdots & u_2 u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_1 u_n & u_2 u_n & \cdots & u_n^2 \end{bmatrix}.$$

This matrix has rank one when $u \neq 0$. In particular,

$$Ax = \sum_{i=1}^{n} \lambda_i \langle x, u_i \rangle u_i.$$

Definition 2.2 (Functions of matrices)**.** For a function $f : \mathbb{R} \to \mathbb{R}$ and a real symmetric matrix $A = \sum_i \lambda_i u_i \otimes u_i$ with an orthonormal eigenbasis, define

$$f(A) := \sum_{i=1}^{n} f(\lambda_i) u_i \otimes u_i.$$

This is independent of the choice of orthonormal basis within each eigenspace.

2.2 Preliminaries from convexity

2.2.1 Convex functions

We start with some more definitions.

Definition 2.3 (Convex and concave functions)**.** A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is *convex* if, for all $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

The endpoint cases are interpreted in the evident way. A function is *concave* if its negative is convex.

Remark 2.2. *Inductively, convexity implies*

$$F\left(\sum_{i=1}^{m} \lambda_i x_i\right) \leq \sum_{i=1}^{m} \lambda_i F(x_i)$$

whenever $\lambda_i \geq 0$ *and* $\sum_i \lambda_i = 1$.

Theorem 2.2 (Jensen's inequality)**.** *Let* μ *be a probability measure, let* $g \in L^1(\mu)$ *be real-valued, and let* $F : \mathbb{R} \to \mathbb{R}$ *be convex. Then*

$$F\left(\int g \, d\mu\right) \leq \int F(g) \, d\mu,$$

where the right-hand side may equal $+\infty$.

Remark 2.3. *Jensen's inequality gives Remark 2.2 by taking a discrete probability space with point masses* λ_i *and values* $g(i) = x_i$*; for vector-valued* x_i*, use the vector-valued version of Jensen's inequality.*

Recall the definitions of a convex set and a convex body from the first section. Given a convex set K, an example of an extended-valued convex function is its *convex indicator*

$$\mathbb{1}_K^{\infty}(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

The *epigraph* of a function f is the set

$$\{(x,t) \in \mathbb{R}^{n+1} : t \geq f(x)\}.$$

It is convex if and only if f is convex.

If $f \in C^2(\mathbb{R})$ is convex, then $f'' \geq 0$. In several variables, the analogous object is the Hessian

$$\nabla^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}.$$

A symmetric matrix A is *positive semidefinite* if $\langle Ax, x\rangle \geq 0$ for every $x \in \mathbb{R}^n$.

Claim 2.1. *If $f \in C^2(\mathbb{R}^n)$ is convex, then $\nabla^2 f(x)$ is positive semidefinite for every x.*

Proof. See the homework problem with hints! □

For later reference, we record again the form of Jensen's inequality that we will use repeatedly.

Theorem 2.3 (Jensen's inequality)**.** *For a probability measure μ, an integrable function f, and a convex function F on the relevant domain,*

$$F\left(\int f\, d\mu\right) \leq \int F(f)\, d\mu.$$

2.2.2 Minkowski functional of a convex body and the radial function

Definition 2.4 (Origin symmetry)**.** A set K is *origin-symmetric* if $K = -K$.

In the definitions of the Minkowski functional and radial function below, we assume that $0 \in \operatorname{int} K$.

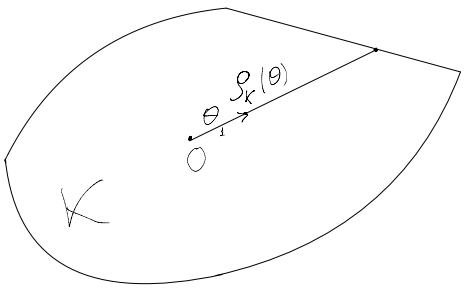
Definition 2.5 (Minkowski functional of a convex body)**.** For a convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior, its Minkowski functional is

$$\|x\|_K := \inf\{\lambda > 0 : x \in \lambda K\}.$$

Definition 2.6 (Radial function of a convex body)**.** For $x \neq 0$, define $\rho_K(x) = \|x\|_K^{-1}$. In particular, for $\theta \in \mathbb{S}^{n-1}$,

$$\rho_K(\theta) = \max\{t \geq 0 : t\theta \in K\}.$$

Remark 2.4. *For a unit vector θ, $\rho_K(\theta)$ is the distance from the origin to the farthest point of K in direction θ. Away from the sphere, our extension satisfies $\rho_K(tx) = t^{-1}\rho_K(x)$ for $t > 0$.*



The Minkowski functional has the following properties:

- $\|tx\|_K = t\|x\|_K$ for $t \geq 0$;
- $\|x\|_K \geq 0$, with equality if and only if $x = 0$;
- if K is symmetric, then $\|-x\|_K = \|x\|_K$;
- $\|x+y\|_K \leq \|x\|_K + \|y\|_K$.

Thus the Minkowski functional of a symmetric convex body is a norm. Conversely, the unit ball $\{x : \|x\| \leq 1\}$ of a norm on $\mathbb{R}^n$ is a symmetric convex body.

2.2.3 Hahn–Banach theorem and supporting hyperplanes

For $\theta \in \mathbb{S}^{n-1}$, write

$$\theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta\rangle = 0\}, \qquad \theta^\perp + t\theta = \{x \in \mathbb{R}^n : \langle x, \theta\rangle = t\}.$$

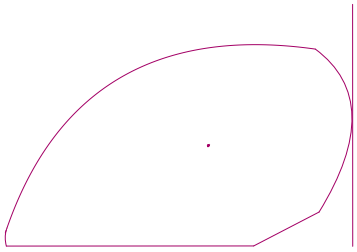
More generally, if H is a k-dimensional linear subspace, its orthogonal complement is

$$H^\perp = \{x \in \mathbb{R}^n : \langle x, y\rangle = 0 \text{ for all } y \in H\},$$

which has dimension $n-k$.

Definition 2.7 (Supporting hyperplane of a convex body)**.** For a convex body K and $\theta \in \mathbb{S}^{n-1}$, the hyperplane $\{x : \langle x, \theta\rangle = t\}$ *supports* K in direction θ if

$$K \subseteq \{x : \langle x, \theta\rangle \leq t\} \quad \text{and} \quad K \cap \{x : \langle x, \theta\rangle = t\} \neq \varnothing.$$



Theorem 2.4 (Supporting hyperplanes and finite-dimensional separation)**.** *For every convex body $K \subseteq \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$, there is a supporting hyperplane $\theta^\perp + t_\theta\theta$. Moreover, every point outside K can be strictly separated from K by a hyperplane.*

Proof. The first assertion follows by maximizing $\langle x, \theta\rangle$ on the compact set K. The separation assertion is the finite-dimensional form of the Hahn–Banach theorem; see the homework. □

Intersections of convex sets are convex; in particular, intersections of half-spaces are convex. Separation gives the converse representation:

Remark 2.5. *For every convex body K,*

$$K = \bigcap_{\theta \in \mathbb{S}^{n-1}} \{x : \langle x, \theta\rangle \leq t_\theta\}.$$

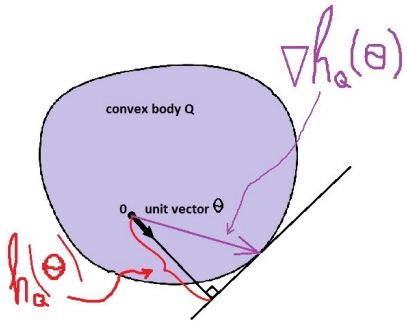
Let us give this t_θ a name...

2.2.4 Support function of a convex set

Definition 2.8 (Support function of a convex body)**.** The support function of a convex body K is

$$h_K(x) := \sup_{y \in K} \langle x, y \rangle, \qquad x \in \mathbb{R}^n.$$

Remark 2.6. *For a* unit *vector θ, $h_K(\theta)$ is the signed distance from the origin to the supporting hyperplane with outer normal θ.*



The support function has a few nice properties:

- $h_K(tx) = t h_K(x)$ for $t \geq 0$;
- $\theta^\perp + h_K(\theta)\theta$ supports K for every $\theta \in \mathbb{S}^{n-1}$;
- h_K is convex;
- if K is an origin-symmetric convex body, then h_K is a norm.

2.2.5 Duality/polarity

Definition 2.9 (Duality/polarity)**.** For any set $K \subseteq \mathbb{R}^n$, its polar is

$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

Writing $H_y = \{x : \langle x, y \rangle \leq 1\}$, we have $K^\circ = \bigcap_{y \in K} H_y$. Thus a polar is always closed and convex, even if K is not. For a convex body containing 0 in its interior, the polar is again a convex body containing 0 in its interior.

Claim 2.2. *If K is a convex body with $0 \in \operatorname{int} K$, then*

$$K^\circ = \{x : h_K(x) \leq 1\}, \qquad h_K(x) = \|x\|_{K^\circ}.$$

If K is symmetric, h_K and $\|\cdot\|_K$ are dual norms.

Example 2.1. *A symmetric interval and a strip are polar to one another. For $a > 0$ and $\theta \in \mathbb{S}^{n-1}$,*

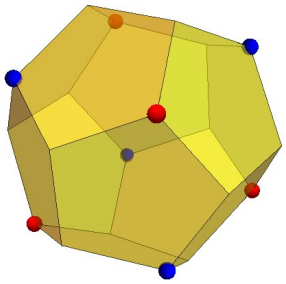
$$[-a\theta, a\theta]^\circ = \{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \le 1/a\}.$$

Definition 2.10. The *convex hull* of $\Omega \subseteq \mathbb{R}^n$ is

$$\operatorname{conv}\Omega = \left\{\sum_{i=1}^k \lambda_i x_i : k \ge 1,\ x_i \in \Omega,\ \lambda_i \ge 0,\ \sum_{i=1}^k \lambda_i = 1\right\}.$$

Definition 2.11. A *polytope* is the convex hull of finitely many points:

$$K = \operatorname{conv}\{v_1, \dots, v_N\}.$$



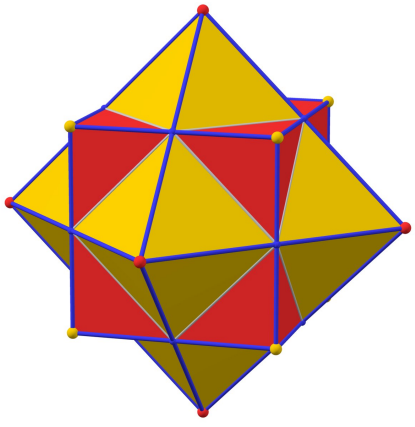
Equivalently, a polytope is a *bounded* intersection of finitely many closed half-spaces. Without boundedness, such an intersection is called a polyhedron.

Remark 2.7. *For $K = \operatorname{conv}\{v_1, \dots, v_N\}$,*

$$K^\circ = \bigcap_{i=1}^N \{x : \langle x, v_i\rangle \le 1\}.$$

If $v_i = a_i\theta_i$ with $a_i > 0$ and $\theta_i \in \mathbb{S}^{n-1}$, this becomes

$$K^\circ = \bigcap_{i=1}^N \{x : \langle x, \theta_i\rangle \le 1/a_i\}.$$



Some other examples:

- $K^\circ = K$ if and only if $K = B_2^n$;
- $(10B_2^n)^\circ = \frac{1}{10}B_2^n$;
- $(B_\infty^n)^\circ = B_1^n$, and more generally $(B_p^n)^\circ = B_q^n$ when $1/p + 1/q = 1$, including the endpoints 1 and ∞.

A few more definitions:

Definition 2.12 (Section of a convex body)**.** If H is an affine k-dimensional subspace, the corresponding section of K is $K \cap H$.

Definition 2.13 (Projection of a convex body)**.** For a linear subspace H, the orthogonal projection of K onto H is

$$K|H = \{x \in H : \text{there exists } y \in H^\perp \text{ with } x + y \in K\}.$$

For an affine subspace, use the analogous orthogonal projection after translating its direction space.

For the following properties, assume that K, L are convex bodies containing the origin in their interiors:

(a) $(K^\circ)^\circ = K$.

(b) If T is invertible, then $(TK)^\circ = (T^\top)^{-1}K^\circ$. In particular, $(aK)^\circ = a^{-1}K^\circ$ for $a > 0$, and origin-centered ellipsoids are closed under polarity.

(c) For a linear subspace H, the polar of $K|H$ *within* H is $K^\circ \cap H$. Equivalently, if the polar is taken in $\mathbb{R}^n$, then $(K|H)^\circ \cap H = K^\circ \cap H$.

(d) $K \subseteq L$ implies $L^\circ \subseteq K^\circ$.

Recall that $K + L = \{x + y : x \in K,\ y \in L\}$.

Claim 2.3. $h_{K+L} = h_K + h_L$.

Proof. Homework! □

Since $\rho_{K^\circ}(\theta) = 1/h_K(\theta)$ for $\theta \in \mathbb{S}^{n-1}$, an equivalent radial identity is

$$\rho_{(K+L)^\circ}(\theta) = \left(\frac{1}{\rho_{K^\circ}(\theta)} + \frac{1}{\rho_{L^\circ}(\theta)}\right)^{-1}.$$

Later in the course, we will discuss a related notion.

Definition 2.14 (Legendre transform)**.** For a proper function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its Legendre transform is the extended-valued function

$$\phi^*(x) := \sup_{y \in \mathbb{R}^n} \Big(\langle x, y \rangle - \phi(y) \Big).$$

Here "proper" means that ϕ is finite somewhere and never takes the value $-\infty$.

The following connection with support functions is important to remember: for a convex body K,

$$(\mathbb{1}_K^\infty)^* = h_K, \qquad h_K^* = \mathbb{1}_K^\infty.$$

The first formula follows immediately from the definition, and the second from separation (or from the biconjugation theorem, which we will discuss later). As before,

$$\mathbb{1}_K^\infty(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

2.3 Preliminaries from probability

For any nonnegative random variable,

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)\, dt. \tag{2}$$

Indeed, $X = \int_0^\infty \mathbb{1}_{\{X>t\}}\, dt$, so Tonelli's theorem gives

$$\mathbb{E}X = \int_0^\infty \mathbb{E}\mathbb{1}_{\{X>t\}}\, dt = \int_0^\infty \mathbb{P}(X > t)\, dt.$$

Lemma 2.1 (Markov's inequality)**.** *Let $X \geq 0$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, for $t > 0$,*

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}X}{t}.$$

Proof. By (2),

$$\mathbb{E}X \geq \int_0^t \mathbb{P}(X > s)\, ds \geq t\mathbb{P}(X > t).$$

□

Definition 2.15. For $1 \leq p < \infty$, the L^p-norm of a random variable X is

$$\|X\|_p = (\mathbb{E}|X|^p)^{1/p}.$$

Theorem 2.5 (Minkowski's inequality)**.** *For random variables X, Y on the same probability space and $1 \leq p < \infty$,*

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p.$$

2.4 Preliminaries from functional analysis

2.4.1 Classical notions and some classical inequalities

The support of a function is the closure of the set where it is nonzero.

Theorem 2.6 (Cauchy–Schwarz)**.** *For vectors a, b in an inner-product space,*

$$|\langle a, b\rangle| \le \|a\|\,\|b\|,$$

where the norm is induced by the inner product.

Theorem 2.7 (Hölder's inequality)**.** *If $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, then*

$$\int |fg|\,d\mu \le \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)}.$$

For finite p, q, the right-hand side is

$$\left(\int |f|^p\,d\mu\right)^{1/p}\left(\int |g|^q\,d\mu\right)^{1/q};$$

at an infinite exponent, use the essential supremum.

2.4.2 Weighted Sobolev spaces

Consider a measure μ on $\mathbb{R}^n$ with a positive, locally Lipschitz density $e^{-V(x)}$. We will mostly work with a smooth convex potential V. Write $C^p(\Omega)$ for the functions whose partial derivatives through order p exist and are continuous on the open set Ω. A locally Lipschitz function is one that is Lipschitz on a neighborhood of each point of its domain; we denote this class on K by $\mathrm{Lip}(K)$.

The differential expression associated with μ is

$$Lu = \Delta u - \langle \nabla u, \nabla V\rangle.$$

For C^2 functions this expression is defined pointwise; when treating L as an operator on $L^2(\mu)$, we restrict to functions in its operator domain, in particular those for which $Lu \in L^2(\mu)$.

The weighted Sobolev space $W^{1,2}(\mu)$ consists of functions $u \in L^2(\mu)$ whose weak first partial derivatives also belong to $L^2(\mu)$. We use the norm

$$\|u\|_{W^{1,2}(\mu)} = \left(\int u^2\,d\mu + \int |\nabla u|^2\,d\mu\right)^{1/2}.$$

Positive fixed coefficients in front of these two terms give equivalent norms.

Remark 2.8. *For the sake of presentation, our differential computations will usually be made for smooth functions with sufficient decay and smooth positive densities. One can extend many of them by cutoff, smoothing, and convergence in the relevant Sobolev norm. Measures supported on convex sets can often be handled by approximating their convex indicators by smooth convex potentials. We will point out any extra hypothesis that affects the mathematical statement, rather than interrupting each computation with routine regularization details.*

Next, recall the Lax–Milgram lemma (see, for example, Evans [64]).

Lemma 2.2 (Lax–Milgram)**.** *Let H be a real Hilbert space, Q a symmetric bilinear form on H, and ℓ a linear functional. Suppose that for positive constants C_1, C_2, C_3,*

- $|Q(f,g)| \le C_1 \|f\|\,\|g\|$ *for all* $f, g \in H$*;*
- $Q(f,f) \ge C_2 \|f\|^2$ *for all* $f \in H$*;*
- $|\ell(f)| \le C_3 \|f\|$ *for all* $f \in H$*.*

Then there is a unique $h \in H$ such that $Q(f,h) = \ell(f)$ for every $f \in H$.

2.4.3 Poincaré inequalities

Finally, recall a basic notion that will appear throughout the course. A probability measure μ satisfies a *Poincaré inequality* if there is a finite constant $C_{\mathrm{Poin}}(\mu)$ such that

$$\mathrm{Var}_\mu(F) \le C_{\mathrm{Poin}}(\mu) \int |\nabla F|^2 \, d\mu \qquad (F \in W^{1,2}(\mu)). \tag{3}$$

The smallest such coefficient is the Poincaré constant. For a finite measure, expectations and variances refer to its normalization to total mass one.

Every log-concave probability measure on $\mathbb{R}^n$ has a finite Poincaré constant on its affine support. In particular, this applies to the uniform probability measure on a convex body. Bounded support alone does *not* guarantee the inequality for an arbitrary absolutely continuous measure: disconnected supports already provide counterexamples.

2.5 Some famous conjectures in high-dimensional convexity

[To do: add this subsection.]

2.6 Homework

Question 2.1 (1 point)**.** *Prove that, for every convex body $K \subseteq \mathbb{R}^n$ and every $x \notin K$, there exist $\theta \in \mathbb{S}^{n-1}$ and $\rho \in \mathbb{R}$ such that*

$$\langle x, \theta \rangle > \rho > \langle y, \theta \rangle \qquad \text{for every } y \in K.$$

This is a finite-dimensional form of the Hahn–Banach separation theorem.

Question 2.2 (1 point). *Prove that a full-dimensional convex hull of finitely many points in $\mathbb{R}^n$ can be expressed as an intersection of finitely many closed half-spaces. What changes if the convex hull has empty interior?*

Question 2.3 (1 point). *Show that the Minkowski functional of an origin-symmetric convex body is a norm on $\mathbb{R}^n$.*

Question 2.4 (1 point). *Show that, for a convex body Q containing the origin in its interior,*

$$|Q| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_Q(\theta)^n \, d\theta = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\theta\|_Q^{-n} \, d\theta.$$

Extend the formula to a convex set containing the origin by allowing extended values for the gauge and radial function, with $1/\infty = 0$ and $1/0 = \infty$.

Question 2.5 (1 point). *a) Show that, for convex bodies K, L,*

$$h_{K+L}(x) = h_K(x) + h_L(x).$$

b) Show that, for $a > 0$,

$$h_K(ax) = a h_K(x) = h_{aK}(x).$$

c) For $v \in \mathbb{R}^n$, show that $h_{[-v,v]}(x) = |\langle v, x\rangle|$, where $[-v, v]$ is the segment joining $-v$ and v.

3 Brunn–Minkowski inequality and friends

Many results related to convexity, high-dimensional analysis, concentration of measure, geometry, and probability are intimately related to the following fundamental result. It will be the cornerstone of this section and is undoubtedly one of the most important results in this course. A great deal has been written on this inequality; we recommend especially Artstein-Avidan, Giannopoulos, and Milman [4], Gardner [70], and Schneider [167].

3.1 Brunn–Minkowski inequality and the isoperimetric inequality

We start by formulating this central result.

Theorem 3.1 (Brunn–Minkowski inequality). *Let $K, L \subseteq \mathbb{R}^n$ be nonempty Borel-measurable sets. Then*

$$|K + L|^{1/n} \geq |K|^{1/n} + |L|^{1/n},$$

where $K + L = \{x + y : x \in K,\ y \in L\}$ is their Minkowski sum.

Remark 3.1. *Borel measurability of K and L guarantees that $K + L$ is Lebesgue measurable: it is the continuous image of the Borel set $K \times L$, and is therefore an analytic set. It need not be Borel. Merely assuming that K and L are Lebesgue measurable does not guarantee measurability of their sum; see the homework.*

Since $|aK| = a^n|K|$ for $a > 0$, the theorem is equivalent to

$$|\lambda K + (1-\lambda)L|^{1/n} \geq \lambda|K|^{1/n} + (1-\lambda)|L|^{1/n}, \qquad 0 \leq \lambda \leq 1. \tag{4}$$

Thus volume to the power $1/n$ is concave under Minkowski addition. We will soon see that many other functionals possess related concavity or convexity properties. The branch of analysis that studies these phenomena and their applications is what we call *convexity*, in its modern understanding. This is closely connected to, but goes beyond, studying individual convex bodies.

For $a, b \geq 0$, $0 < \lambda < 1$, and $p > 0$, the weighted arithmetic–geometric mean inequality gives

$$(\lambda a^p + (1-\lambda)b^p)^{1/p} \geq a^\lambda b^{1-\lambda}.$$

Consequently, theorem 3.1 implies the dimension-free multiplicative form

$$|\lambda K + (1-\lambda)L| \geq |K|^\lambda |L|^{1-\lambda}. \tag{5}$$

Conversely, the validity of (5) for all pairs of sets and all λ implies theorem 3.1, by rescaling the two sets to have the same volume. This useful application of homogeneity is left as homework.

We can now prove the aforementioned isoperimetric inequality.

Theorem 3.2 (Isoperimetric inequality)**.** *For every Borel set $K \subseteq \mathbb{R}^n$ with $0 < |K| < \infty$,*

$$\frac{|\partial K|}{|K|^{(n-1)/n}} \geq \frac{|\partial B_2^n|}{|B_2^n|^{(n-1)/n}} = n|B_2^n|^{1/n}.$$

Here perimeter is the outer Minkowski perimeter defined earlier; for convex bodies it agrees with the usual surface area.

Proof. By Brunn–Minkowski,

$$\begin{aligned} |\partial K| &= \liminf_{\varepsilon \downarrow 0} \frac{|K + \varepsilon B_2^n| - |K|}{\varepsilon} \\ &\geq \liminf_{\varepsilon \downarrow 0} \frac{\left(|K|^{1/n} + \varepsilon |B_2^n|^{1/n}\right)^n - |K|}{\varepsilon} \\ &= n|K|^{(n-1)/n}|B_2^n|^{1/n}. \end{aligned}$$

Divide by $|K|^{(n-1)/n}$ and use $|\partial B_2^n| = n|B_2^n|$. □

Let us also mention the equality case for convex bodies.

Claim 3.1. *If K, L are convex bodies, then*

$$|K + L|^{1/n} = |K|^{1/n} + |L|^{1/n}$$

if and only if $K = tL + v$ for some $t > 0$ and $v \in \mathbb{R}^n$.

See [167] for a proof. For general measurable sets, the equality statement requires additional qualifications concerning null sets and degenerate cases.

3.2 Proof of the Brunn–Minkowski inequality (full generality due to Lazar Lyusternik in 1935)

Following the early work of Brunn and Minkowski, the inequality was proved in its full generality by Lyusternik in 1935, followed by work of Henstock and Macbeath [88], and Hadwiger and Ohmann [87]. Below is the classical box-decomposition proof.

Step 1: coordinate boxes. Suppose

$$K = [0, a_1] \times \cdots \times [0, a_n], \qquad L = [0, b_1] \times \cdots \times [0, b_n],$$

with positive side lengths. Their sum is the box with side lengths $a_i + b_i$, so it suffices to show

$$\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n}.$$

Indeed, AM–GM gives

$$\begin{aligned} &\prod_{i=1}^{n} \left(\frac{a_i}{a_i + b_i} \right)^{1/n} + \prod_{i=1}^{n} \left(\frac{b_i}{a_i + b_i} \right)^{1/n} \\ &\qquad \leq \frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{a_i + b_i} + \frac{1}{n} \sum_{i=1}^{n} \frac{b_i}{a_i + b_i} = 1. \end{aligned}$$

Translations do not affect this argument.

Step 2: finite unions of boxes. Suppose K and L are finite unions of coordinate boxes with pairwise disjoint interiors. We induct on the total number of boxes. The case of two boxes is Step 1. For the induction step, suppose, after interchanging K and L if necessary, that K contains at least two boxes. Two boxes with disjoint interiors can be separated by a coordinate hyperplane

$$H = \{x : \langle x, \theta \rangle = t\}, \qquad \theta \in \{e_1, \ldots, e_n\}.$$

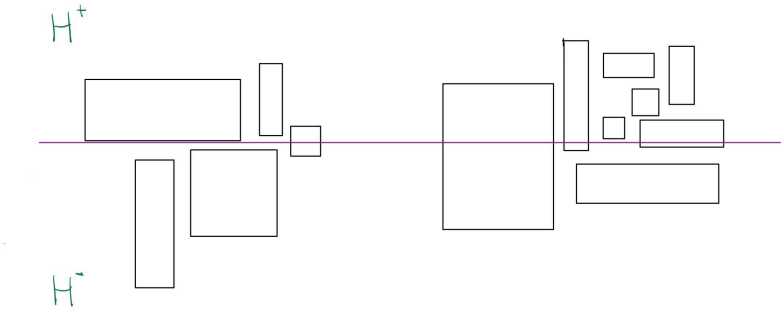
Choose H so that at least one box lies on each side. Although some other boxes may be cut, each half of K is then a union of at most one fewer boxes than K. Set

$$H^+ = \{x : \langle x, \theta \rangle > t\}, \qquad H^- = \{x : \langle x, \theta \rangle \leq t\}.$$

By translating L in direction θ, we can arrange that

$$\frac{|K \cap H^+|}{|K|} = \frac{|L \cap H^+|}{|L|} = a \in (0, 1).$$

This follows from continuity of the volume cut off by a moving hyperplane.



The two sums $(K \cap H^+) + (L \cap H^+)$ and $(K \cap H^-) + (L \cap H^-)$ lie on opposite sides of the hyperplane $\langle x, \theta \rangle = 2t$, up to its boundary. Each pair involves fewer boxes in total, so the induction hypothesis yields

$$\begin{aligned} |K + L| &\geq |(K \cap H^+) + (L \cap H^+)| + |(K \cap H^-) + (L \cap H^-)| \\ &\geq \Big(|K \cap H^+|^{1/n} + |L \cap H^+|^{1/n}\Big)^n \\ &\qquad + \Big(|K \cap H^-|^{1/n} + |L \cap H^-|^{1/n}\Big)^n \\ &= a\Big(|K|^{1/n} + |L|^{1/n}\Big)^n + (1-a)\Big(|K|^{1/n} + |L|^{1/n}\Big)^n \\ &= \Big(|K|^{1/n} + |L|^{1/n}\Big)^n. \end{aligned}$$

Step 3: approximation. Approximate compact sets from outside by finite unions of small coordinate boxes. Their sums converge to the compact Minkowski sum, so continuity of volume from above gives the inequality for compact sets. Inner approximation by compact subsets then gives the result for Borel sets of finite positive volume. Exhaustion handles infinite volume; if one set has volume zero, a translate of the other set lies in the sum. This completes the proof. □

3.3 Steiner symmetrization and another proof of the Brunn–Minkowski inequality (for convex bodies)

Steiner symmetrization was introduced by Jacob Steiner in his work on the isoperimetric inequality in 1837. Given $\theta \in \mathbb{S}^{n-1}$, replace each one-dimensional section parallel to θ by the interval of the same length centered on $\theta^\perp$. The union of these intervals is the Steiner symmetral, denoted $S_\theta(K)$. It has the same volume as K and, in several senses, better isoperimetric properties. We will use this technique more than once.

Definition 3.1 (Steiner symmetrization)**.** For a bounded Borel set $K \subseteq \mathbb{R}^n$, put

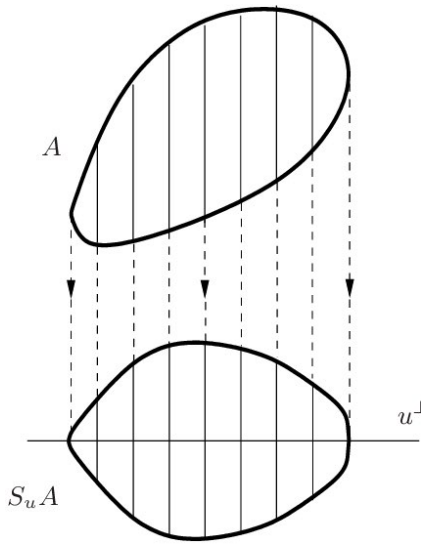
$$\ell_K(y) = \Big|K \cap (y + \mathbb{R}\theta)\Big|_1, \qquad y \in \theta^\perp.$$

Its Steiner symmetrization is the union, over nonempty fibers, of the centered intervals

$$S_\theta(K) = \bigcup_{y \in K|\theta^\perp} \{y + t\theta : |t| \le \ell_K(y)/2\}.$$

Endpoint conventions and zero-length fibers do not affect the volume. For convex bodies this is the usual compact Steiner symmetral.

Remark 3.2. *Every fiber of a Borel set is measurable. Fubini's theorem also gives the measurability of its fiber-length function. We will use Steiner symmetrization primarily for convex bodies.*



Claim 3.2. $|K| = |S_\theta(K)|$.

Proof. Integrating the lengths of the fibers gives

$$|K| = \int_{\theta^\perp} \ell_K(y)\, dy = \int_{\theta^\perp} \left|S_\theta(K) \cap (y + \mathbb{R}\theta)\right|_1 dy = |S_\theta(K)|.$$

□

We also need a notion of distance between convex bodies.

Definition 3.2 (Hausdorff distance)**.** For compact nonempty sets $K, L \subseteq \mathbb{R}^n$, define

$$d_H(K, L) = \inf\{\varepsilon \ge 0 : K \subseteq L + \varepsilon B_2^n,\ L \subseteq K + \varepsilon B_2^n\}.$$

Properties of Steiner symmetrization

For convex bodies K, L, the following hold.

- $S_\theta(K)$ is convex.
- $\mathrm{circ}(S_\theta(K)) \le \mathrm{circ}(K)$, where

$$\mathrm{circ}(K) = \inf\{s > 0 : \exists y \in \mathbb{R}^n,\ K \subseteq y + sB_2^n\}.$$

- $\mathrm{inrad}(S_\theta(K)) \ge \mathrm{inrad}(K)$, where

$$\mathrm{inrad}(K) = \sup\{s > 0 : \exists y \in \mathbb{R}^n,\ y + sB_2^n \subseteq K\}.$$

- $S_\theta(aK) = aS_\theta(K)$ for $a \ge 0$.
- Steiner symmetrization is continuous in the Hausdorff metric on convex bodies.
- $S_\theta(K) + S_\theta(L) \subseteq S_\theta(K + L)$.
- $|\partial S_\theta(K)| \le |\partial K|$.
- $\mathrm{diam}(S_\theta(K)) \le \mathrm{diam}(K)$, where $\mathrm{diam}(K) = \sup_{x,y\in K} |x - y|$.

These properties are left as homework.

At last, we state another very important fact: suitably chosen successive Steiner symmetrizations converge to a ball.

Claim 3.3. *There is a sequence of directions $\{\theta_j\} \subseteq \mathbb{S}^{n-1}$ such that, for every convex body K,*

$$S_{\theta_j} \cdots S_{\theta_1}(K) \longrightarrow RB_2^n \quad \textit{in Hausdorff distance}, \qquad R = \left(\frac{|K|}{|B_2^n|}\right)^{1/n}.$$

In particular, one can use the same sequence for two prescribed convex bodies.

Sketch of the compactness argument. Choose $r > 0$ with $K \subset rB_2^n$. Every successive symmetral is still contained in this centered ball. By the Blaschke selection theorem, this family has compact closure in the Hausdorff metric; volume is continuous there.

A useful quantity to minimize on this closure is the moment of inertia

$$J(L) = \int_L |x|^2\, dx.$$

Centering a fiber decreases its second moment, strictly unless that fiber was already centered. Thus $J(S_\theta L) \le J(L)$, with equality, for convex bodies, precisely when L is symmetric about $\theta^\perp$. If L minimizes J in the closure of all finite successive symmetrals, continuity shows that

$S_\theta L$ belongs to the same closure. Minimality therefore forces equality for every θ, and L must be a centered ball.

One obtains an actual sequence by choosing successive finite batches of symmetrizations that approach this minimum. The same argument applied to $J(K)+J(L)$ gives simultaneous convergence for any prescribed pair. A standard diagonal argument over a countable dense family of convex bodies gives a sequence that works for all convex bodies; see [4]. The radius follows from preservation of volume. □

Proof of Brunn–Minkowski via Steiner symmetrization

For a direction θ_1,

$$|K+L| = |S_{\theta_1}(K+L)| \geq |S_{\theta_1}(K) + S_{\theta_1}(L)|.$$

Applying another symmetrization gives

$$\begin{aligned} |S_{\theta_1}K + S_{\theta_1}L| &= |S_{\theta_2}(S_{\theta_1}K + S_{\theta_1}L)| \\ &\geq |S_{\theta_2}S_{\theta_1}K + S_{\theta_2}S_{\theta_1}L|. \end{aligned}$$

Choose a common sequence along which the two bodies converge to $R_1B_2^n$ and $R_2B_2^n$. Continuity of Minkowski addition and volume yields

$$|K+L| \geq |(R_1+R_2)B_2^n| = (R_1+R_2)^n|B_2^n|.$$

Since $|K| = R_1^n|B_2^n|$ and $|L| = R_2^n|B_2^n|$, we obtain

$$|K+L|^{1/n} \geq R_1|B_2^n|^{1/n} + R_2|B_2^n|^{1/n} = |K|^{1/n} + |L|^{1/n}.$$

This proves Brunn–Minkowski for convex bodies. □

3.4 Mixed volumes and Minkowski's first inequality

Definition 3.3. For convex bodies $K, L \subset \mathbb{R}^n$, define

$$V_j(K,L) = \frac{(n-j)!}{n!}\,\frac{d^j}{dt^j}|K+tL|\bigg|_{t=0}, \qquad 0 \leq j \leq n.$$

These are the mixed volumes with $n-j$ copies of K and j copies of L; derivatives at zero are taken from the right.

We briefly outline some facts about mixed volumes.

- $V_0(K,L) = |K|$ and $V_n(K,L) = |L|$.
- $V_j(K,B_2^n)$ are often called *quermassintegrals*. Up to dimension-dependent normalizations, they are the intrinsic volumes of dimension $n-j$. For polytopes, intrinsic volumes are sums of face volumes weighted by the corresponding exterior angles; they are not generally the unweighted total areas of the faces.

- $V_1(K, B_2^n) = |\partial K|/n$. More generally, $nV_1(K, L)$ is the anisotropic perimeter associated with L.
- The function $|K + tL|$ is a polynomial of degree n for $t \geq 0$:

$$|K + tL| = \sum_{j=0}^{n} \frac{t^j}{j!} \frac{d^j}{dt^j} |K + tL| \Big|_{t=0} = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} V_j(K, L) t^j = \sum_{j=0}^{n} \binom{n}{j} V_j(K, L) t^j.$$

  For $L = B_2^n$, this is the Steiner polynomial. All derivatives of order greater than n vanish.
- The mixed volume $V_{n-1}(K, B_2^n)$ is proportional to the mean width:

$$w(K) = \frac{2}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} h_K(\theta)\, d\theta, \qquad V_{n-1}(K, B_2^n) = \frac{|B_2^n|}{2} w(K).$$

- There is a symmetry property: $V_j(K, L) = V_{n-j}(L, K)$.
- In the plane, $V_1(K, L) = V_1(L, K)$, and

$$|K + tL| = |K| + 2tV_1(K, L) + t^2|L|.$$

We leave the proofs of these deeper facts to the reader; [4, 167] contain much more detailed discussions. Here is a classical consequence of Brunn–Minkowski.

Lemma 3.1 (Minkowski's first inequality)**.** *For convex bodies K, L,*

$$V_1(K, L) \geq |K|^{(n-1)/n} |L|^{1/n}.$$

Proof. Differentiate the Brunn–Minkowski lower bound for $|K + tL|$ at $t = 0$, just as in the proof of the isoperimetric inequality. The details are homework. □

Remark 3.3 (Minkowski's second inequality)**.** *Brunn–Minkowski implies that $F(t) = |K + tL|^{1/n}$ is concave. A differentiable concave function lies below its tangent line; its second derivative, when defined, is nonpositive. The first observation gives Minkowski's first inequality. Applying the second observation at zero gives*

$$V_2(K, L)|K| \leq V_1(K, L)^2;$$

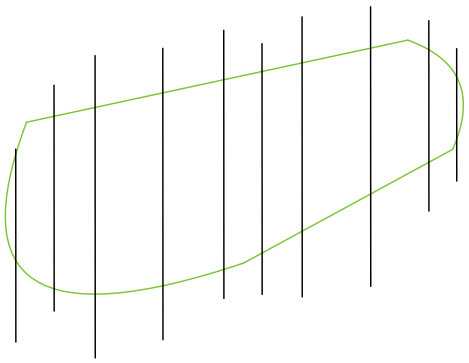
see Question 3.11.

Remark 3.4. *More generally, the Alexandrov–Fenchel inequalities imply that the mixed volumes $V_j(K, L)$ form a log-concave sequence. Their normalization is chosen so that this statement has no extra binomial coefficients; see [167].*

3.5 Brunn's concavity principle

Definition 3.4 (Section function)**.** For a convex body $K \subset \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$, define

$$A_{\theta,K}(t) = |K \cap (\theta^\perp + t\theta)|_{n-1}.$$



Theorem 3.3 (Brunn)**.** *For $n \geq 2$, the function $A_{\theta,K}^{1/(n-1)}$ is concave on its support interval.*

Proof. Write $K_t = K \cap (\theta^\perp + t\theta)$. Convexity gives

$$\lambda K_s + (1-\lambda) K_t \subseteq K_{\lambda s + (1-\lambda)t}. \tag{6}$$

Indeed, the convex combination of two points of K remains in K, and the corresponding affine hyperplanes satisfy

$$\lambda(\theta^\perp + s\theta) + (1-\lambda)(\theta^\perp + t\theta) = \theta^\perp + (\lambda s + (1-\lambda)t)\theta.$$

Thus the scalar product of the resulting point with θ is $\lambda s + (1-\lambda)t$. After identifying the parallel section hyperplanes with $\mathbb{R}^{n-1}$, Brunn–Minkowski gives

$$\begin{aligned} |K_{\lambda s+(1-\lambda)t}|_{n-1}^{1/(n-1)} &\geq |\lambda K_s + (1-\lambda) K_t|_{n-1}^{1/(n-1)} \\ &\geq \lambda |K_s|_{n-1}^{1/(n-1)} + (1-\lambda)|K_t|_{n-1}^{1/(n-1)}. \end{aligned}$$

□

Remark 3.5. *The same argument works for a k-dimensional linear subspace H: the function*

$$y \longmapsto |K \cap (H + y)|_k^{1/k}, \qquad y \in H^\perp,$$

is concave on its support. We leave the details as homework.

3.6 Log-concave functions and measures, Borell's theorem and the Prékopa–Leindler inequality

We start with two seemingly different definitions.

Definition 3.5 (Log-concave function)**.** A nonnegative function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$. Equivalently, $\log f$ is concave on its positivity set, which must be convex; we use $\log 0 = -\infty$.

Thus a nonzero log-concave function can be written as $f = e^{-V}$ with V an extended-valued convex function. Its support is convex. Products of log-concave functions are log-concave. Examples include

- $f = \mathbb{1}_K$ for a convex set K;
- $f(x) = e^{-|x|^2/2}$;
- $f(x) = e^{-\|x\|_M^q} \mathbb{1}_K(x)$, where $q \geq 1$, K is convex, and M is a convex body containing the origin in its interior.

Definition 3.6 (Log-concave measure)**.** A Borel measure μ on $\mathbb{R}^n$ is log-concave if

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}$$

for Borel sets K, L and $0 < \lambda < 1$. Measures supported on an affine subspace are allowed; when discussing densities in $\mathbb{R}^n$, we assume full-dimensional support. We use the completion of the measure for the measurable Minkowski sums that occur here.

These notions are closely related.

Theorem 3.4 (Borell)**.** *A nonzero locally finite Borel measure is log-concave if and only if, on the affine hull of its support, it has a log-concave density with respect to the corresponding Lebesgue measure.*

Remark 3.6. *Lebesgue measure has the constant log-concave density* 1*. Thus Borell's theorem contains the multiplicative Brunn–Minkowski inequality, and homogeneity recovers its additive form. It also applies to Gaussian measure, giving*

$$\gamma(\lambda K + (1-\lambda)L) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}.$$

One standard proof uses the following celebrated functional version of Brunn–Minkowski.

Theorem 3.5 (Prékopa–Leindler inequality)**.** *Fix $0 < \lambda < 1$. Let $f, g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be measurable potentials such that*

$$h(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)g(y)$$

for all x, y. Then

$$\int e^{-h} \ge \left(\int e^{-f}\right)^{\lambda} \left(\int e^{-g}\right)^{1-\lambda}.$$

We assume the two integrals on the right are finite; the zero-integral and endpoint cases are immediate.

Remark 3.7. *Equivalently, for nonnegative measurable F, G, H, the assumption*

$$H(\lambda x + (1-\lambda)y) \ge F(x)^{\lambda} G(y)^{1-\lambda}$$

implies $\int H \ge (\int F)^{\lambda} (\int G)^{1-\lambda}$.

Proof of the density-to-measure direction of theorem 3.4. Let f be a log-concave density and $d\mu = f\,dx$. Define

$$H(z) = f(z)\mathbb{1}_{\lambda K + (1-\lambda)L}(z), \quad F(x) = f(x)\mathbb{1}_K(x), \quad G(y) = f(y)\mathbb{1}_L(y).$$

Log-concavity of f implies

$$H(\lambda x + (1-\lambda)y) \ge F(x)^{\lambda} G(y)^{1-\lambda}.$$

Prékopa–Leindler therefore gives

$$\mu(\lambda K + (1-\lambda)L) \ge \mu(K)^{\lambda} \mu(L)^{1-\lambda}.$$

The converse direction of Borell's theorem is left as homework. □

Proof of the Prékopa–Leindler inequality

Recall the layer-cake formula: for a nonnegative measurable function f,

$$\int f\,d\mu = \int_0^{\infty} \mu(\{f > s\})\,ds. \tag{7}$$

This is Fubini's theorem applied to the subgraph $\{(x, s) : 0 < s < f(x)\}$, and is the function version of (2). We prove the formulation in remark 3.7 by induction on dimension.

Step 1: $n = 1$**.** First suppose F, G are bounded and nonzero. Divide F and G by their respective suprema a, b, and divide H by $a^\lambda b^{1-\lambda}$. This preserves the hypothesis and the desired conclusion. We may therefore assume

$$\sup F = \sup G = 1.$$

This normalization matters: for every $0 < s < 1$, both superlevel sets below are nonempty. The hypothesis gives

$$\{H > s\} \supseteq \lambda\{F > s\} + (1-\lambda)\{G > s\}. \tag{8}$$

Apply the one-dimensional Brunn–Minkowski inequality and integrate:

$$\begin{aligned}
\int_{\mathbb{R}} H &= \int_0^\infty |\{H > s\}|\, ds \\
&\geq \int_0^1 |\lambda\{F > s\} + (1-\lambda)\{G > s\}|\, ds \\
&\geq \lambda \int_0^1 |\{F > s\}|\, ds + (1-\lambda) \int_0^1 |\{G > s\}|\, ds \\
&= \lambda \int_{\mathbb{R}} F + (1-\lambda) \int_{\mathbb{R}} G \\
&\geq \left(\int_{\mathbb{R}} F\right)^\lambda \left(\int_{\mathbb{R}} G\right)^{1-\lambda}.
\end{aligned}$$

The final step is AM–GM. Undoing the normalization proves the original statement. For unbounded functions apply the bounded result to $\min(F, m)$ and $\min(G, m)$, with the same H, and let $m \to \infty$.

Step 2: induction. Assume the result is known in dimension $n-1$. Write $x = (\bar{x}, x_n)$ and define

$$H_n(x_n) = \int_{\mathbb{R}^{n-1}} H(\bar{x}, x_n)\, d\bar{x},$$

with F_n, G_n defined similarly. For fixed x_n, y_n, apply the induction hypothesis to

$$H\Big(\lambda(\bar{x}, x_n) + (1-\lambda)(\bar{y}, y_n)\Big) \geq F(\bar{x}, x_n)^\lambda G(\bar{y}, y_n)^{1-\lambda}.$$

It gives

$$H_n(\lambda x_n + (1-\lambda) y_n) \geq F_n(x_n)^\lambda G_n(y_n)^{1-\lambda}.$$

The one-dimensional result and Fubini now yield

$$\int_{\mathbb{R}^n} H = \int_{\mathbb{R}} H_n \geq \left(\int_{\mathbb{R}} F_n\right)^\lambda \left(\int_{\mathbb{R}} G_n\right)^{1-\lambda} = \left(\int_{\mathbb{R}^n} F\right)^\lambda \left(\int_{\mathbb{R}^n} G\right)^{1-\lambda}.$$

For presentation, we may first use bounded, compactly supported functions, so all fibers are integrable; truncation and the usual measurable approximation give the general version. □

3.7 Log-concavity of marginals and convolutions of log-concave measures

Definition 3.7 (Marginal)**.** Let H be a linear subspace of $\mathbb{R}^n$. The marginal $\pi_\mu(H)$ is the pushforward of μ under orthogonal projection onto $H^\perp$:

$$\pi_\mu(H)(\Omega) = \mu(\Omega + H), \qquad \Omega \subseteq H^\perp.$$

If X has law μ, then $\pi_\mu(H)$ is the law of its projection onto $H^\perp$. If μ has density f, the marginal has density

$$f_{\mu,H}(x) = \int_H f(x+y)\,dy = \int_{x+H} f(z)\,dz.$$

Example 3.1. *If μ is uniform on a Borel set A with $0 < |A| < \infty$, and $\dim H = k$, then*

$$f_{\mu,H}(x) = \frac{|A \cap (x+H)|_k}{|A|}.$$

Theorem 3.6. *Every marginal of a log-concave measure is log-concave.*

Proof. Let f be the density. Our goal is to show

$$f_{\mu,H}(\lambda x + (1-\lambda)y) \geq f_{\mu,H}(x)^\lambda f_{\mu,H}(y)^{1-\lambda}.$$

The change of variables $z = x + u$ identifies the affine fiber $x + H$ with H, so $f_{\mu,H}(x) = \int_H f(x+u)\,du$. Fix $x, y \in H^\perp$ and $\lambda \in [0,1]$. On H, set

$$A(z) = f(x+z), \quad B(z) = f(y+z), \quad C(z) = f(\lambda x + (1-\lambda)y + z).$$

For $u, v \in H$, log-concavity gives

$$\begin{aligned} C(\lambda u + (1-\lambda)v) &= f\Big(\lambda(x+u) + (1-\lambda)(y+v)\Big) \\ &\geq f(x+u)^\lambda f(y+v)^{1-\lambda} = A(u)^\lambda B(v)^{1-\lambda}. \end{aligned}$$

Prékopa–Leindler on H therefore gives

$$\begin{aligned} f_{\mu,H}(\lambda x + (1-\lambda)y) &= \int_H C \\ &\geq \left(\int_H A\right)^\lambda \left(\int_H B\right)^{1-\lambda} = f_{\mu,H}(x)^\lambda f_{\mu,H}(y)^{1-\lambda}. \end{aligned}$$

The same argument takes place in the affine support when the original measure is lower-dimensional. □

Viewing a convolution as a marginal in a higher-dimensional space gives the following.

Corollary 3.1. *If f, g are integrable log-concave functions, then their convolution $f * g$ is log-concave.*

Proof. The function $(x, y) \mapsto f(x - y)g(y)$ is log-concave, and integrating in y gives $f * g$. The details are homework. □

Thus the class of log-concave random vectors is closed under linear projections and under sums of *independent* random vectors.

3.8 Borell–Brascamp–Lieb inequality

Definition 3.8. For $p > 0$, a nonnegative function F is p-concave if F^p is concave on its positivity set. For $p < 0$, this means that F^p is convex there. The case $p = 0$ means log-concavity. Equivalently, F dominates the corresponding weighted power mean of its endpoint values.

Prékopa–Leindler is one member of a larger family.

Theorem 3.7 (Borell–Brascamp–Lieb)**.** *Fix $0 < \lambda < 1$ and $p > -1/n$, $p \neq 0$. Suppose f, g, h are nonnegative potentials, with the powers and integrals below well defined. The pointwise hypotheses below are required on pairs where the corresponding endpoint densities $f^{1/p}$ and $g^{1/p}$ are positive. If $p > 0$ and*

$$h(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)g(y),$$

then

$$\left(\int h^{1/p} \right)^{p/(np+1)} \geq \lambda \left(\int f^{1/p} \right)^{p/(np+1)} + (1 - \lambda) \left(\int g^{1/p} \right)^{p/(np+1)}.$$

If $-1/n < p < 0$, reverse both displayed inequalities. In the negative-power formulation, potentials may take the value $+\infty$, with $(+\infty)^{1/p} = 0$; in particular, the pointwise condition concerns finite endpoint potentials.

The limiting case $p = 0$, expressed in terms of the densities, is Prékopa–Leindler. At $p = -1/n$, the conclusion becomes the minimum bound $\int H \geq \min\{\int F, \int G\}$. See the equivalent density formulation in Question 3.9.

Applying this theorem in place of Prékopa–Leindler gives the following extension of theorem 3.6.

Theorem 3.8. *If F is p-concave and one integrates over k-dimensional fibers, the resulting marginal density is $p/(1+kp)$-concave, provided $p \geq -1/k$, with the usual limiting conventions at the endpoints.*

The proof follows the same pattern as before and is left as homework; see [42].

3.9 Linearizations of geometric and functional inequalities

We will repeatedly exploit a simple idea. Suppose a functional $\mathcal{F}$ is concave:

$$\mathcal{F}((1-t)f+tg) \geq (1-t)\mathcal{F}(f)+t\mathcal{F}(g).$$

For fixed f, g, set

$$\alpha(t) = \mathcal{F}((1-t)f+tg) - (1-t)\mathcal{F}(f) - t\mathcal{F}(g).$$

Then the following hold whenever the indicated derivatives exist.

1. $\alpha'(0) \geq 0$, because $\alpha(t) \geq 0$ for $t \in [0,1]$ and $\alpha(0) = 0$.
2. $\alpha''(t) \leq 0$, because the subtracted terms are affine in t and $\mathcal{F}$ is concave along this line.
3. More generally, if $\mathcal{F}$ attains a maximum at f_0, then along any admissible two-sided variation $f_0 + \varepsilon f$, the first derivative is zero and the second derivative is nonpositive.

These observations uncover new inequalities inside existing concavity statements.

Our first goal is to put Prékopa–Leindler into this form. Its hypothesis is

$$h((1-t)x+ty) \leq (1-t)f(x)+tg(y).$$

What is the largest possible h satisfying all these inequalities?

Definition 3.9 (Infimal convolution)**.** For $0 < t < 1$, define

$$f\square_t g(z) = \inf_{(1-t)x+ty=z} \{(1-t)f(x)+tg(y)\}.$$

We set $f\square_0 g = f$ and $f\square_1 g = g$. We also use the unweighted infimal convolution

$$f\square g(z) = \inf_{x+y=z} \{f(x)+g(y)\}.$$

Since $h = f\square_t g$ satisfies the hypothesis, Prékopa–Leindler gives

$$\int e^{-f\square_t g} \geq \left(\int e^{-f}\right)^{1-t} \left(\int e^{-g}\right)^{t}.$$

The endpoint identities $f\square_0 g = f$ and $f\square_1 g = g$ show that this operation interpolates between f and g. We would like a transformation $\mathcal{T}$ that turns the inequality into ordinary linear interpolation:

$$\int e^{-\mathcal{T}((1-t)f+tg)} \geq \left(\int e^{-\mathcal{T}(f)}\right)^{1-t} \left(\int e^{-\mathcal{T}(g)}\right)^{t}.$$

In other words, we want a convenient linear coordinate system for this nonlinear interpolation.

Example 3.2. *For convex bodies K, L,*

$$\mathbb{1}_K^\infty \square \mathbb{1}_L^\infty = \mathbb{1}_{K+L}^\infty.$$

Indeed, the infimum is zero precisely when $z = x + y$ with $x \in K$ and $y \in L$.

Since $h_{K+L} = h_K + h_L$, a promising transformation should send $\mathbb{1}_K^\infty$ to h_K and infimal convolution to addition. The Legendre transform does exactly this.

3.10 Legendre transform

Definition 3.10. For $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its Legendre transform is

$$f^*(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - f(y)\}.$$

We will work with proper functions, meaning that f is finite somewhere and never takes the value $-\infty$.

Example 3.3. *1. For a convex body K,*

$$\begin{aligned}(\mathbb{1}_K^\infty)^*(x) &= \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - \mathbb{1}_K^\infty(y)\} \\ &= \sup_{y \in K} \langle x, y\rangle = h_K(x).\end{aligned}$$

2. For $f(y) = |y|$,

$$f^*(x) = \sup_{r \geq 0} \{r|x| - r\} = \begin{cases} 0, & |x| \leq 1, \\ +\infty, & |x| > 1, \end{cases} = \mathbb{1}_{B_2^n}^\infty(x).$$

3. The quadratic $f(y) = |y|^2/2$ is its own Legendre transform:

$$f^*(x) = \sup_y \{\langle x, y\rangle - |y|^2/2\} = |x|^2/2.$$

For $x \neq 0$, one may first maximize over vectors parallel to x and write

$$\sup_{t \geq 0} \{t|x|^2 - t^2|x|^2/2\}.$$

This quadratic is maximized at $t = 1$, so the maximizing vector is $y = x$. Completing the square gives the same answer and also covers $x = 0$.

4. If $f(y) = C$ is constant, then

$$f^*(x) = \begin{cases} -C, & x = 0, \\ +\infty, & x \neq 0. \end{cases}$$

The value at zero records the constant C.

5. Let

$$f(y) = \begin{cases} -\sqrt{1 - |y|^2}, & |y| \leq 1, \\ +\infty, & |y| > 1. \end{cases}$$

For $x \neq 0$, the maximizing y is parallel to x, and

$$f^*(x) = \sup_{0 \leq r \leq 1} \{r|x| + \sqrt{1 - r^2}\}.$$

Differentiate the function inside the supremum:

$$\frac{d}{dr}\Big(r|x| + \sqrt{1-r^2}\Big) = |x| - \frac{r}{\sqrt{1-r^2}}.$$

It vanishes at $r = |x|/\sqrt{1+|x|^2}$, which lies in the required interval. Substituting gives

$$\begin{aligned} f^*(x) &= \frac{|x|^2}{\sqrt{1+|x|^2}} + \sqrt{1 - \frac{|x|^2}{1+|x|^2}} \\ &= \frac{|x|^2+1}{\sqrt{1+|x|^2}} = \sqrt{1+|x|^2}. \end{aligned}$$

The same formula holds at $x = 0$. In one dimension this is the upper branch of a hyperbola with asymptotes $y = \pm x$.

Lemma 3.2 (Properties of the Legendre transform)**.** *1. f^* is convex and lower semicontinuous, as a supremum of continuous affine functions.*

2. *If f is proper, lower semicontinuous, and convex, then $f^{**} = f$.*
3. *$(f + a)^* = f^* - a$ for $a \in \mathbb{R}$.*
4. *If $f \le g$, then $f^* \ge g^*$.*
5. *$(af)^*(x) = af^*(x/a)$ for $a > 0$.*

Proof. Homework! In the second property, a finite convex function on all of $\mathbb{R}^n$ is automatically continuous, so the lower semicontinuity assumption causes no extra difficulty in that case. □

Proposition 3.1 (The Legendre transform linearizes infimal convolution)**.** *For proper functions for which the expressions are defined,*

$$(f \square g)^* = f^* + g^*.$$

In particular, $(f \square_t g)^* = (1-t)f^* + tg^*$.

Proof. A direct calculation gives

$$\begin{aligned} (f\square g)^*(x) &= \sup_y \Big\{ \langle x, y\rangle - \inf_{a+b=y} (f(a) + g(b)) \Big\} \\ &= \sup_{a,b} \{\langle x, a+b\rangle - f(a) - g(b)\} \\ &= f^*(x) + g^*(x). \end{aligned}$$

The weighted identity follows in the same way. □

Remark 3.8. *It is useful to picture what happens to epigraphs under infimal convolution. For closed convex functions, taking the Legendre transform back recovers the lower semicontinuous closure of the infimal convolution; in the smooth coercive situations below, no closure correction is needed.*

Proposition 3.2 (Legendre transform of smooth functions)**.** *Let $V \in C^2(\mathbb{R}^n)$ have positive definite Hessian. On the image of ∇V, its Legendre transform is smooth and*

1. $V(x) + V^*(\nabla V(x)) = \langle x, \nabla V(x)\rangle$;
2. $\nabla V^*(\nabla V(x)) = x$, *and* $\nabla V(\nabla V^*(y)) = y$ *for* $y \in \nabla V(\mathbb{R}^n)$;
3. $\nabla^2 V^*(\nabla V(x)) = (\nabla^2 V(x))^{-1}$.

Proof. Fenchel's inequality is $V(x) + V^*(y) \geq \langle x, y\rangle$. For $y = \nabla V(x)$, convexity gives

$$V(z) \geq V(x) + \langle \nabla V(x), z - x\rangle,$$

so x attains the supremum defining $V^*(y)$, proving the first identity. Differentiating it with respect to x gives

$$\nabla V(x) + \nabla^2 V(x)\nabla V^*(\nabla V(x)) = \nabla V(x) + \nabla^2 V(x)x.$$

Cancel the invertible Hessian to obtain the second identity, then differentiate once more to obtain the third. □

Remark 3.9. *For the sake of presentation, our differential calculations will use smooth potentials with positive definite Hessians, sufficient growth, and enough decay to justify differentiation under the integral and integration by parts. Strict convexity alone does not guarantee an invertible Hessian. One can generally pass to the intended convex setting by adding a small quadratic term, smoothing, and using truncation and limiting arguments. Gradient identities are understood on the natural dual domain; superlinear growth makes that domain all of $\mathbb{R}^n$.*

Remark 3.10. *The duality identities also give $V^{**} = V$ in this smooth setting, in agreement with lemma 3.2.*

3.11 Generalized log-Sobolev inequality

Recall that Prékopa–Leindler can be written as

$$\int e^{-f \square_t g} \geq \left(\int e^{-f}\right)^{1-t} \left(\int e^{-g}\right)^{t}.$$

Using Legendre transforms gives

$$\int e^{-((1-t)f + tg)^*} \geq \left(\int e^{-f^*}\right)^{1-t} \left(\int e^{-g^*}\right)^{t}.$$

In fact, this inequality does not require f, g themselves to be convex: the definition of the conjugate directly gives

$$((1-t)f + tg)^*((1-t)x + ty) \le (1-t)f^*(x) + tg^*(y),$$

and Prékopa–Leindler applies. This formulation was noted by Cordero-Erausquin and Klartag [53]. Thus $f \mapsto \log \int e^{-f^*}$ is a concave functional on its natural domain.

Remark 3.11. *By contrast, Hölder's inequality says that $f \mapsto \log \int e^{-f}$ is convex.*

Set

$$\alpha(t) = \log \int e^{-((1-t)f+tg)^*} - (1-t)\log \int e^{-f^*} - t \log \int e^{-g^*}.$$

We have $\alpha(t) \ge 0$, $\alpha(0) = 0$, and hence $\alpha'(0) \ge 0$. To compute the derivative, we need the following lemma.

Lemma 3.3. *Let V_t be a smooth family of convex functions in the regular setting described after proposition 3.2. Put $y = \nabla V_t^*(x)$, and let dots denote derivatives in t. Then*

$$\frac{d}{dt} V_t^*(x) = -\dot{V}_t(y),$$
$$\frac{d^2}{dt^2} V_t^*(x) = -\ddot{V}_t(y) + \left\langle (\nabla^2 V_t(y))^{-1} \nabla \dot{V}_t(y), \nabla \dot{V}_t(y) \right\rangle.$$

Proof. We prove the first identity; the second is homework. Differentiate

$$V_t(z) + V_t^*(\nabla V_t(z)) = \langle \nabla V_t(z), z \rangle$$

with respect to t. The chain rule gives

$$\dot{V}_t(z) + (\partial_t V_t^*)(\nabla V_t(z)) + \langle \nabla V_t^*(\nabla V_t(z)), \nabla \dot{V}_t(z) \rangle = \langle \nabla \dot{V}_t(z), z \rangle.$$

Since $\nabla V_t^*(\nabla V_t(z)) = z$, the last terms cancel. Substitute $z = \nabla V_t^*(x)$ to finish. □

Write $Z_F = \int e^{-F}$. The lemma gives

$$\begin{aligned}\alpha'(0) &= -\frac{1}{\int e^{-f^*}} \int e^{-f^*}\, \partial_t\Big(((1-t)f + tg)^*\Big)\Big|_{t=0} + \log \frac{\int e^{-f^*}}{\int e^{-g^*}} \\ &= \frac{1}{\int e^{-f^*}} \int (g-f)(\nabla f^*) e^{-f^*} + \log \frac{\int e^{-f^*}}{\int e^{-g^*}} \ge 0.\end{aligned}$$

For convex f, g, put $F = f^*$ and $G = g^*$. Rearranging,

$$\int G^*(\nabla F) e^{-F} \ge \int F^*(\nabla F) e^{-F} + Z_F \log \frac{Z_G}{Z_F}. \tag{9}$$

In the equal-mass case this resembles Minkowski's first inequality.

Theorem 3.9 (Minkowski's first inequality for functions)**.** *If F, G are convex and $\int e^{-F} = \int e^{-G}$, then*

$$\int G^*(\nabla F)e^{-F} \geq \int F^*(\nabla F)e^{-F}.$$

For fixed F, the left-hand side is minimized by $G = F$.

Remark 3.12. *Compare this with $V_1(K, L) \geq |K|$ when $|K| = |L|$.*

The duality formula and integration by parts give

$$\int F^*(\nabla F)e^{-F} = \int (\langle \nabla F, x\rangle - F)e^{-F},$$
$$\int \langle \nabla F, x\rangle e^{-F} = -\int \langle \nabla e^{-F}, x\rangle = n\int e^{-F}.$$

Substituting into (9) yields the following.

Theorem 3.10 (Generalized log-Sobolev inequality)**.** *For convex F, G with finite nonzero exponential integrals,*

$$\int G^*(\nabla F)e^{-F} \geq n\int e^{-F} - \int Fe^{-F} + \left(\int e^{-F}\right)\log\frac{\int e^{-G}}{\int e^{-F}}.$$

Corollary 3.2. *If $\int e^{-F} = \int e^{-G}$, then*

$$\int G^*(\nabla F)e^{-F} \geq n\int e^{-F} - \int Fe^{-F}.$$

Remark 3.13. *The derivation of log-Sobolev inequalities by linearizing Prékopa–Leindler goes back to Bobkov and Ledoux.*

Remark 3.14. *Recall that $\nabla F^* \circ \nabla F = \mathrm{Id}$ in the regular setting. Together with the duality identity, this also recovers $F^{**} = F$.*

3.12 Reformulations and notable special cases of the generalized log-Sobolev inequality

Remark 3.15. *Equality holds in theorem 3.10 when $F = G$.*

Remark 3.16. *The functional inequality recovers Minkowski's first inequality by approximating the convex indicators $F = \mathbb{1}_K^\infty$ and $G = \mathbb{1}_L^\infty$. The equal-mass condition becomes $|K| = |L|$, and the first variation tends to*

$$\int_{\partial K} h_L(n_x)\, d\mathcal{H}^{n-1}(x) = nV_1(K, L).$$

The corresponding expression for $L = K$ is $n|K|$. Thus the limit gives $V_1(K, L) \geq |K|$. One should use smooth approximations, or the first variation of the indicator integrals: the gradient of the extended-valued function $\mathbb{1}_K^\infty$ is not an ordinary boundary vector field.

Put $\phi = e^{-F}$. Then $\nabla F = -\nabla\phi/\phi$. If $\int e^{-G} = 1$, theorem 3.10 becomes

$$\int G^*\left(-\frac{\nabla\phi}{\phi}\right)\phi \geq n\int \phi + \int \phi\log\phi - \left(\int\phi\right)\log\int\phi.$$

Definition 3.11 (Entropy)**.** For a nonnegative function ϕ, define

$$\mathrm{Ent}_\mu(\phi) = \int \phi\log\phi\, d\mu - \left(\int \phi\, d\mu\right)\log\left(\int \phi\, d\mu\right),$$

with $0\log 0 = 0$. We write $\mathrm{Ent}(\phi)$ when μ is Lebesgue measure.

Remark 3.17. *If μ is a probability measure, Jensen's inequality for $t\log t$ gives $\mathrm{Ent}_\mu(\phi) \geq 0$. This need not hold for Lebesgue measure.*

Theorem 3.11 (Reformulation of generalized log-Sobolev)**.** *For a log-concave function ϕ and a convex G with $\int e^{-G} = 1$,*

$$\int G^*\left(-\frac{\nabla\phi}{\phi}\right)\phi \geq n\int\phi + \mathrm{Ent}(\phi).$$

Taking $G = \mathbb{1}_{B_2^n}^\infty + \log|B_2^n|$, whose conjugate is $G^*(x) = |x| - \log|B_2^n|$, gives the following.

Corollary 3.3 (L_1 entropy–Sobolev inequality, Bobkov–Ledoux)**.** *For a log-concave ϕ,*

$$\mathrm{Ent}(\phi) + C_n\int\phi \leq \int|\nabla\phi|, \qquad C_n = n + \log|B_2^n|.$$

For smooth positive functions this is an ordinary gradient integral; approximation gives the corresponding total-variation interpretation at jumps.

Next take $G(x) = |x|^2/2 + n\log\sqrt{2\pi}$, so that $G^*(x) = |x|^2/2 - n\log\sqrt{2\pi}$.

Corollary 3.4 (Classical Lebesgue log-Sobolev inequality, first form)**.**

$$\mathrm{Ent}(\phi) + n\log(\sqrt{2\pi}e)\int\phi \leq \frac{1}{2}\int\frac{|\nabla\phi|^2}{\phi}.$$

The integral on the right is the Lebesgue Fisher information, up to the factor $1/2$. Substituting $\phi = f^2$ gives

Corollary 3.5 (Classical Lebesgue log-Sobolev inequality, second form)**.**

$$\mathrm{Ent}(f^2) + n\log(\sqrt{2\pi}e)\int f^2 \leq 2\int|\nabla f|^2.$$

Remark 3.18. *The two classical inequalities hold without a log-concavity assumption. A direct differentiation of infimal convolution proves this extension; see the homework.*

For $1 < p < \infty$, let $q = p/(p-1)$ and

$$Z_q = \int_{\mathbb{R}^n} e^{-|x|^q/q}\,dx = |\mathbb{S}^{n-1}| \int_0^\infty r^{n-1} e^{-r^q/q}\,dr.$$

Taking $G(x) = |x|^q/q + \log Z_q$, whose conjugate is $G^*(y) = |y|^p/p - \log Z_q$, gives

Corollary 3.6 (L_p entropy–Sobolev inequality)**.** *For nonnegative f with f^p log-concave,*

$$\mathrm{Ent}(f^p) + (n + \log Z_q) \int f^p \le p^{p-1} \int |\nabla f|^p.$$

Finally, we formulate the Gaussian version.

Theorem 3.12 (Gaussian log-Sobolev inequality)**.** *For standard Gaussian measure γ and $g \in W^{1,2}(\gamma)$,*

$$\mathrm{Ent}_\gamma(g^2) \le 2 \int |\nabla g|^2\,d\gamma.$$

Remark 3.19. *No log-concavity assumption on g is needed; see the homework.*

Remark 3.20. *The theorem is equivalent to corollary 3.5 under*

$$g(x) = (2\pi)^{n/4} e^{|x|^2/4} f(x).$$

The cancellation after integration by parts is homework.

Remark 3.21. *The function $g = 1$ gives equality. More generally, exponential affine functions give equality whenever the integrals are finite.*

Remark 3.22. *A related classical inequality is the sharp Lebesgue L_1 Sobolev inequality, for $n \ge 2$:*

$$n|B_2^n|^{1/n} \left(\int_{\mathbb{R}^n} |f|^{n/(n-1)}\,dx \right)^{(n-1)/n} \le \int_{\mathbb{R}^n} |\nabla f|\,dx.$$

It is equivalent to the Euclidean isoperimetric inequality, via the coarea formula and approximation of indicator functions.

Remark 3.23. *There is also a spherical log-Sobolev inequality. If σ is normalized surface measure on S^{n-1}, then*

$$\mathrm{Ent}_\sigma(f^2) \le \frac{2}{n-1} \int_{S^{n-1}} |\nabla_{S^{n-1}} f|^2\,d\sigma.$$

By applying it on spheres of radius $\sqrt{n}$ and passing to Gaussian coordinate marginals as $n \to \infty$, one obtains theorem 3.12.

Theorem 3.13 (Generalized log-Sobolev inequality for log-concave measures)**.** *Let $d\mu = e^{-V}\,dx$ be log-concave. If F, G are convex and $\int e^{-G}\,d\mu = 1$, then*

$$\mathrm{Ent}_\mu(e^{-F}) + n \int e^{-F}\,d\mu - \int \langle \nabla V, x \rangle e^{-F}\,d\mu \le \int G^*(\nabla F) e^{-F}\,d\mu.$$

Proof. Homework! Use Prékopa–Leindler relative to the log-concave density and weighted integration by parts. □

3.13 The p-Beckner inequality

We mention the following result without proof.

Theorem 3.14 (p-Beckner inequality)**.** *For $f \in W^{1,2}(\gamma)$ and $1 \le p < 2$,*

$$\int f^2 \, d\gamma - \left(\int |f|^p \, d\gamma \right)^{2/p} \le (2-p) \int |\nabla f|^2 \, d\gamma.$$

At $p = 1$, this gives Gaussian Poincaré for nonnegative f. Apply it to $f + c$ for a sufficiently large constant c and then use approximation to obtain, for arbitrary f,

$$\mathrm{Var}_\gamma(f) \le \int |\nabla f|^2 \, d\gamma.$$

We will prove this independently below. At the other endpoint, dividing by $2 - p$ and letting $p \uparrow 2$ gives

$$\mathrm{Ent}_\gamma(f^2) \le 2 \int |\nabla f|^2 \, d\gamma;$$

see the homework.

Remark 3.24. *Gaussian log-Sobolev also implies the full Beckner family, through the associated hypercontractive semigroup. One should not interpret this as pointwise monotonicity in p of the displayed deficit divided by $2 - p$; that monotonicity is not valid for arbitrary f.*

Remark 3.25. *Analogous inequalities hold on the sphere, with the appropriate dimension-dependent constants and normalized surface measure.*

Question (Siva): Is there a Lebesgue Beckner inequality?
Question 2 (Siva): What happens if we substitute $\phi = f^2 e^{-V}$ into the Lebesgue generalized log-Sobolev inequality?

Below we fix a log-concave probability measure $d\mu = e^{-G} \, dx$, with G smooth and convex.

Definition 3.12 (Variance)**.**

$$\mathrm{Var}_\mu(\phi) = \int \phi^2 \, d\mu - \left(\int \phi \, d\mu \right)^2.$$

Variance is unchanged by adding a constant and vanishes for constant functions. It measures the squared L^2 distance from the best constant approximation.

3.14 A few words about the Laplace operator with respect to log-concave measures

Definition 3.13 (Laplace operator associated with μ)**.** For a sufficiently regular function u, set

$$L_\mu u = \Delta u - \langle \nabla G, \nabla u \rangle.$$

Example 3.4. *For Lebesgue measure this is Δ. For Gaussian measure it is $L_\gamma u = \Delta u - \langle x, \nabla u\rangle$, the Ornstein–Uhlenbeck operator.*

Lemma 3.4 (Integration by parts)**.** *For smooth u, v with compact support, or with sufficient decay to make the boundary terms vanish,*

$$\int u L_\mu v \, d\mu = -\int \langle \nabla u, \nabla v\rangle \, d\mu.$$

Proof. Classical integration by parts gives

$$\begin{aligned}\int u L_\mu v \, d\mu &= \int u e^{-G} \Delta v \, dx - \int u \langle \nabla G, \nabla v\rangle e^{-G} \, dx \\ &= -\int \langle \nabla (u e^{-G}), \nabla v\rangle \, dx - \int u \langle \nabla G, \nabla v\rangle e^{-G} \, dx \\ &= -\int \langle \nabla u, \nabla v\rangle \, d\mu.\end{aligned}$$

The two terms involving ∇G cancel. □

3.15 A short and nonstandard proof sketch of integration by parts

There is a simple change-of-variables proof, useful later for higher-order identities.

Lemma 3.5 (Integration by parts)**.** *For $d\mu = e^{-v}\, dx$ and $L_\mu g = \Delta g - \langle \nabla v, \nabla g\rangle$,*

$$\int f L_\mu g \, d\mu = -\int \langle \nabla f, \nabla g\rangle \, d\mu,$$

under the same regularity and decay assumptions. When $v = 0$, this is ordinary Lebesgue integration by parts.

Proof. For presentation, take g smooth with bounded Hessian and f compactly supported, so that $x = y + t\nabla g(y)$ is a diffeomorphism for sufficiently small $|t|$. Then

$$\int f(x) \, d\mu(x) = \int f(y + t\nabla g(y)) e^{-v(y + t\nabla g(y))} \det(\mathrm{Id} + t\nabla^2 g(y)) \, dy.$$

The left-hand side is independent of t. Differentiate at zero, using $\frac{d}{dt} \det(\mathrm{Id} + tA)\Big|_{t=0} = \operatorname{tr} A$, to obtain

$$0 = \int \Big(\langle \nabla f, \nabla g\rangle - f \langle \nabla v, \nabla g\rangle + f \Delta g \Big) \, d\mu.$$

Rearranging gives the claim. Cutoffs and approximation recover the usual broader function classes. □

3.16 A word about eigenfunctions

The spectral gap of $-L_\mu$ is

$$\lambda_1 = \inf_{\substack{u \in W^{1,2}(\mu) \\ \mathrm{Var}_\mu(u) > 0}} \frac{\int |\nabla u|^2 \, d\mu}{\mathrm{Var}_\mu(u)} = \inf_{\substack{u \in W^{1,2}(\mu), \int u \, d\mu = 0 \\ u \neq 0}} \frac{\int |\nabla u|^2 \, d\mu}{\int u^2 \, d\mu}.$$

The equality follows by subtracting the mean. If the infimum is attained, a minimizing function satisfies $L_\mu u = -\lambda_1 u$, and λ_1 is the first positive eigenvalue. In general the spectral gap need not be an eigenvalue; we return to this distinction later.

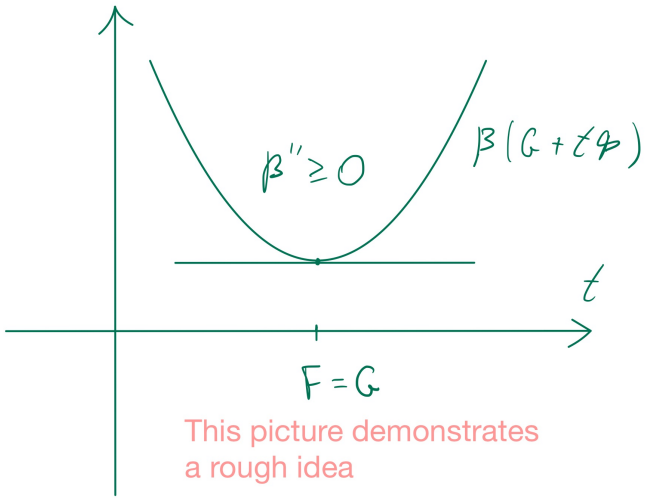
For a full-dimensional log-concave probability measure the gap is positive. With our convention for the Poincaré constant,

$$C_{\mathrm{Poin}}(\mu) = \frac{1}{\lambda_1}, \qquad \mathrm{Var}_\mu(u) \leq \frac{1}{\lambda_1} \int |\nabla u|^2 \, d\mu.$$

We will discuss the analytic details later in the course.

3.17 Deriving the Brascamp–Lieb inequality from generalized log-Sobolev

We obtained generalized log-Sobolev by taking a first derivative of Prékopa–Leindler. We now take a second derivative at an equality case. The nonnegativity of the resulting second variation will give the Brascamp–Lieb inequality.



For convex F, G with $\int e^{-G} = 1$, generalized log-Sobolev says

$$\int G^*(\nabla F) e^{-F} \geq n \int e^{-F} - \int F e^{-F} - \left(\int e^{-F} \right) \log \int e^{-F},$$

with equality at $F = G$. Put $F = G + t\phi$, $d\mu = e^{-G}\,dx$, and define the deficit

$$\begin{aligned}\beta(t) = &\int G^*(\nabla G + t\nabla\phi)e^{-G-t\phi} - n\int e^{-G-t\phi} \\ &+ \int (G + t\phi)e^{-G-t\phi} + \left(\int e^{-G-t\phi}\right)\log\int e^{-G-t\phi}.\end{aligned}$$

We work first with a positive definite Hessian and compactly supported smooth perturbations, so $G + t\phi$ is convex for small $|t|$. Then $\beta \geq 0$ and $\beta(0) = 0$, hence $\beta'(0) = 0$ and $\beta''(0) \geq 0$. Approximation extends the final inequality to the stated class.

First expand

$$\begin{aligned}G^*(\nabla G + t\nabla\phi) &= G^*(\nabla G) + t\langle \nabla G^*(\nabla G), \nabla\phi\rangle \\ &\quad + \frac{t^2}{2}\langle \nabla^2 G^*(\nabla G)\nabla\phi, \nabla\phi\rangle + o(t^2), \\ e^{-G-t\phi} &= e^{-G}\left(1 - t\phi + \frac{t^2}{2}\phi^2\right) + o(t^2).\end{aligned}$$

Let $m = \int \phi\,d\mu$. Since μ is a probability measure,

$$\begin{aligned}\log\int e^{-G-t\phi} &= \log\left(1 - tm + \frac{t^2}{2}\int \phi^2\,d\mu + o(t^2)\right) \\ &= -tm + \frac{t^2}{2}\left(\int \phi^2\,d\mu - m^2\right) + o(t^2) \\ &= -tm + \frac{t^2}{2}\operatorname{Var}_\mu(\phi) + o(t^2). \qquad (10)\end{aligned}$$

Using the gradient and Hessian identities from proposition 3.2, the first term of β expands as

$$\begin{aligned}&\int G^*(\nabla G + t\nabla\phi)e^{-G-t\phi} \\ &= \int \left(G^*(\nabla G) + t\langle x, \nabla\phi\rangle + \frac{t^2}{2}\langle (\nabla^2 G)^{-1}\nabla\phi, \nabla\phi\rangle\right)\left(1 - t\phi + \frac{t^2}{2}\phi^2\right)\,d\mu + o(t^2) \\ &= \int G^*(\nabla G)\,d\mu + t\int \Big(\langle x, \nabla\phi\rangle - \phi G^*(\nabla G)\Big)\,d\mu \\ &\quad + \frac{t^2}{2}\int \Big(\langle (\nabla^2 G)^{-1}\nabla\phi, \nabla\phi\rangle + \phi^2 G^*(\nabla G) - 2\phi\langle x, \nabla\phi\rangle\Big)\,d\mu + o(t^2).\end{aligned}$$

The remaining three terms give

$$\begin{aligned}
&- n \int e^{-G-t\phi} + \int (G + t\phi) e^{-G-t\phi} + \left(\int e^{-G-t\phi} \right) \log \int e^{-G-t\phi} \\
&= -n \int \left(1 - t\phi + \frac{t^2}{2}\phi^2 \right) d\mu + \int (G + t\phi) \left(1 - t\phi + \frac{t^2}{2}\phi^2 \right) d\mu \\
&\quad + (1 - tm) \left(-tm + \frac{t^2}{2} \operatorname{Var}_\mu(\phi) \right) + o(t^2) \\
&= -n + \int G \, d\mu + t \left(n \int \phi \, d\mu - \int G\phi \, d\mu \right) \\
&\quad + \frac{t^2}{2} \left(\int (G - n)\phi^2 \, d\mu - \operatorname{Var}_\mu(\phi) \right) + o(t^2).
\end{aligned}$$

Here the quadratic contribution from $Z(t) \log Z(t)$ is $\frac{1}{2} \int \phi^2 \, d\mu + \frac{1}{2} m^2$, which explains the variance in the last line after combining all terms.

Now check the cancellations.

1. The duality formula $G + G^*(\nabla G) = \langle \nabla G, x \rangle$, together with integration by parts, gives

$$\int G^*(\nabla G) \, d\mu = n - \int G \, d\mu.$$

   Thus the constant terms cancel, as expected.

2. By lemma 3.4,

$$\begin{aligned}
\int \langle \nabla \phi, x \rangle \, d\mu &= - \int \phi L_\mu(|x|^2/2) \, d\mu \\
&= -n \int \phi \, d\mu + \int \phi \langle \nabla G, x \rangle \, d\mu.
\end{aligned}$$

   Together with duality, this cancels the coefficient of t and confirms $\beta'(0) = 0$.

3. The coefficient of $t^2/2$ must be nonnegative:

$$\begin{aligned}
0 \leq \int \Big(\langle (\nabla^2 G)^{-1} \nabla \phi, \nabla \phi \rangle + \phi^2 G^*(\nabla G) - 2\phi \langle \nabla \phi, x \rangle \Big) \, d\mu \\
+ \int (G - n)\phi^2 \, d\mu - \operatorname{Var}_\mu(\phi).
\end{aligned} \tag{11}$$

   The remaining cancellation is the following claim.

Claim 3.4.

$$\int \Big(\phi^2 G^*(\nabla G) - 2\phi \langle \nabla \phi, x \rangle \Big) \, d\mu = \int (n - G)\phi^2 \, d\mu.$$

Proof. By duality and integration by parts,

$$\begin{aligned}
&\int \left(\phi^2 G^*(\nabla G) - 2\phi\langle \nabla\phi, x\rangle\right) d\mu \\
&= \int \phi^2(\langle \nabla G, x\rangle - G)\, d\mu - \int \langle \nabla(\phi^2), x\rangle\, d\mu \\
&= \int \phi^2(\langle \nabla G, x\rangle - G)\, d\mu + \int \phi^2(n - \langle \nabla G, x\rangle)\, d\mu \\
&= \int (n - G)\phi^2\, d\mu.
\end{aligned}$$

□

Combining the claim with (11) yields the theorem.

Theorem 3.15 (Brascamp–Lieb inequality, 1976)**.** *Let $G \in C^2(\mathbb{R}^n)$ have positive definite Hessian and $\int e^{-G} = 1$. For $d\mu = e^{-G}\,dx$ and every locally Lipschitz ϕ with the indicated finite integrals,*

$$\mathrm{Var}_\mu(\phi) \le \int \langle (\nabla^2 G)^{-1}\nabla\phi, \nabla\phi\rangle\, d\mu.$$

Remark 3.26 (Brascamp–Lieb as a local form of Prékopa–Leindler)**.** *One can instead differentiate the concavity statement*

$$\frac{d^2}{dt^2}\log \int e^{-(f+tg)^*} \le 0.$$

With $G = f^*$ and $\phi = g \circ \nabla f^*$, its second derivative is exactly the Brascamp–Lieb deficit, with the opposite sign. Conversely, applying Brascamp–Lieb along a smooth convex interpolation and integrating gives the corresponding Prékopa–Leindler inequality. Approximation supplies the more general formulation; see the homework.

Remark 3.27. *Equality holds for $\phi = \langle \nabla G, \theta\rangle$ and for this function plus a constant, whenever the integrals are finite. Notice that*

$$L_\mu\langle x, \theta\rangle = -\langle \nabla G, \theta\rangle.$$

Remark 3.28. *It is tempting to perturb this equality case by εf and hope for another inequality. The linear terms cancel, but the quadratic terms give Brascamp–Lieb again: in this sense it is the "end of the line" for this linearization scheme. See the homework.*

For Gaussian measure, $\nabla^2 G = \mathrm{Id}$, so we obtain

Corollary 3.7 (Gaussian Poincaré inequality)**.**

$$\mathrm{Var}_\gamma(\phi) \le \int |\nabla\phi|^2\, d\gamma.$$

This proves the special case discussed after Beckner's inequality.

Remark 3.29. *Linear functions attain equality and satisfy $L_\gamma\langle x,\theta\rangle = -\langle x,\theta\rangle$. Thus the Gaussian spectral gap is 1, with linear first eigenfunctions.*

Remark 3.30. *Gaussian Poincaré can also be obtained by linearizing Gaussian log-Sobolev around a constant. That calculation is substantially shorter and is left as homework.*

Remark 3.31 (An important observation about Brascamp–Lieb)**.** *Let V, W be convex, and normalize $d\nu = Z^{-1}e^{-V-W}\,dx$ to be a probability measure. Since $\nabla^2(V+W) \succeq \nabla^2 V \succ 0$, inversion reverses the matrix order. Therefore*

$$\mathrm{Var}_\nu(\phi) \le \int \langle (\nabla^2(V+W))^{-1}\nabla\phi, \nabla\phi\rangle\, d\nu \le \int \langle (\nabla^2 V)^{-1}\nabla\phi, \nabla\phi\rangle\, d\nu.$$

Approximate $W = \mathbb{1}_K^\infty$ by convex penalties to restrict the inequality to a convex body.

Corollary 3.8. *If $d\mu = e^{-V}\,dx$, $\nabla^2 V \succ 0$, and K is convex with $\mu(K) > 0$, then*

$$⨍_K \phi^2\,d\mu - \left(⨍_K \phi\,d\mu\right)^2 \le ⨍_K \langle (\nabla^2 V)^{-1}\nabla\phi, \nabla\phi\rangle\, d\mu.$$

Here $⨍_K f\,d\mu = \mu(K)^{-1}\int_K f\,d\mu$.

Thus Brascamp–Lieb automatically restricts to convex sets.

Corollary 3.9 (Extension of Gaussian Poincaré)**.** *If $d\mu = e^{-V}\,dx$ is a probability measure and $\nabla^2 V \succeq k\,\mathrm{Id}$ for $k > 0$, then*

$$\mathrm{Var}_\mu(\phi) \le \frac{1}{k}\int |\nabla\phi|^2\,d\mu.$$

Consequently $C_{\mathrm{Poin}}(\mu) \le 1/k$, or equivalently $\lambda_1 \ge k$.

3.18 Returning to Poincaré inequalities: Payne–Weinberger and the circle

For a log-concave probability measure, the Poincaré inequality reads

$$\mathrm{Var}_\mu(f) \le \frac{1}{\lambda_1}\int |\nabla f|^2\,d\mu.$$

Theorem 3.16 (Payne–Weinberger)**.** *If K is a convex body of diameter R, then for every locally Lipschitz f,*

$$\frac{1}{|K|}\int_K f^2 - \left(\frac{1}{|K|}\int_K f\right)^2 \le \frac{R^2}{\pi^2|K|}\int_K |\nabla f|^2.$$

Theorem 3.17 (Poincaré inequality on the circle)**.** *For a 2π-periodic C^1 function ϕ,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^2\,d\theta-\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi\,d\theta\right)^2\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|\dot\phi|^2\,d\theta.$$

Proof. Write the Fourier expansion

$$\phi(\theta)=\sum_{k\in\mathbb{Z}}\widehat\phi(k)e^{ik\theta},\qquad\widehat\phi(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(\theta)e^{-ik\theta}\,d\theta.$$

Then $\widehat{\dot\phi}(k)=ik\widehat\phi(k)$. Parseval gives

$$\frac{1}{2\pi}\int|\dot\phi|^2=\sum_{k\neq0}k^2|\widehat\phi(k)|^2\ge\sum_{k\neq0}|\widehat\phi(k)|^2=\frac{1}{2\pi}\int\phi^2-|\widehat\phi(0)|^2.$$

□

Remark 3.32. *Equality holds for $\phi(\theta)=c+a\cos\theta+b\sin\theta$.*

Remark 3.33. *If ϕ is antipodally even on the circle, meaning $\phi(\theta+\pi)=\phi(\theta)$, all odd Fourier modes vanish. Hence*

$$\sum_{k\neq0}|\widehat\phi(k)|^2=\sum_{|k|\ge2}|\widehat\phi(k)|^2\le\frac{1}{4}\sum_{|k|\ge2}k^2|\widehat\phi(k)|^2,$$

and the Poincaré constant improves to $1/4$. Reflection symmetry $\phi(-\theta)=\phi(\theta)$ alone does not give this improvement: $\cos\theta$ is a counterexample.

3.19 Colesanti's inequality via Brascamp–Lieb (Cordero-Erausquin's approach)

We have seen parallels between functional and volumetric inequalities:

Functional	**Volumetric**
$\log\int e^{-(f+tg)^*}$ is concave	$\log\lvert K+tL\rvert$ is concave
Generalized log-Sobolev	Minkowski's first inequality
Brascamp–Lieb	Minkowski's second inequality

Minkowski's second inequality, written in terms of the volume polynomial, is

$$|K|\,\frac{d^2}{dt^2}|K+tL|\bigg|_{t=0}\le\frac{n-1}{n}\left(\frac{d}{dt}|K+tL|\bigg|_{t=0}\right)^2.$$

The factor $(n-1)/n$ comes from differentiating the power $s^{1/n}$, rather than $\log s$.

One might try to substitute the convex indicator of K into Brascamp–Lieb and interpret its derivatives as boundary geometry. This is only a formal analogy: $\nabla\mathbb{1}_K^\infty$ and its Hessian are not ordinary functions. Cordero-Erausquin's approach instead uses a homogeneous potential and integrates in body polar coordinates; see [53]. This leads to the following boundary inequality.

Theorem 3.18 (Colesanti)**.** *Let $K \subset \mathbb{R}^n$, $n \geq 2$, be a C^2 convex body with positive definite second fundamental form $\mathrm{I\!I}$, and let $\phi \in C^1(\partial K)$ satisfy $\int_{\partial K} \phi \, dS = 0$. Then*

$$\int_{\partial K} \operatorname{tr}(\mathrm{I\!I})\phi^2 \, dS \leq \int_{\partial K} \left\langle \mathrm{I\!I}^{-1} \nabla_{\partial K}\phi, \nabla_{\partial K}\phi \right\rangle \, dS.$$

Proof. Translate K so that $0 \in \operatorname{int} K$. Put

$$V(x) = \frac{1}{2}\|x\|_K^2 = \frac{1}{2} h_{K^\circ}(x)^2, \qquad s(x) = \langle x, n_x \rangle \quad (x \in \partial K),$$

and extend $f = \phi/s$ from ∂K to a homogeneous function of degree one by $F(rx) = rf(x)$, $r > 0$. The body polar-coordinate formula is

$$dx = r^{n-1} s(x) \, dr \, dS(x).$$

Consequently, $\int F e^{-V} = 0$ because $\int_{\partial K} fs \, dS = \int_{\partial K} \phi \, dS = 0$.

At a point $x \in \partial K$, homogeneity and the geometry of the level set $\{V = 1/2\}$ give

$$(\nabla^2 V)x = \frac{n_x}{s}, \qquad \langle \nabla^2 V Y, Z \rangle = \frac{\mathrm{I\!I}(Y, Z)}{s} \quad (Y, Z \in T_x \partial K).$$

Thus, using the decomposition into the radial direction x and the tangent space,

$$\langle (\nabla^2 V)^{-1} \nabla F, \nabla F \rangle = f^2 + s \langle \mathrm{I\!I}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle.$$

The left side is homogeneous of degree zero. Apply Brascamp–Lieb to F for the probability measure proportional to $e^{-V} dx$, and integrate radially. Since

$$\int_0^\infty r^{n+1} e^{-r^2/2} \, dr = n \int_0^\infty r^{n-1} e^{-r^2/2} \, dr,$$

the resulting inequality is

$$(n-1) \int_{\partial K} \frac{\phi^2}{s} \, dS \leq \int_{\partial K} s^2 \left\langle \mathrm{I\!I}^{-1} \nabla_{\partial K}(\phi/s), \nabla_{\partial K}(\phi/s) \right\rangle \, dS.$$

To rewrite it, let $x_T = x - s n_x$ be the tangential component of x and $H = \operatorname{tr}(\mathrm{I\!I})$. Then

$$\nabla_{\partial K} s = \mathrm{I\!I} x_T, \qquad \operatorname{div}_{\partial K} x_T = n - 1 - sH.$$

Expand the square on the right and integrate its cross term by parts on ∂K. The terms containing $\nabla_{\partial K} s$ cancel, leaving

$$\int_{\partial K} s^2 \left\langle \mathrm{I\!I}^{-1} \nabla_{\partial K}(\phi/s), \nabla_{\partial K}(\phi/s) \right\rangle \, dS = \int_{\partial K} \left(\langle \mathrm{I\!I}^{-1} \nabla_{\partial K}\phi, \nabla_{\partial K}\phi \rangle + \frac{n-1}{s}\phi^2 - H\phi^2 \right) dS.$$

Substitution proves the claim. □

Remark 3.34. *For presentation, the calculation is written in the smooth setting. The possible lack of smoothness of V and F at the origin is handled by regularization and cutoffs; smooth approximation gives the stated boundary regularity. Notice the polar in h_{K°: using $V = h_K^2/2$ gives the boundary inequality for K° instead. The same inequality also follows from the second variation of volume and Brunn–Minkowski.*

3.20 Dimensional extensions of generalized log-Sobolev and Brascamp–Lieb inequalities

For negative p, the Borell–Brascamp–Lieb inequality gives a power-convex version of the dual functional.

Corollary 3.10. *In the regular setting, if $-1/n < p < 0$ and f, g are convex with positive conjugates, then*

$$t \longmapsto \left(\int_{\mathbb{R}^n} \bigl(((1-t)f + tg)^* \bigr)^{1/p} \right)^{p/(np+1)}$$

is convex wherever the integrals are finite.

The negative exponent reverses the inequality. This is the power analogue of the concavity of $\log \int e^{-f^*}$.

The same second-variation idea yields a dimensional Brascamp–Lieb inequality. For a strictly convex potential, the endpoint formulation below is equivalent, after a change of test function, to the refinement of Bolley, Gentil, and Guillin [30]. Its uncentered version away from the endpoint is a case of Nguyen's variance inequality [155].

Corollary 3.11 (Dimensional Brascamp–Lieb inequality)**.** *Let $q \le -n$ and let $d\mu = e^{-V}\,dx$ be a probability measure. Assume that*

$$H_q := \nabla^2 V - \frac{\nabla V \otimes \nabla V}{q}$$

is positive definite. For every sufficiently regular g satisfying

$$\int g e^{V/q}\,d\mu = 0,$$

we have

$$\int g^2 e^{2V/q}\,d\mu \le \frac{-q}{1-q} \int e^{2V/q} \langle H_q^{-1} \nabla g, \nabla g \rangle\,d\mu.$$

As usual, we first work with smooth functions for which the integrals and derivatives are justified, and then use approximation. The matrix condition is equivalent to convexity of $W = e^{-V/q}$, with positive definite Hessian in the regular case.

Remark 3.35. *The centering condition and the weight on the right are essential. For $\beta = -q > n$, the corresponding uncentered statement is*

$$\begin{aligned} \operatorname{Var}_\mu(g e^{-V/\beta}) \le \frac{\beta}{\beta+1} \int e^{-2V/\beta} \left\langle \left(\nabla^2 V + \frac{\nabla V \otimes \nabla V}{\beta} \right)^{-1} \nabla g, \nabla g \right\rangle d\mu \\ + \frac{n}{(\beta+1)(\beta-n)} \left(\int g e^{-V/\beta}\,d\mu \right)^2. \end{aligned}$$

This is the $r = 0$ case of Nguyen's theorem. Under centering, the last term vanishes, and the resulting inequality remains valid at $\beta = n$. To see the endpoint, use the quasiconcavity furnished by the endpoint Borell–Brascamp–Lieb inequality at a stationary point; the details are homework. When $\nabla^2 V$ is positive definite, substituting $g = e^{V/n}h$ and applying the rank-one inverse formula gives precisely the dimensional remainder in Bolley–Gentil–Guillin's inequality. No uncentered endpoint assertion is made here.

Remark 3.36. *To recover ordinary Brascamp–Lieb as $\beta \to \infty$, keep a mean-zero function h fixed and set $g = e^{V/\beta}h$. Then $ge^{-V/\beta} = h$, and the right side tends to $\int \langle (\nabla^2 V)^{-1}\nabla h, \nabla h\rangle \, d\mu$, under the usual integrability assumptions.*

Theorem 3.19 (Bolley, Cordero-Erausquin, Fujita, Gentil, Guillin [31])**.** *Let $n \geq 2$, $0 \leq t \leq 1$, and let g, w, h be positive measurable functions, allowing $+\infty$ where appropriate. Suppose*

$$h((1-t)x + ty) \leq (1-t)g(x) + tw(y), \qquad \int g^{-n} = \int w^{-n} = 1.$$

Then

$$\int h^{1-n} \geq (1-t)\int g^{1-n} + t\int w^{1-n}.$$

Corollary 3.12 (Convex Sobolev inequality [31])**.** *Let $n \geq 2$ and let $w : \mathbb{R}^n \to (0,\infty)$ satisfy*

$$\liminf_{|x|\to\infty} \frac{w(x)}{|x|^\gamma} > 0 \quad \text{for some } \gamma > \frac{n}{n-1}.$$

For a positive sufficiently regular g with $\int g^{-n} = \int w^{-n} = 1$, and with the indicated integrals finite,

$$\int w^*(\nabla g)g^{-n} \geq \frac{1}{n-1}\int w^{1-n}.$$

Remark 3.37. *For $1 < p < n$, take $q = p/(p-1)$, choose the normalizing constant in $w(x) = C_q(1+|x|^q/q)$, and substitute $g = f^{p/(p-n)}$. This recovers the sharp Sobolev inequality*

$$\|f\|_{p^*} \leq \frac{\|h_p\|_{p^*}}{\|\nabla h_p\|_p}\|\nabla f\|_p, \qquad p^* = \frac{np}{n-p}, \quad h_p(x) = (1+|x|^{p/(p-1)})^{(p-n)/p}.$$

The constants and the passage from positive test functions to the usual Sobolev class follow by normalization and approximation.

3.21 Homework

Question 3.1 (2 points)**.** *a) Show that if $K, L \subseteq \mathbb{R}^n$ are Borel, then $K + L$ is Lebesgue measurable. Explain why it is an analytic set; it need not be Borel.*
b) Find Lebesgue-measurable sets K, L whose sum is not Lebesgue measurable.

Question 3.2 (1 point)**.** *Let S_u denote Steiner symmetrization about $u^\perp$, and let K, L be convex bodies. Show that*
a) $S_u(aK) = aS_u(K)$ for $a > 0$;
b) $K \subseteq L$ implies $S_u(K) \subseteq S_u(L)$. Use this and suitable inner and outer dilations of a convex body to prove continuity in the Hausdorff metric;
c) $S_u(K) + S_u(L) \subseteq S_u(K + L)$.

Question 3.3 (1 point)**.** *Recall that* $\operatorname{diam}(A) = \max_{x,y\in A} |x - y|$ *for a compact set A. Prove*

$$\operatorname{diam}(S_u(K)) \le \operatorname{diam}(K).$$

Deduce the isodiametric inequality*: among sets of prescribed volume, a Euclidean ball minimizes diameter. First treat convex bodies; for a bounded measurable set, pass to its closed convex hull, which does not increase diameter.*

Question 3.4 (1 point)**.** *Prove that Steiner symmetrization decreases the perimeter of a convex body. Deduce another proof of the isoperimetric inequality for convex bodies.*

Question 3.5 (1 point)**.** *For a convex body K, recall*

$$r(K) = \sup\{t > 0 : \exists y,\ y + tB_2^n \subseteq K\}, \qquad R(K) = \inf\{t > 0 : \exists y,\ K \subseteq y + tB_2^n\}.$$

a) Prove $r(S_uK) \ge r(K)$.
b) Prove $R(S_uK) \le R(K)$.
Conclude that, for fixed volume, the Euclidean ball maximizes inradius and minimizes circumradius.

Question 3.6 (2 points)**.** *Prove Urysohn's inequality. With mean width defined by*

$$w(K) = \frac{2}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} h_K(\theta)\, d\theta,$$

show that $|K| = |B_2^n|$ implies $w(K) \ge 2$.
Hint: use Brunn–Minkowski and symmetrization. For Steiner symmetrization, compare S_uK with the Minkowski average of K and its reflection about $u^\perp$.

Question 3.7 (1 point)**.** *Assume the multiplicative Brunn–Minkowski inequality*

$$|\lambda K + (1 - \lambda)L| \ge |K|^\lambda |L|^{1-\lambda}$$

is known for all nonempty Borel sets K, L and all $0 < \lambda < 1$. Use homogeneity to deduce

$$|\lambda K + (1 - \lambda)L|^{1/n} \ge \lambda |K|^{1/n} + (1 - \lambda)|L|^{1/n}.$$

Hint: first rescale the sets to have equal volume.

Question 3.8 (1 point). *For $a, b > 0$ and $0 \le \lambda \le 1$, show that*

$$(\lambda a^p + (1-\lambda)b^p)^{1/p} \longrightarrow a^\lambda b^{1-\lambda} \quad \text{as } p \to 0.$$

Question 3.9 (2 points). *a) Let $p \ge -1/n$. Suppose nonnegative measurable f, g, h satisfy*

$$h(\lambda x + (1-\lambda)y) \ge \left(\lambda f(x)^p + (1-\lambda)g(y)^p\right)^{1/p}$$

whenever $f(x)g(y) > 0$. Show that

$$\int h \ge \left(\lambda \left(\int f\right)^{p/(np+1)} + (1-\lambda)\left(\int g\right)^{p/(np+1)}\right)^{(np+1)/p}.$$

Use the geometric mean at $p = 0$ and the minimum of the two integrals at $p = -1/n$. At zero endpoint values use the usual convention that the pointwise lower bound is zero.
Hint: try a subdivision argument analogous to the box proof of Brunn–Minkowski.
b) Deduce that a measure with a p-concave density supported on a full-dimensional convex set is $p/(np+1)$-concave.
c) Deduce that integrating a p-concave density over k-dimensional fibers gives a $p/(kp+1)$-concave marginal, for $p \ge -1/k$. Relate this to Brunn's principle.

Question 3.10 (2 points). *A function f is* unconditional *if $f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) = f(x)$ for all choices $\varepsilon_i \in \{-1, 1\}$. A set is unconditional if its indicator is unconditional.*

Suppose K is an unconditional convex body and V an unconditional convex function. For $d\mu = e^{-V}\,dx$, prove that $t \mapsto \log \mu(e^t K)$ is concave on $\mathbb{R}$.
Hint: restrict integration to the positive orthant and use the change of variables $x_i = e^{s_i}$ in Prékopa–Leindler.

Question 3.11 (1 point). *a) Prove Minkowski's first inequality $V_1(K, L) \ge |K|^{(n-1)/n}|L|^{1/n}$.*
b) For $n \ge 2$, prove Minkowski's quadratic inequality

$$V_2(K, L)|K| \le V_1(K, L)^2.$$

Hint: differentiate the concave function $|K + tL|^{1/n}$ twice at zero.

Question 3.12 (1 point). *This question was added at Alex's request. Give an example of a rough, nonconvex measurable set K of finite volume for which the outer Minkowski perimeter limit does not exist:*

$$\liminf_{\varepsilon \downarrow 0} \frac{|K + \varepsilon B_2^n| - |K|}{\varepsilon} < \limsup_{\varepsilon \downarrow 0} \frac{|K + \varepsilon B_2^n| - |K|}{\varepsilon}.$$

Question 3.13 (1 point). *Let $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex.*
a) Show that V is continuous on the relative interior of its effective domain $\{V < \infty\}$. Why can continuity fail at the boundary?
b) State and apply Alexandrov's theorem: V has a second-order Taylor expansion almost everywhere in the interior of its effective domain (with relative Lebesgue measure if the domain is lower-dimensional). This is twice differentiability almost everywhere; it does not assert that V is C^2 on a neighborhood of almost every point.

Question 3.14 (1 point). *a) Let $V \in C^2(\mathbb{R}^n)$. Show that*

$$V\left(\frac{z_1+z_2}{2}\right) + \beta(z_1,z_2) = \frac{V(z_1)+V(z_2)}{2}, \tag{12}$$

where $z(t) = ((1-t)z_1 + (1+t)z_2)/2$ and

$$\beta(z_1,z_2) = \frac{1}{8}\int_{-1}^{1} (1-|t|)\langle \nabla^2 V(z(t))(z_1-z_2), z_1-z_2\rangle \, dt. \tag{13}$$

If V is convex, this quantity is nonnegative.
b) Deduce that a C^2 function is convex if and only if its Hessian is positive semidefinite everywhere.

Question 3.15 (2 points). *a) Show that for convex bodies K, L, the function $t \mapsto |K+tL|$ is a polynomial of degree n on $t \geq 0$.*
b) Deduce the expansion

$$|K+tL| = \sum_{j=0}^{n} \binom{n}{j} V_j(K,L) t^j.$$

Question 3.16 (2 points). *For a convex body K, the outer unit normal $n_K(x)$ is uniquely defined for $\mathcal{H}^{n-1}$-almost every $x \in \partial K$. Define its surface area measure by*

$$S_K(\Omega) = \mathcal{H}^{n-1}(\{x \in \partial K : n_K(x) \in \Omega\}), \qquad \Omega \subseteq \mathbb{S}^{n-1} \text{ Borel.}$$

The exceptional boundary points with more than one normal have zero surface measure.
a) Prove

$$V_1(K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L \, dS_K, \qquad |K| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K \, dS_K.$$

b) Use Minkowski's first inequality and its equality case to show that $S_K = S_L$ implies $K = L + v$ for some $v \in \mathbb{R}^n$.

Question 3.17 (2 points). *Recall the orthogonal projection*

$$K|\theta^\perp = \{x \in \theta^\perp : \exists t \in \mathbb{R},\ x + t\theta \in K\}.$$

a) Prove Cauchy's projection formula

$$\left|K \mid \theta^\perp\right|_{n-1} = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle| \, dS_K(u).$$

Here the left-hand side denotes the $(n-1)$-dimensional volume of the projected body $K|\theta^\perp$. The formula holds for every convex body.
Hint: use elementary geometry for polytopes and approximation, or Questions 2.5 and 3.16.
b) Suppose K, L are origin-symmetric convex bodies with equal projection volumes in every direction. Show that $K = L$.
Hint: in addition to part a), use injectivity of the cosine transform on even finite measures, then the uniqueness statement in Question 3.16.

Question 3.18 (1 point)**.** *Let the support function h of a planar convex body be C^2 on S^1, parametrized by angle u, and assume $h + \ddot{h} > 0$. Prove that its surface area measure has density*

$$f_K(u) = h(u) + \ddot{h}(u).$$

Explain why $h + \ddot{h}$ is translation invariant.

Question 3.19 (10 points)**.** *Prove, perhaps using elementary harmonic analysis, that for every pair of smooth π-periodic functions ψ, h with $h > 0$ and $h + \ddot{h} > 0$,*

$$\left(\int_{-\pi}^{\pi} (h^2 - \dot{h}^2)\, du\right)\left(\int_{-\pi}^{\pi} \left(\psi^2 - \dot{\psi}^2 + \psi^2 \frac{h + \ddot{h}}{h}\right) du\right) \le 2\left(\int_{-\pi}^{\pi} (h\psi - \dot{h}\dot{\psi})\, du\right)^2. \qquad (14)$$

I can provide motivation upon request. The assumption is π-periodicity, not merely 2π-periodicity.

Question 3.20 (2 points)**.** *Prove the Rogers–Shephard inequality. For a convex body $K \subset \mathbb{R}^n$, define its difference body $K - K = \{x - y : x, y \in K\}$. Show that*

$$|K - K| \le \binom{2n}{n} |K|.$$

Hint: Brunn–Minkowski shows that $x \mapsto |K \cap (x + K)|^{1/n}$ is concave on $K - K$. It has value $|K|^{1/n}$ at zero, so

$$|K \cap (x + K)|^{1/n} \ge |K|^{1/n}(1 - \|x\|_{K-K}), \qquad x \in K - K.$$

Use this bound and Fubini to obtain

$$|K|^2 = \int_{K-K} |K \cap (x + K)|\, dx \ge \binom{2n}{n}^{-1} |K|\, |K - K|.$$

Question 3.21 (2 points)**.** *Prove Grünbaum's inequality: if K is a convex body with barycenter at the origin, then for every $\theta \in \mathbb{S}^{n-1}$,*

$$|\{x \in K : \langle x, \theta\rangle \ge 0\}| \ge \left(\frac{n}{n+1}\right)^n |K| \ge \frac{|K|}{e}.$$

Question 3.22 (3 points)**.** *Prove Busemann's theorem. Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $n \ge 2$. Show that*

$$x \longmapsto \frac{|x|}{|x^\perp \cap K|_{n-1}}, \qquad x \ne 0,$$

extended by zero at $x = 0$, is convex. Deduce that it is a norm. Its unit ball is the intersection body *of K.*

Question 3.23. *Derive the Santaló area formula for a planar convex body with sufficiently smooth support function h:*

$$|K| = \frac{1}{2}\int_{-\pi}^{\pi}(h^2 - \dot{h}^2)\,dt.$$

Hint: use Questions 3.16 and 3.18, then integration by parts.

Question 3.24 (2 points)**.** *Prove, using Fourier series, that if h, ψ are 2π-periodic C^1 functions and $\int_{-\pi}^{\pi}(h^2 - \dot{h}^2) \geq 0$, then*

$$\left(\int_{-\pi}^{\pi}(h^2 - \dot{h}^2)\right)\left(\int_{-\pi}^{\pi}(\psi^2 - \dot{\psi}^2)\right) \leq \left(\int_{-\pi}^{\pi}(h\psi - \dot{h}\dot{\psi})\right)^2.$$

The zero case is immediate. For the positive case, separate the constant Fourier coefficient, the first harmonics, and the higher harmonics, and use Cauchy–Schwarz on the higher coefficients. Explain how this gives another proof of planar Minkowski's quadratic inequality, using Questions 3.16 and 3.23. The positivity hypothesis holds when h is the support function of a convex body. It cannot simply be omitted: $h(\theta) = \cos(2\theta)$ and $\psi(\theta) = \sin(2\theta)$ give a counterexample.

Question 3.25 (1 point)**.** *Prove the general version of Brunn's principle: if $K \subset \mathbb{R}^n$ is a convex body and H is a k-dimensional linear subspace, then*

$$y \longmapsto |K \cap (y + H)|_k^{1/k}$$

is concave on its support in $H^\perp$. Here $1 \leq k \leq n-2$; the case $k = n-1$ was proved in class.

Question 3.26. *Show that the convolution of integrable log-concave functions is log-concave. Hint: represent the convolution as a marginal of a log-concave function on $\mathbb{R}^{2n}$.*

Question 3.27 (1 point)**.** *Give an alternative proof of Gaussian Poincaré,*

$$\int f^2\,d\gamma - \left(\int f\,d\gamma\right)^2 \leq \int |\nabla f|^2\,d\gamma,$$

using the expansion of f in Hermite polynomials, the orthogonal polynomial system for Gaussian measure.

Question 3.28 (1 point)**.** *Prove corollary 3.11 from the Borell–Brascamp–Lieb inequality, including the endpoint $q = -n$. First work in the regular setting. Put $\beta = -q$, $W = e^{V/\beta}$, and choose a Legendre perturbation*

$$W_t = (W^* + tF)^*, \qquad g = F(\nabla W).$$

For $I(t) = \int W_t^{-\beta}\,dx$, compute $I'(0)$ and $I''(0)$. If $\beta > n$, use convexity of $I^{-1/(\beta-n)}$ to obtain also the uncentered correction term stated in the remark. At $\beta = n$, use quasiconcavity of I and the centering condition $I'(0) = 0$ to conclude $I''(0) \leq 0$.

Question 3.29 (1 point)**.** *Deduce Gaussian Poincaré from Gaussian log-Sobolev by linearizing around a constant. This is a simpler version of the argument used in class.*

Question 3.30 (1 point)**.** *Prove that the Gaussian log-Sobolev inequality and the classical Lebesgue log-Sobolev inequality in corollary 3.5 are equivalent.*

Question 3.31 (1 point)**.** *By differentiating infimal convolution directly, prove Gaussian log-Sobolev without assuming log-concavity of f:*

$$\mathrm{Ent}_\gamma(f^2) \le 2\int |\nabla f|^2 \, d\gamma.$$

First treat sufficiently regular functions with the needed decay, then indicate the approximation argument for $W^{1,2}(\gamma)$.

Question 3.32 (1 point)**.** *Deduce a Sobolev inequality from the log-Sobolev inequality for Lebesgue measure. Author confirmation required: specify the intended exponent and sharpness of the constant, particularly whether the target is the sharp L_1 inequality stated above.*

Question 3.33 (1 point)**.** *Show that Gaussian Beckner implies Gaussian log-Sobolev by dividing by $2-p$ and letting $p \uparrow 2$. In particular, verify that*

$$\lim_{p\uparrow 2} \frac{\int f^2 \, d\gamma - (\int |f|^p \, d\gamma)^{2/p}}{2-p} = \frac{1}{2}\,\mathrm{Ent}_\gamma(f^2).$$

Question 3.34 (1 point)**.** *Deduce the Euclidean isoperimetric inequality from the sharp L_1 Sobolev inequality, using smooth approximations to indicator functions.*

Question 3.35 (1 point)**.** *Let $d\mu = e^{-V}\,dx$ be log-concave, and let f, g be smooth convex functions with $\int e^{-f}\,d\mu = \int e^{-g}\,d\mu$. Prove*

$$\int g^*(\nabla f) e^{-f}\,d\mu \ge n\int e^{-f}\,d\mu - \int \langle \nabla V, x\rangle e^{-f}\,d\mu - \int f e^{-f}\,d\mu.$$

Question 3.36 (3 points)**.** *Can Gaussian Beckner inequalities for $1 \le p < 2$ be obtained directly by linearizing suitable geometric inequalities?*

Question 3.37 (2 points)**.** *Prove the following dual formulation of the extension of Borell–Brascamp–Lieb due to Bolley, Cordero-Erausquin, Fujita, Gentil, and Guillin. Let $n \ge 2$, and let f, g be convex with positive conjugates satisfying*

$$\int (f^*)^{-n} = \int (g^*)^{-n} = 1.$$

Show that

$$\int \Big(((1-t)f + tg)^*\Big)^{1-n} \ge (1-t)\int (f^*)^{1-n} + t\int (g^*)^{1-n}.$$

Assume the integrals are finite and use the natural extended-value conventions.

Question 3.38 (Generalized Sobolev, 2 points)**.** *Prove the generalized Sobolev inequality of Bolley, Cordero-Erausquin, Fujita, Gentil, and Guillin. Let $n \geq 2$, and let F, G be positive convex functions with*

$$\int F^{-n} = \int G^{-n} = 1, \qquad \liminf_{|x|\to\infty} \frac{G(x)}{|x|^\gamma} > 0 \quad \text{for some } \gamma > \frac{n}{n-1}.$$

Assuming sufficient regularity and integrability, prove

$$\int G^*(\nabla F) F^{-n} \geq \frac{1}{n-1} \int G^{1-n}.$$

Question 3.39 (Cordero-Erausquin's proof of Colesanti's inequality, 4 points)**.** *Let K be a C^2 convex body with positive definite second fundamental form $\mathrm{I\!I}$. Prove that if $f \in C^1(\partial K)$ and $\int_{\partial K} f \, dS = 0$, then*

$$\int_{\partial K} \left(\mathrm{tr}(\mathrm{I\!I}) f^2 - \langle \mathrm{I\!I}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \right) dS \leq 0.$$

Compare this with Question 3.11, part b). The body polar-coordinate formula, for $0 \in \operatorname{int} K$, is

$$\int_{\mathbb{R}^n} F(x)\, dx = \int_{\partial K} \int_0^\infty F(ty) t^{n-1} \langle y, n_y \rangle \, dt \, dS(y).$$

For integration over K, replace the radial upper limit by 1. Hint: translate K to contain the origin, take $V = \| \cdot \|_K^2/2 = h_{K^\circ}^2/2$, and apply Brascamp–Lieb to the homogeneous extension

$$F(ty) = t \frac{f(y)}{\langle y, n_y \rangle}, \qquad y \in \partial K, \quad t > 0.$$

Fill in the Hessian calculation and the surface integration by parts in the proof of theorem 3.18.

Question 3.40 (1 point)**.** *Show that a C^1 function ϕ on the circle satisfying the antipodal symmetry $\phi(\theta + \pi) = \phi(\theta)$ obeys*

$$\int_{-\pi}^{\pi} \phi^2 - \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} \phi \right)^2 \leq \frac{1}{4} \int_{-\pi}^{\pi} \dot{\phi}^2.$$

Here antipodal symmetry means π-periodicity, not reflection symmetry on $[-\pi, \pi]$.

Question 3.41 (1 point)**.** *Show that if ϕ is 2π-periodic and C^1, and $\phi(0) = 0$, then*

$$\int_{-\pi}^{\pi} \phi^2 \leq 4 \int_{-\pi}^{\pi} \dot{\phi}^2.$$

Hint: regard the circle cut at zero as an interval of length 2π with zero boundary values.

Question 3.42 (1 point)**.** *Explain why Brascamp–Lieb is the "end of the line" for this linearization procedure. For $d\mu = e^{-V}\,dx$, substitute $f = \langle \nabla V, \theta\rangle + \varepsilon\phi$ into*

$$\mathrm{Var}_\mu(f) \le \int \langle (\nabla^2 V)^{-1}\nabla f, \nabla f\rangle \, d\mu.$$

Check that the constant and linear terms cancel and that the quadratic term is exactly Brascamp–Lieb for ϕ again.

Question 3.43 (1 point)**.** *Let $d\mu = e^{-V}\,dx$ be a log-concave probability measure and $Lu = \Delta u - \langle \nabla V, \nabla u\rangle$. Define the spectral gap by*

$$\lambda_1 = \inf_{\substack{f \in W^{1,2}(\mu),\ \int f\,d\mu = 0 \\ f \neq 0}} \frac{\int |\nabla f|^2\,d\mu}{\int f^2\,d\mu}.$$

Show that the same infimum is obtained by using the variance in the denominator and allowing all nonconstant f. If the infimum is attained by f_1, prove by varying $f_1 + \varepsilon g$ that $Lf_1 = -\lambda_1 f_1$ weakly, and hence classically when elliptic regularity applies.
Hint: attainment requires a compactness hypothesis and does not hold for every log-concave measure. You may assume compactness of the embedding of the relevant form domain into $L^2(\mu)$ when proving existence of a minimizer.

Question 3.44 (1 point)**.** *For a symmetric matrix A, prove*

$$\det(\mathrm{Id} + tA) = 1 + t\,\mathrm{tr}\,A + \frac{t^2}{2}\Big((\mathrm{tr}\,A)^2 - \mathrm{tr}(A^2)\Big) + o(t^2).$$

Recall that $\mathrm{tr}(A^2) = \|A\|_{HS}^2$ for symmetric A. The quadratic coefficient is the difference of the two displayed terms, not $\|A\|_{HS}^2$ alone.

Question 3.45 (3 points)**.** *Show the following improvement of Gaussian log-Sobolev. If $d\mu = e^{-V}\,d\gamma$ is a probability measure, then, assuming the required integrability,*

$$-\int V\,d\mu \le \frac{\int |x|^2\,d\mu - n}{2} + \frac{n}{2}\log\left(2 + \frac{\int |\nabla V|^2\,d\mu - \int |x|^2\,d\mu}{n}\right).$$

Hint: optimize the Lebesgue log-Sobolev inequality over dilations and rewrite it relative to Gaussian measure.

Question 3.46 (1 point)**.** *Prove the following improvement of Brascamp–Lieb for unconditional functions. Suppose f, w are unconditional, w is convex, and $d\mu = Ce^{-w}\,dx$ is a probability measure. Then*

$$\mathrm{Var}_\mu(f) \le \int \langle (\nabla^2 w + T)^{-1}\nabla f, \nabla f\rangle \, d\mu,$$

where

$$T = \operatorname{diag}\left(\frac{\partial_1 w}{x_1}, \ldots, \frac{\partial_n w}{x_n}\right).$$

Work first in the smooth setting where the indicated matrix is positive definite. On the coordinate hyperplanes, use the continuous extensions when available; degeneracies can be handled by adding a small quadratic term.
Hint: use the multiplicative Prékopa–Leindler inequality for unconditional functions, as in Question 3.10.

Question 3.47 (2 points, important question)**.** *a) Prove the second identity in lemma 3.3. If v_t is a smooth convex family in the regular setting and $y = \nabla v_t^*(x)$, show that*

$$\frac{d^2}{dt^2} v_t^*(x) = -\ddot{v}_t(y) + \langle (\nabla^2 v_t(y))^{-1} \nabla \dot{v}_t(y), \nabla \dot{v}_t(y) \rangle.$$

b) Deduce Brascamp–Lieb directly from Prékopa–Leindler, without passing through generalized log-Sobolev. Differentiate

$$\frac{d^2}{dt^2} \log \int e^{-(f+tg)^*} \leq 0$$

at $t = 0$ and verify that it is equivalent to

$$\mathrm{Var}_\mu(\phi) \leq \int \langle (\nabla^2 V)^{-1} \nabla \phi, \nabla \phi \rangle \, d\mu,$$

where $V = f^*$, $d\mu = e^{-V} dx$ is normalized to have mass one, and $\phi = g \circ \nabla f^*$.

4 The Blaschke-Santaló inequality and friends

4.1 The formulation of the Blaschke-Santaló inequality

Let K be an origin-symmetric convex body. Recall that

$$K^\circ = \{x \in \mathbb{R}^n : \ \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$$

is its polar. If T is an invertible linear operator, then $(TK)^\circ = (T^{-1})^\top K^\circ$. The volume product $|K|\,|K^\circ|$ is invariant under invertible linear transformations:

$$|TK|\,|(TK)^\circ| = |\det T|\,|K|\,|\det T^{-1}|\,|K^\circ| = |K|\,|K^\circ|.$$

In particular, for every ellipsoid E centered at the origin,

$$|E|\,|E^\circ| = |B_2^n|^2 = (1 + o(1)) \frac{1}{\pi n} \left(\frac{2\pi e}{n}\right)^n,$$

and for every parallelepiped P centered at the origin,

$$|P|\,|P^\circ| = |B_\infty^n|\,|B_1^n| = \frac{4^n}{n!} = (1 + o(1)) \frac{1}{\sqrt{2\pi n}} \left(\frac{4e}{n}\right)^n.$$

We formulate the celebrated

Theorem 4.1 (Blaschke-Santaló inequality [166]). *For every symmetric convex body K,*

$$|K|\,|K^\circ| \le |B_2^n|^2.$$

And what about an estimate from below?

Conjecture 4.1 (Mahler, 1937, symmetric version). *For every symmetric convex body K in $\mathbb{R}^n$,*

$$|K|\,|K^\circ| \ge \frac{4^n}{n!} = |B_\infty^n|\,|B_1^n|.$$

Mahler proved this in dimension 2; see also the homework. Iriyeh and Shibata proved it in dimension 3, and their proof was later simplified by Fradelizi, Hubard, Meyer, Roldán-Pensado and Zvavitch.

Remark 4.1 (Answering Siva's question). *Obtaining an isoperimetric inequality in the opposite direction is less simple, since the isoperimetric ratio is not invariant under general linear transformations. Indeed, needle-shaped convex bodies can have arbitrarily large perimeter relative to $|K|^{(n-1)/n}$. However, we can obtain an affine-invariant quantity by taking the infimum of this ratio over invertible linear transformations. The following theorem is due to Keith Ball: if K is a symmetric convex body and $|K| = |B_\infty^n| = 2^n$, then*

$$\inf_{T:\,|\det T|=1} |\partial(TK)|_{n-1} \le |\partial B_\infty^n|_{n-1}.$$

The cube often appears as an optimizer, or a conjectured optimizer, in reverse isoperimetric inequalities; Mahler's conjecture is one example of this theme.

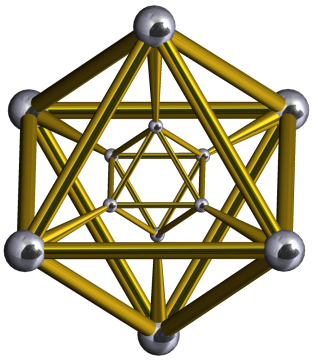
4.2 Hanner polytopes

The following interesting construction of a class of convex bodies appears in [84].

Definition 4.1 (Hanner polytopes). Hanner polytopes are defined inductively. In dimension 1, the symmetric interval $[-1,1]$ is a Hanner polytope. If $H \subset \mathbb{R}^k$ and $K \subset \mathbb{R}^{n-k}$ are Hanner polytopes, place them in the complementary coordinate subspaces of $\mathbb{R}^n$. Then both

$$H \times K = \{(x,y) : x \in H,\ y \in K\} \quad \text{and} \quad \operatorname{conv}\Big((H \times \{0\}) \cup (\{0\} \times K)\Big)$$

are Hanner polytopes in $\mathbb{R}^n$.



The family of Hanner polytopes in $\mathbb{R}^n$ includes B_1^n and B_∞^n. Up to invertible linear transformations, there is one type in dimension 2, the square, and two types in dimension 3, the cube B_∞^3 and the cross-polytope B_1^3. In dimension 4, there are four types:

$$B_\infty^4, \qquad B_1^4, \qquad B_1^3 \times [-1,1], \qquad \Big(B_1^3 \times [-1,1]\Big)^\circ.$$

Here the last polar is $\operatorname{conv}\Big((B_\infty^3 \times \{0\}) \cup \{\pm e_4\}\Big)$. Notice that $B_1^2 \times B_1^2$ is a linear image of the 4-dimensional cube, so it does not give an additional type.

It is a homework problem to show that every Hanner polytope $H \subset \mathbb{R}^n$ satisfies

$$|H|\,|H^\circ| = \frac{4^n}{n!} = |B_\infty^n|\,|B_1^n|.$$

Thus there are many conjectured minimizers in Mahler's conjecture. Kim [93] showed that all Hanner polytopes are local minimizers of the volume product.

4.3 About Bourgain-Milman's theorem

Bourgain and Milman [39] proved the following "isomorphic version" of Mahler's conjecture.

Theorem 4.2 (Bourgain-Milman, 1987)**.** *There is an absolute constant $c > 0$ such that, for every symmetric convex body $K \subset \mathbb{R}^n$,*

$$|K|\,|K^\circ| \geq \frac{c^n}{n^n}.$$

Equivalently, after changing the absolute constant, the lower bound can be written as

$$|K|\,|K^\circ| \geq \widetilde{c}^{\,n} |B_\infty^n|\,|B_1^n|.$$

Corollary 4.1. *There are absolute constants $c_1, c_2 > 0$ such that*

$$n\Big(|K|\,|K^\circ|\Big)^{1/n} \in [c_1, c_2]$$

for every symmetric convex body $K \subset \mathbb{R}^n$.

The Blaschke-Santaló inequality gives the asymptotic upper bound $2\pi e + o(1)$. Mahler's conjecture would give the asymptotic lower bound $4e + o(1)$. The strongest asymptotic lower constant discussed here is due to Kuperberg [123], who obtained $\pi e + o(1)$. See also Nazarov [152], Berndtsson [21], and Giannopoulos, Paouris and Vritsiou [72].

The Bourgain-Milman theorem has many applications to problems in which the dimensional dependence is studied up to an absolute constant.

4.4 Volume product of non-symmetric convex bodies and related questions and results

If K is a convex body and $0 \notin \operatorname{int} K$, then $|K^\circ| = \infty$. Indeed, a separating hyperplane gives a direction $u \neq 0$ such that $\langle u, x\rangle \leq 0$ for every $x \in K$. Consequently, K° contains the ray $\{tu : t \geq 0\}$. Since K is bounded, K° also contains a Euclidean ball about the origin. By convexity, it contains the convex hull of that ball and the ray, which has infinite volume. The ball is needed in this argument: an unbounded set by itself need not have infinite volume.

Thus the inequality fails for non-symmetric bodies in arbitrary positions. We need to choose a center. Consider

$$\inf_{z\in\mathbb{R}^n} |K - z|\,|(K - z)^\circ| = |K| \inf_{z\in\operatorname{int} K} |(K - z)^\circ|.$$

This quantity is finite. Moreover, the polar volume is strictly convex as a function of z in $\operatorname{int} K$ and tends to infinity as z approaches the boundary. Hence the infimum is attained at a unique point; see the homework.

Definition 4.2 (Santaló point)**.** The unique minimizing point is called the *Santaló point* of K and is denoted by $\operatorname{san}(K)$. If $\operatorname{san}(K) = 0$, we say that K is in *Santaló position*.

Theorem 4.3 (Non-symmetric Blaschke-Santaló inequality)**.** *For every convex body K,*

$$|K| \inf_{z\in\mathbb{R}^n} |(K - z)^\circ| = |K|\,|(K - \operatorname{san}(K))^\circ| \leq |B_2^n|^2.$$

Many special centers besides the Santaló point can be associated with a non-symmetric convex body. One of the most commonly used is the following.

Definition 4.3 (Center of mass)**.** The *center of mass*, or *barycenter*, of a convex body K is

$$\operatorname{bar} K = \frac{1}{|K|}\int_K x\,dx.$$

More generally, the barycenter of a probability measure μ with finite first moment is $\int x\,d\mu(x)$. For a nonzero finite measure, this integral must be divided by its total mass. In general, the Santaló point and the barycenter do not coincide, and they can be very far apart.

Does the Blaschke-Santaló inequality still hold with another choice of center? It certainly holds if K°, rather than K, is in Santaló position. It turns out that K is in Santaló position if and only if K° has barycenter at the origin; see the homework and the computation below. Applying this observation to the polar gives the following.

Claim 4.1. *For every convex body K,*

$$|K|\,|(K - \operatorname{bar} K)^\circ| \leq |B_2^n|^2.$$

A notable proof of the non-symmetric Blaschke-Santaló inequality was found by Lehec [133, 134], using explicitly that the body has barycenter at the origin.

The following theorem from [75] connects these two formulations and provides an estimate for every convex body containing the origin in its interior.

Theorem 4.4 (Fradelizi, Gozlan, Sadovsky and Zugmeyer)**.** *For every convex body K with $0 \in \operatorname{int} K$,*

$$|K|\,|K^\circ| \le |B_2^n|^2 \Big(1 - \langle \operatorname{bar} K, \operatorname{san}(K^\circ)\rangle\Big)^{n+1}.$$

Remark 4.2 (An analytic characterization of the Santaló point)**.** *For a convex body K containing the origin in its interior, $\rho_{K^\circ} = 1/h_K$ on the sphere. Polar coordinates give*

$$|L| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_L(\theta)^n \, d\theta, \qquad |K^\circ| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(\theta)^{-n} \, d\theta.$$

Also, $h_{K-z}(\theta) = h_K(\theta) - \langle \theta, z\rangle$. Hence, for $z \in \operatorname{int} K$,

$$F(z) := |(K-z)^\circ| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \Big(h_K(\theta) - \langle \theta, z\rangle\Big)^{-n} \, d\theta.$$

Differentiation gives

$$\nabla F(z) = \int_{\mathbb{S}^{n-1}} \frac{\theta}{\Big(h_K(\theta) - \langle \theta, z\rangle\Big)^{n+1}} \, d\theta = (n+1)\int_{(K-z)^\circ} x \, dx.$$

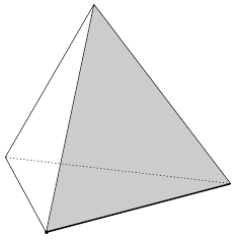
Thus $z = \operatorname{san}(K)$ exactly when the last vector is zero. In particular, $\operatorname{san}(K) = 0$ if and only if $\operatorname{bar}(K^\circ) = 0$.

Finally, we mention the famous

Conjecture 4.2 (Non-symmetric Mahler conjecture)**.** *For every convex body K,*

$$|K|\,|(K - \operatorname{san}(K))^\circ| \ge \frac{(n+1)^{n+1}}{(n!)^2}.$$

The constant is the volume product of an n-dimensional simplex whose barycenter is at the origin. Equivalently, the same lower bound is conjectured for $|K|\,|K^\circ|$ whenever $0 \in \operatorname{int} K$.



4.5 A connection with the slicing problem

Klartag [97] found a connection between Conjecture 4.2 and the sharp version of Bourgain's slicing problem [37, 38, 113].

Conjecture 4.3 (Bourgain, sharp version of the slicing problem)**.** *Let K be a convex body, and let* $\mathrm{Cov}(K)$ *be the covariance matrix of the uniform probability distribution on K. The affine-invariant quantity*

$$\frac{\det \mathrm{Cov}(K)}{|K|^2}$$

is maximized by simplices.

All nondegenerate simplices are affine images of a regular simplex, so they give the same value.

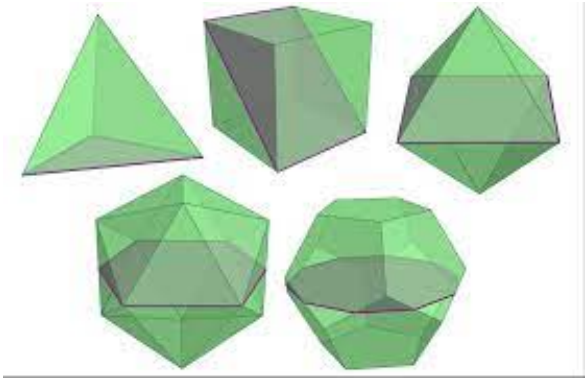
Theorem 4.5 (Klartag [97])**.** *Conjecture 4.3 implies Conjecture 4.2.*

An affirmative answer to the sharp conjecture would also imply an affirmative answer to the following simple-sounding problem, formulated here in the historical setting of this course.

Conjecture 4.4 (Bourgain, the slicing problem)**.** *There is an absolute constant $c > 0$ such that every convex body $K \subset \mathbb{R}^n$ with $|K| = 1$ has a hyperplane section satisfying*

$$|K \cap (\theta^\perp + t\theta)|_{n-1} \geq c$$

for some $\theta \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$.



The usual slicing conjecture is equivalent to a dimension-free upper bound for the isotropic constant

$$L_K = \frac{\Big(\det \mathrm{Cov}(K)\Big)^{1/(2n)}}{|K|^{1/n}}.$$

The sharp conjecture asks in addition for the exact maximizing body in each dimension. Thus it implies the usual conjecture, but the two statements should not be identified. The implication from the sharp slicing conjecture to Mahler's conjecture is much subtler.

Definition 4.4. A probability measure μ with finite second moments is *isotropic* if

$$\int x \, d\mu(x) = 0 \qquad \text{and} \qquad \mathrm{Cov}(\mu) = \mathrm{Id}\,.$$

Here, for $X \sim \mu$,

$$\mathrm{Cov}(\mu) = \Big(\mathbb{E}[(X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)]\Big)_{i,j},$$

which reduces to $(\mathbb{E}X_iX_j)_{i,j}$ when the barycenter is zero.

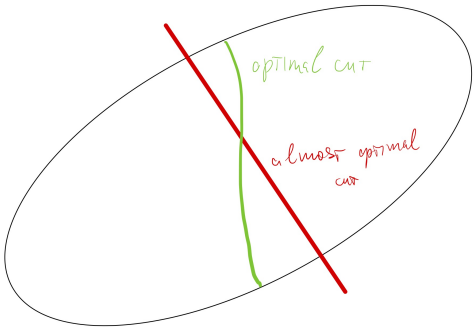
Remark 4.3. *Every probability measure with finite second moments and positive-definite covariance can be put in isotropic position by an invertible affine map: if $m = \mathbb{E}X$ and $\Sigma = \mathrm{Cov}(X)$, then $\Sigma^{-1/2}(X - m)$ is isotropic.*

One can also show that the usual slicing conjecture follows from the KLS conjecture; see, for example, Eldan and Klartag [61].

Conjecture 4.5 (Kannan, Lovász and Simonovits [92])**.** *If μ is an isotropic log-concave probability measure, then every locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\int f^2 \, d\mu - \left(\int f \, d\mu\right)^2 \le C \int |\nabla f|^2 \, d\mu,$$

where $C > 0$ is an absolute constant independent of the dimension and of μ.



Geometrically, KLS says that, up to an absolute factor, the best isoperimetric cut of a convex body is achieved by a hyperplane. In particular, if we seek a cut dividing the volume in half, a hyperplane cut has relative perimeter within an absolute factor of the minimum. See the homework for the connection between this formulation and the Poincaré inequality.

4.6 Proof of the symmetric Blaschke-Santaló inequality, and an interesting open problem

Claim 4.2. *For every symmetric convex body $K \subset \mathbb{R}^n$,*

$$|K|\,|K^\circ| \le |B_2^n|^2.$$

The proof uses Steiner symmetrization. We first show:

Lemma 4.1. *For every symmetric convex body K and $u \in \mathbb{S}^{n-1}$,*

$$|K^\circ| \le |(S_u K)^\circ|.$$

Proof. We may assume $u = e_n$ and write points of $\mathbb{R}^n$ as $(x, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Since the fibers of K are intervals,

$$S_{e_n} K = \left\{ \left(x, \frac{s-t}{2} \right) : (x,s), (x,t) \in K \right\}.$$

Consequently,

$$(S_{e_n} K)^\circ = \left\{ (y, r) : \langle x, y \rangle + r\frac{s-t}{2} \le 1 \text{ whenever } (x,s), (x,t) \in K \right\}.$$

For a set $L \subset \mathbb{R}^n$, denote its horizontal slice by $L(r) = \{x \in \mathbb{R}^{n-1} : (x, r) \in L\}$. If $y \in K^\circ(r)$ and $z \in K^\circ(-r)$, then

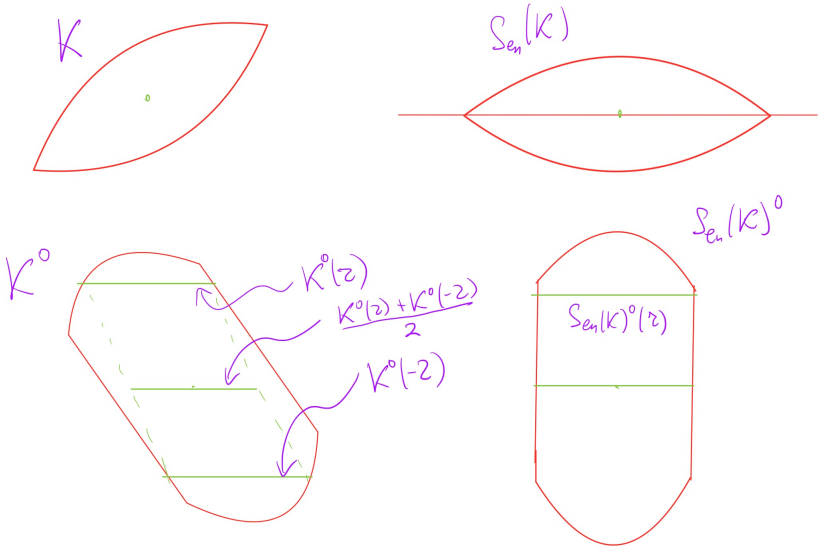
$$\langle x, y \rangle + sr \le 1, \qquad \langle x, z \rangle - tr \le 1$$

for every pair $(x,s), (x,t) \in K$. Averaging these two inequalities yields

$$\left\langle x, \frac{y+z}{2} \right\rangle + \frac{s-t}{2} r \le 1.$$

Thus

$$\frac{K^\circ(r) + K^\circ(-r)}{2} \subseteq (S_{e_n} K)^\circ(r).$$



Symmetry gives $|K^\circ(r)|_{n-1} = |K^\circ(-r)|_{n-1}$. The Brunn-Minkowski inequality in dimension $n-1$ therefore implies

$$|(S_{e_n}K)^\circ(r)|_{n-1} \geq \left|\frac{K^\circ(r)+K^\circ(-r)}{2}\right|_{n-1} \geq \sqrt{|K^\circ(r)|_{n-1}\,|K^\circ(-r)|_{n-1}} = |K^\circ(r)|_{n-1}.$$

Integrating in r and applying Fubini's theorem gives

$$|K^\circ| \leq |(S_{e_n}K)^\circ|.$$

The case $n = 1$ is immediate. □

To deduce the Blaschke-Santaló inequality, choose successive Steiner symmetrizations of K that converge to RB_2^n, where $R = |K|^{1/n}/|B_2^n|^{1/n}$. The volume stays fixed, whereas the polar volume increases. Polarity and volume are continuous along this sequence, so

$$|K|\,|K^\circ| \leq |RB_2^n|\,|(RB_2^n)^\circ| = |B_2^n|^2. \qquad \square$$

We mention another open question in the historical setting of these notes.

Conjecture 4.6 (Cordero-Erausquin)**.** *If μ is an even log-concave measure and K is a symmetric convex body, then*

$$\mu(K)\mu(K^\circ) \leq \mu(B_2^n)^2.$$

In particular, for every symmetric convex body L,

$$|K \cap L|\,|K^\circ \cap L| \leq |B_2^n \cap L|^2.$$

Klartag established weighted versions for a special class of measures [100]; the inequality also holds for rotation-invariant log-concave measures [69].

Question: Is it at least true that $\mu(RB_2^n)\mu(R^{-1}B_2^n) \leq \mu(B_2^n)^2$ for every even log-concave μ?

Answer: Yes! This statement is connected to the result of Cordero-Erausquin and Rotem [51] on the B-conjecture for rotation-invariant measures. We discuss the equivalence more fully in Remark 7.11.

4.7 A fun inequality on the circle

Recall from Remark 4.2 that, for a convex body K containing the origin in its interior,

$$|K^\circ| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(\theta)^{-n}\,d\theta.$$

We now express the area of a planar body K in terms of its support function. First suppose K is a polygon containing the origin. If its ith side has length a_i and distance h_i from the origin (see Figure 1), triangulation gives $|K| = \frac{1}{2}\sum_i a_i h_i$.

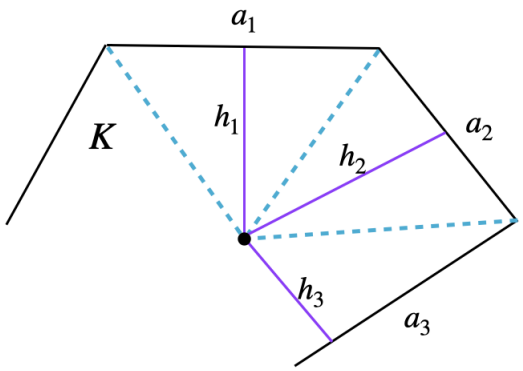


Figure 1: Decomposing a polygon into triangles with a common vertex at the origin.

What happens as the polygon approximates a smooth body? The heights become the support function (see Remark 2.6), and the side lengths become the surface area measure. For a smooth strictly convex planar body of positive curvature, the surface area measure in the normal-angle coordinate has density $1/k_K(\theta)$, where k_K is the curvature. The relation $1/k_K = h_K + \ddot{h}_K$ (see the homework) gives

$$|K| = \frac{1}{2}\int_{\mathbb{S}^1} \frac{h_K(\theta)}{k_K(\theta)}\,d\theta = \frac{1}{2}\int_{\mathbb{S}^1} h_K(\theta)\big(h_K(\theta) + \ddot{h}_K(\theta)\big)\,d\theta. \tag{15}$$

The support function is periodic, so integration by parts gives $\int_{-\pi}^{\pi} h_K\ddot{h}_K = -\int_{-\pi}^{\pi} \dot{h}_K^2$. Therefore

$$|K| = \frac{1}{2}\int_{\mathbb{S}^1} \big(h_K^2 - \dot{h}_K^2\big)\,d\theta. \tag{16}$$

This is sometimes called Santaló's area formula. For the sake of presentation we derived it in the smooth case; approximation by smooth convex bodies yields the formula for general planar convex bodies, whose support functions are Lipschitz.

For symmetric K, the Blaschke-Santaló inequality now reads

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (h_K^2 - \dot{h}_K^2)\,d\theta\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} h_K^{-2}\,d\theta\right) \le 1.$$

In fact, the support-function assumption is unnecessary.

Proposition 4.1. *If $f \in C^1(\mathbb{R})$ is π-periodic, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\dot{f}^2 - f^2)\,d\theta + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{-2}\,d\theta\right)^{-1} \ge 0.$$

If the inverse-power integral is infinite, its reciprocal is interpreted as zero.

Proof. We first treat positive smooth f. A function on the circle is a support function exactly when its 1-homogeneous extension is convex; in dimension 2, this is equivalent to $f + \ddot{f} \geq 0$ in the distributional sense. Thus the result follows from Blaschke-Santaló when f is already a support function.

For general positive f, extend it to $F(x) = |x| f(x/|x|)$ on $\mathbb{R}^2$ and set $G = F^{**}$. Write $g = G|_{\mathbb{S}^1}$. Then g is the largest support function below f. It is positive and π-periodic, and $g + \ddot{g}$ is a nonnegative measure. On every open interval where $g < f$, this measure vanishes, so $\ddot{g} + g = 0$ there. Geometrically, such an interval consists of normal directions belonging to a single vertex of the convex body determined by the inequalities $\langle x, \theta \rangle \leq f(\theta)$. Equivalently, the measure $g + \ddot{g}$ is supported on the contact set $\{g = f\}$.

The inverse-power term satisfies

$$\left(\int_{-\pi}^{\pi} f^{-2} \, d\theta \right)^{-1} \geq \left(\int_{-\pi}^{\pi} g^{-2} \, d\theta \right)^{-1}. \tag{17}$$

For the derivative term, put $u = f - g$. Integration by parts, with $g + \ddot{g}$ understood as a measure, gives

$$\begin{aligned} \int_{-\pi}^{\pi} (\dot{f}^2 - f^2) \, d\theta - \int_{-\pi}^{\pi} (\dot{g}^2 - g^2) \, d\theta &= \int_{-\pi}^{\pi} (\dot{u}^2 - u^2) \, d\theta - 2 \int_{\mathbb{S}^1} u \, d(g + \ddot{g}) \\ &= \int_{-\pi}^{\pi} (\dot{u}^2 - u^2) \, d\theta. \end{aligned}$$

The last term is nonnegative. Indeed, $u = 0$ on the contact set, and each component I of its complement has length at most π, by periodicity. On each such interval, the Dirichlet Poincaré inequality gives

$$\int_I u^2 \leq \frac{|I|^2}{\pi^2} \int_I \dot{u}^2 \leq \int_I \dot{u}^2.$$

Summing over the components proves the required comparison. Together with (17) and the inequality for g, this proves the claim for positive smooth functions.

Finally, smooth approximation and the replacement $f \mapsto \sqrt{f^2 + \varepsilon^2}$ remove positivity and extra smoothness. The inverse-power terms converge monotonically as $\varepsilon \downarrow 0$, and the derivative terms converge in L^2. □

The non-symmetric Blaschke-Santaló inequality leads to a version in which a first harmonic is added to choose the appropriate center.

Theorem 4.6. *If $f \in C^1(\mathbb{R})$ is 2π-periodic, there are $t, s \in \mathbb{R}$ such that, with $f_{t,s}(\theta) = f(\theta) + t \cos\theta + s \sin\theta$,*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (\dot{f}_{t,s}^2 - f_{t,s}^2) \, d\theta + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f_{t,s}|^{-2} \, d\theta \right)^{-1} \geq 0.$$

Again, an infinite inverse-power integral has reciprocal zero.

Explanation of the reduction. For positive f, use the preceding convexification construction and translate the resulting convex body to its Santaló point. Add the same first harmonic to f and its convexified support function g. Both functions remain positive. On each interval where $g < f$, the translated support function satisfies $\ddot{g} + g = 0$ and is positive, so the interval has length less than π. The same Dirichlet Poincaré argument therefore applies, and the non-symmetric Blaschke-Santaló inequality gives the conclusion.

To see why signed functions are also allowed, set $A(f) = \int_{-\pi}^{\pi}(f^2 - \dot{f}^2)$. Adding a first harmonic does not change A. If $A(f) \le 0$, the conclusion is immediate. Otherwise, the usual circle Poincaré inequality shows that f has nonzero mean; after changing its sign, assume its mean is $a > 0$.

Choose t, s to maximize $m = \min_\theta f_{t,s}(\theta)$; a maximizer exists because this minimum tends to $-\infty$ as $t^2 + s^2 \to \infty$. At a maximizer, the directions of minimum points have the origin in their convex hull: otherwise a small first-harmonic perturbation would raise all the minimum values. We may therefore select two or three minimum points whose consecutive arcs have length at most π. For $u = f_{t,s} - m$, Dirichlet Poincaré on these arcs gives $A(u) \le 0$. Consequently,

$$0 < A(f) = A(f_{t,s}) = A(u) + 4\pi m a - 2\pi m^2.$$

This is impossible if $m \le 0$. Thus a first-harmonic translate is positive, reducing to the positive case above. □

Remark 4.4. *Chen, Howard, Lutwak, Yang and Zhang [47] gave an independent analytic proof of a more general inequality, obtaining in particular a new proof of the planar Blaschke-Santaló inequality. For positive f, their formulation chooses t, s so that the first moments of $f_{t,s}^{-3}$ vanish; this is the analytic analogue of choosing the Santaló point.*

Now reparametrize the π-periodic inequality by $\phi(\theta) = f(\theta/2)$. Then ϕ is 2π-periodic, and we obtain:

Theorem 4.7. *For every 2π-periodic $\phi \in C^1(\mathbb{R})$,*

$$4\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\dot{\phi}^2\,d\theta\right) \ge \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^2\,d\theta - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\phi|^{-2}\,d\theta\right)^{-1},$$

with the same convention for an infinite inverse-power integral.

Remark 4.5. *This resembles the p-Beckner inequality. On the circle, with normalized angular measure, the corresponding inequality for $p \in [1, 2)$ is*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^2\,d\theta - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\phi|^p\,d\theta\right)^{2/p} \le (2-p)\frac{1}{2\pi}\int_{-\pi}^{\pi}\dot{\phi}^2\,d\theta.$$

The inequality just proved has exactly the same form with $p = -2$. Thus the symmetric planar Blaschke-Santaló inequality, together with the convexification argument above, yields this negative-exponent extension. Conversely, it implies Blaschke-Santaló for symmetric planar bodies by substituting their support functions and undoing the reparametrization.

A homework question. Is there an interval of negative p for which the p-Beckner inequality holds on the circle?

Remark 4.6. *For positive functions, the substitutions $g = \phi^{-2}$ and $f = \phi^2$ give, respectively,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{g}\,d\theta - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g\,d\theta\right)^{-1} \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\dot{g}^2}{g^3}\,d\theta$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f\,d\theta - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{-1}\,d\theta\right)^{-1} \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\dot{f}^2}{f}\,d\theta.$$

4.8 Functional version of the Blaschke-Santaló inequality

We now present a functional version of the Blaschke-Santaló inequality, introduced by Ball [11]. We shall see that it implies the geometric inequality. Our proof will also use the geometric version, following Artstein-Avidan, Klartag and Milman [6]. For simplicity we focus on the symmetric statement, although a non-symmetric functional inequality is also available [6]. We recommend Lehec's proof [133], which does not rely on the geometric Blaschke-Santaló inequality.

Theorem 4.8 (Ball [11]; Artstein-Avidan, Klartag and Milman [6]; Lehec [133])**.** *Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an even measurable function with $0 < \int e^{-\psi} < \infty$. Then*

$$\left(\int_{\mathbb{R}^n} e^{-\psi}\,dx\right)\left(\int_{\mathbb{R}^n} e^{-\psi^*}\,dx\right) \le \left(\int_{\mathbb{R}^n} e^{-|x|^2/2}\,dx\right)^2 = (2\pi)^n.$$

Recall that ψ^* is the smallest function ϕ for which $\psi(x) + \phi(y) \ge \langle x, y\rangle$ for every x, y. Therefore:

Corollary 4.2. *If $f, g : \mathbb{R}^n \to [0, \infty)$ are even measurable functions satisfying*

$$f(x)g(y) \le e^{-\langle x,y\rangle} \qquad \text{for all } x, y \in \mathbb{R}^n,$$

then, whenever the product of integrals is well-defined,

$$\left(\int f\right)\left(\int g\right) \le (2\pi)^n.$$

Proof of Theorem 4.8. First we reduce to a convex potential. Replacing ψ by its lower-semicontinuous convex envelope ψ^{**} increases $e^{-\psi}$ and leaves its conjugate unchanged. If the conjugate integral vanishes, there is nothing to prove. In the remaining case this reduction gives a proper even convex potential with integrable exponential. For the sake of presentation we work with this nondegenerate case; truncation and approximation give the same conclusion in the limiting cases.

For a constant c, $(\psi + c)^* = \psi^* - c$, so adding a constant does not change the product. Since an even convex function is minimized at the origin, we may normalize $\psi(0) = 0$. Then $\psi \geq 0$, $\psi^* \geq 0$, and $\psi^*(0) = 0$.

The layer-cake formula and the change of variables $t = e^{-s}$ give

$$\int_{\mathbb{R}^n} e^{-\psi(x)}\,dx = \int_0^1 |\{e^{-\psi} > t\}|\,dt = \int_0^\infty e^{-s}|\{\psi < s\}|\,ds.$$

Similarly,

$$\int_{\mathbb{R}^n} e^{-\psi^*(x)}\,dx = \int_0^\infty e^{-t}|\{\psi^* < t\}|\,dt.$$

The key observation is the following.

Claim 4.3. *For $s, t > 0$,*

$$\{\psi^* < t\} \subseteq (s+t)\{\psi < s\}^\circ.$$

Proof. If $x \in \{\psi < s\}$ and $y \in \{\psi^* < t\}$, the defining inequality for the Legendre transform gives

$$\langle x, y\rangle \leq \psi(x) + \psi^*(y) < s + t.$$

□

Define on $(0, \infty)$

$$a(s) = e^{-s}|\{\psi < s\}|, \qquad b(t) = e^{-t}|\{\psi^* < t\}|, \qquad c(r) = |B_2^n|\,2^{n/2}e^{-r}r^{n/2},$$

and extend these functions by zero to the negative half-line. We claim that $c((s+t)/2) \geq \sqrt{a(s)b(t)}$. Indeed,

$$c\left(\frac{s+t}{2}\right)^2 = |B_2^n|^2 e^{-s-t}(s+t)^n,$$

while the inclusion above and the geometric Blaschke-Santaló inequality imply

$$\begin{aligned} a(s)b(t) &= e^{-s-t}|\{\psi < s\}|\,|\{\psi^* < t\}| \\ &\leq e^{-s-t}(s+t)^n|\{\psi < s\}|\,|\{\psi < s\}^\circ| \\ &\leq e^{-s-t}(s+t)^n|B_2^n|^2. \end{aligned}$$

Sublevel sets can be replaced by their closures without changing the relevant volumes; degenerate cases follow by a limit.

Applying the one-dimensional Prékopa-Leindler inequality to a, b, c, we conclude

$$\begin{aligned} \left(\int e^{-\psi}\right)\left(\int e^{-\psi^*}\right) &= \left(\int_0^\infty a(s)\,ds\right)\left(\int_0^\infty b(t)\,dt\right) \\ &\leq \left(\int_0^\infty c(r)\,dr\right)^2 \\ &= 2^n|B_2^n|^2\left(\int_0^\infty e^{-r}r^{n/2}\,dr\right)^2 = (2\pi)^n. \end{aligned}$$

The last equality follows from $|B_2^n| = \pi^{n/2}/\Gamma(1 + n/2)$. □

Remark 4.7. *Setting $\psi(x) = \|x\|_K^2/2$, so that $\psi^*(x) = \|x\|_{K^\circ}^2/2$, and integrating in polar coordinates recovers the geometric Blaschke-Santaló inequality; see the homework.*

Remark 4.8. *For convex potentials, equality holds exactly for*

$$\psi(x) = \frac{1}{2}\langle Ax, x\rangle + c,$$

where A is positive definite and $c \in \mathbb{R}$; see [6]. Thus every centered nondegenerate Gaussian, with any normalization, is an equality case.

We also mention the following theorem; see the homework.

Theorem 4.9 (Fradelizi-Meyer [69])**.** *Let $f, g : \mathbb{R}^n \to [0,\infty)$ be even measurable functions, and let $\rho : [0,\infty) \to [0,\infty)$ be measurable. Suppose*

$$f(x)g(y) \le \rho(\langle x, y\rangle)^2 \qquad \text{whenever } \langle x, y\rangle > 0.$$

Then

$$\left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right) \le \left(\int_{\mathbb{R}^n} \rho(|x|^2)\,dx\right)^2,$$

with the usual integrability conventions.

4.9 Linearizing Theorem 4.8

Take $\psi_\varepsilon(x) = |x|^2/2 + \varepsilon f(x)$, where f is even. For the computation, assume f is smooth with bounded Hessian and sufficient integrability. Then ψ_ε is convex for sufficiently small $|\varepsilon|$, and differentiation under the integrals is justified. The resulting inequality extends to even $W^{1,2}(\gamma)$ functions by approximation.

The functional Blaschke-Santaló inequality gives

$$\left(\int e^{-(|x|^2/2+\varepsilon f)}\,dx\right)\left(\int e^{-(|x|^2/2+\varepsilon f)^*}\,dx\right) \le (2\pi)^n.$$

The variation formulas for the Legendre transform, evaluated at $q(x) = |x|^2/2$, give

$$\frac{d}{d\varepsilon}\psi_\varepsilon^*\Big|_{\varepsilon=0} = -f, \qquad \frac{d^2}{d\varepsilon^2}\psi_\varepsilon^*\Big|_{\varepsilon=0} = |\nabla f|^2,$$

because $\nabla q = x$ and $\nabla^2 q = \mathrm{Id}$. Hence

$$\psi_\varepsilon^*(x) = \frac{|x|^2}{2} - \varepsilon f(x) + \frac{\varepsilon^2}{2}|\nabla f(x)|^2 + o(\varepsilon^2).$$

Expanding the exponentials and dividing by $(2\pi)^n$, we obtain

$$\begin{aligned} 1 &\geq \left(\int \left(1 - \varepsilon f + \frac{\varepsilon^2}{2} f^2 \right) d\gamma \right) \left(\int \left(1 + \varepsilon f + \frac{\varepsilon^2}{2} (f^2 - |\nabla f|^2) \right) d\gamma \right) + o(\varepsilon^2) \\ &= 1 + \varepsilon^2 \left(\int f^2 \, d\gamma - \left(\int f \, d\gamma \right)^2 - \frac{1}{2} \int |\nabla f|^2 \, d\gamma \right) + o(\varepsilon^2). \end{aligned}$$

The constant terms agree, and the linear terms cancel. Comparing the quadratic terms yields:

Theorem 4.10. *For every even $f \in W^{1,2}(\gamma)$,*

$$\int f^2 \, d\gamma - \left(\int f \, d\gamma \right)^2 \leq \frac{1}{2} \int |\nabla f|^2 \, d\gamma.$$

Remark 4.9. *The constant improves from 1 in the usual Gaussian Poincaré inequality, which also follows from Beckner's inequality (Theorem 3.14), to 1/2 when the function is even.*

Remark 4.10. *Using the appropriately centered non-symmetric functional Blaschke-Santaló inequality, one can replace evenness by the condition $\int \nabla f \, d\gamma = 0$. Integration by parts gives*

$$\int \partial_i f \, d\gamma = \int \langle \nabla f, \nabla x_i \rangle \, d\gamma = - \int f L_\gamma x_i \, d\gamma = \int f x_i \, d\gamma.$$

Thus the condition says exactly that f is orthogonal to all linear functions. After removing the constant and linear eigenspaces, the next eigenvalue of $-L_\gamma$ is 2, and this is the source of the improved constant.

Remark 4.11. *The same result follows from Fourier analysis in Gaussian space, using the decomposition into Hermite polynomials.*

Recall that the Brascamp-Lieb inequality allows us to restrict the usual Gaussian Poincaré inequality to any convex set K of positive Gaussian measure:

$$\frac{1}{\gamma(K)} \int_K f^2 \, d\gamma - \left(\frac{1}{\gamma(K)} \int_K f \, d\gamma \right)^2 \leq \frac{1}{\gamma(K)} \int_K |\nabla f|^2 \, d\gamma.$$

Can we also restrict the improved inequality in Theorem 4.10?

4.10 A brief excursion into mass transport

We make a brief excursion into mass transport. For a more detailed, but still concise, introduction, we recommend Klartag [109]. Another insightful survey is by Ball [12]. See also the books by Villani [175], Figalli and Glaudo [66], and Bogachev, Kolesnikov and Shaposhnikov [29].

Definition 4.5. Let μ, ν be Borel probability measures on $\mathbb{R}^n$. A measurable map $T : \operatorname{supp}\mu \to \mathbb{R}^n$ *transports* μ to ν, written $T_{\#}\mu = \nu$, if

$$\nu(A) = \mu(T^{-1}(A))$$

for every Borel set A. Equivalently,

$$\int \phi \, d\nu = \int \phi \circ T \, d\mu$$

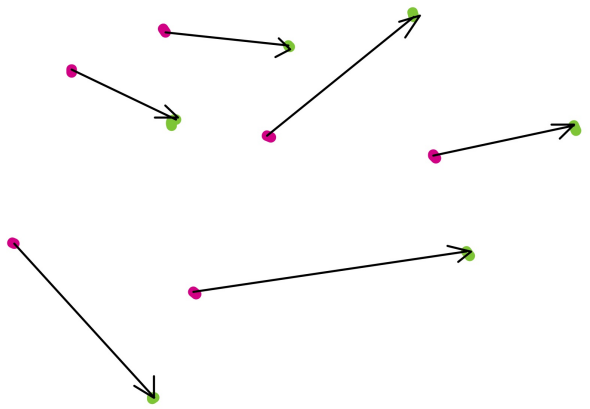
for every integrable test function ϕ.

Let us consider some examples.

Example 4.1. *For a discrete probability measure* $\mu = \sum_i \lambda_i \delta_{x_i}$,

$$T_{\#}\mu = \sum_i \lambda_i \delta_{T(x_i)}.$$

Thus a bijection on the support moves the atoms while preserving their weights. A general map may also merge atoms. It cannot split an atom, so arbitrary atomic probability measures need not be related by a transport map, much less a bijection.



Example 4.2. *Let* γ_1 *be the standard Gaussian probability measure on* $\mathbb{R}$*, and let*

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \, ds.$$

Then $\Phi_{\#}\gamma_1 = \operatorname{Unif}[0,1]$*. Indeed, for* $0 < x < y < 1$,

$$y - x = \int_{\Phi^{-1}(x)}^{\Phi^{-1}(y)} \Phi'(t) \, dt = \gamma_1\Big([\Phi^{-1}(x), \Phi^{-1}(y)]\Big).$$

This verifies the pushforward identity on intervals, and hence on all Borel sets.

Remark 4.12. *Suppose $T_{\#}\mu = \nu$, with densities ρ_μ and ρ_ν, and suppose for this calculation that T is a smooth invertible map. The change-of-variables formula gives*

$$\rho_\mu(x) = \rho_\nu(T(x))\,|\det DT(x)|.$$

Here $DT = (\partial_j T_i)_{i,j}$ is the Jacobian matrix and $|\det DT|$ is the Jacobian factor. This formula should not be read as an equality between densities evaluated at the same point.

Definition 4.6 (Optimal transport)**.** For probability measures with finite second moments, a map $\widehat{T}$ with $\widehat{T}_{\#}\mu = \nu$ is *optimal for quadratic cost* if

$$\int |\widehat{T}(x) - x|^2\, d\mu(x) \le \int |T(x) - x|^2\, d\mu(x)$$

for every map T transporting μ to ν.

Theorem 4.11 (Brenier)**.** *Let μ, ν be Borel probability measures with finite second moments, and assume that μ is absolutely continuous. A map T with $T_{\#}\mu = \nu$ is optimal for quadratic cost if and only if $T = \nabla F$ μ-almost everywhere for a convex potential F, finite on the relevant part of the source domain.*

Compactness of the set of transport plans, together with Brenier's theorem, gives the following existence statement. The passage through plans matters: compactness alone does not guarantee that a limiting plan is a map.

Corollary 4.3. *Under the assumptions of Theorem 4.11, there exists a convex potential F such that*

$$(\nabla F)_{\#}\mu = \nu.$$

The optimal map is unique μ-almost everywhere. If the source has positive density on a connected open domain, the potential is unique there up to an additive constant.

Remark 4.13. *When F and the densities are smooth enough for change of variables, the transport equation becomes*

$$\rho_\mu(x) = \rho_\nu(\nabla F(x))\det \nabla^2 F(x).$$

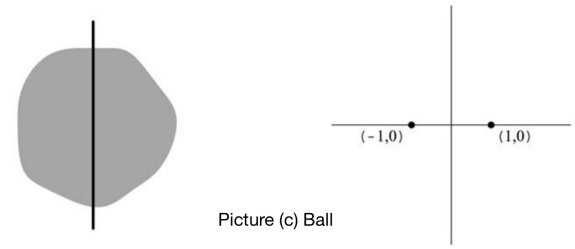
The Hessian appears because $D(\nabla F) = \nabla^2 F$. This is a form of the Monge-Ampère equation.

Here are some further examples.

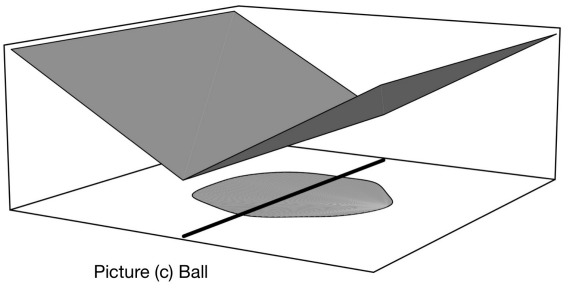
Example 4.3. *Suppose we transport the uniform probability measure on a symmetric convex body K to $\nu = \frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{-e_1}$, where δ_y denotes the unit point mass at y. Each half of K, cut by $x_1 = 0$, has mass $1/2$. It is therefore optimal to move every point to the closer of e_1 and $-e_1$:*

$$T(x) = \begin{cases} e_1, & x_1 > 0, \\ -e_1, & x_1 < 0. \end{cases}$$

The value on $x_1 = 0$ is irrelevant, since this hyperplane has source measure zero. Observe that $T = \nabla F$ almost everywhere for $F(x) = |\langle x, e_1\rangle|$.



Picture (c) Ball



Picture (c) Ball

The transport illustrations above are from Ball's exposition. Restricting and normalizing the source in this example is essential: Lebesgue measure on all of $\mathbb{R}^n$ cannot be transported to a measure of finite total mass.

Example 4.4. *More generally, a convex piecewise-affine potential*

$$F(x) = \max_i \{\langle x, \theta_i \rangle + a_i\}$$

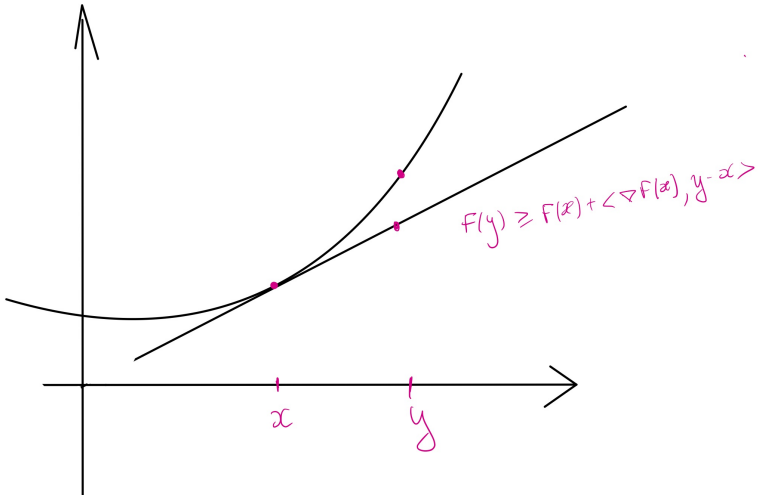
has gradient equal to one of the vectors θ_i almost everywhere. For an absolutely continuous source probability measure μ, it therefore transports μ to an atomic measure supported on those vectors. The mass at θ_i equals the μ-measure of the region where the ith affine function is maximal. To prescribe the atomic masses, one must choose the offsets a_i appropriately. Taking all $a_i = 0$ gives the simple picture determined by $F(x) = \max_i \langle x, \theta_i \rangle$, with target weights determined by the resulting cells.

We now consider a discrete version of the easy direction of Brenier's theorem.

Lemma 4.2. *Let $x_1, \dots, x_N, y_1, \dots, y_N \in \mathbb{R}^n$. Suppose there is a convex function F differentiable at each x_i such that $\nabla F(x_i) = y_i$. Then, for every permutation σ,*

$$\sum_{i=1}^{N} |x_i - y_{\sigma(i)}|^2 \geq \sum_{i=1}^{N} |x_i - y_i|^2.$$

Proof. Convexity gives $F(x_j) - F(x_i) \geq \langle \nabla F(x_i), x_j - x_i \rangle$.



Apply this with $j = \sigma^{-1}(i)$ and sum. Since a permutation leaves the sum of the $F(x_i)$ unchanged,

$$\sum_{i=1}^{N} \langle y_i, x_{\sigma^{-1}(i)} - x_i \rangle \le 0,$$

or, after reindexing,

$$\sum_{i=1}^{N} \langle x_i, y_{\sigma(i)} \rangle \le \sum_{i=1}^{N} \langle x_i, y_i \rangle.$$

Expanding the squared distances proves the claim, because the sums of the $|x_i|^2$ and $|y_i|^2$ terms are unchanged by the permutation. □

Lastly, we state an important theorem of Caffarelli [43]; see also Kolesnikov [115].

Theorem 4.12 (Caffarelli's contraction theorem)**.** *Let γ be standard Gaussian measure and let μ be a probability measure that is strongly log-concave with Gaussian curvature at least 1. In the smooth case, this means $d\mu = e^{-v}dx$ with $\nabla^2 v \succeq \mathrm{Id}$; equivalently, the density $d\mu/d\gamma$ is log-concave. The Brenier map T satisfying $T_{\#}\gamma = \mu$ has a 1-Lipschitz representative:*

$$|T(x) - T(y)| \le |x - y| \qquad (x, y \in \mathbb{R}^n).$$

Remark 4.14. *For a convex set K with $\gamma(K) > 0$, the normalized restriction*

$$d\gamma_K = \frac{\mathbb{1}_K}{\gamma(K)}\, d\gamma$$

is strongly log-concave in this sense. The corresponding Brenier map from γ to γ_K is therefore a contraction. The indicator is handled by approximating its convex potential by smooth convex potentials and passing to the limit.

4.11 Restricting the symmetric Gaussian Poincaré inequality onto a symmetric convex set

Combining Theorem 4.10, obtained from Blaschke-Santaló, with Caffarelli's contraction theorem gives:

Theorem 4.13. *If $K \subset \mathbb{R}^n$ is symmetric and convex, $\gamma(K) > 0$, and $f \in W^{1,2}(\gamma_K)$ is even, then*

$$\frac{1}{\gamma(K)} \int_K f^2 \, d\gamma - \left(\frac{1}{\gamma(K)} \int_K f \, d\gamma \right)^2 \leq \frac{1}{2\gamma(K)} \int_K |\nabla f|^2 \, d\gamma. \tag{18}$$

Proof. Let $T_\# \gamma = \gamma_K$ be the Brenier map. It is a contraction by Theorem 4.12. Moreover, it is odd: the reflected map $x \mapsto -T(-x)$ is also an optimal transport between the same symmetric measures, and uniqueness gives $T(-x) = -T(x)$ almost everywhere. Hence $f \circ T$ is even.

The pushforward identity gives

$$\mathrm{Var}_{\gamma_K}(f) = \mathrm{Var}_{\gamma}(f \circ T).$$

By the chain rule and the contraction property,

$$\int |\nabla (f \circ T)|^2 \, d\gamma \leq \int |\nabla f|^2 \circ T \, d\gamma = \int |\nabla f|^2 \, d\gamma_K.$$

Applying Theorem 4.10 to the even function $f \circ T$, we obtain

$$\mathrm{Var}_{\gamma_K}(f) \leq \frac{1}{2} \int |\nabla (f \circ T)|^2 \, d\gamma \leq \frac{1}{2} \int |\nabla f|^2 \, d\gamma_K,$$

which is (18). We may first make the calculation for smooth functions and then use Sobolev approximation. □

Remark 4.15. *Klartag [100] used Caffarelli's theorem to prove the following weighted functional Blaschke-Santaló inequality. If μ is an even log-concave measure and ψ is even, then*

$$\left(\int e^{-\psi} \, d\mu \right) \left(\int e^{-\psi^*} \, d\mu \right) \leq \left(\int e^{-|x|^2/2} \, d\mu \right)^2,$$

with the natural integrability assumptions. Taking $d\mu = \mathbb{1}_K \, dx$ and repeating the linearization in Subsection 4.9 yields Theorem 4.13, and hence the Gaussian B-theorem (Theorem 4.14) discussed next.

4.12 The B-conjecture

Conjecture 4.7 (B-conjecture)**.** *Let μ be an even log-concave measure on $\mathbb{R}^n$ and let K be a symmetric convex set. Then $\mu(e^t K)$ is log-concave in $t \in \mathbb{R}$. Equivalently, its midpoint form is*

$$\mu(\sqrt{ab}\, K) \geq \sqrt{\mu(aK)\mu(bK)}, \qquad a, b > 0.$$

Remark 4.16. *By the Prékopa–Leindler inequality, for any convex set K we know that*

$$\mu\left(\frac{a+b}{2}K\right) \geq \sqrt{\mu(aK)\mu(bK)}.$$

For symmetric K, the origin belongs to K, so its dilates are nested. The arithmetic–geometric mean inequality shows that this conclusion is weaker than the B-conjecture.

4.13 The B-theorem for the Gaussian measure due to Cordero-Erausquin, Fradelizi, Maurey

In the case of the standard Gaussian measure, conjecture 4.7 was verified:

Theorem 4.14 (B-theorem, Cordero-Erausquin, Fradelizi, Maurey [52])**.** *Let γ be the standard Gaussian measure and let K be a symmetric convex set. Then $\gamma(e^t K)$ is log-concave in $t \in \mathbb{R}$.*

Proof. It is enough to prove

$$\frac{d^2}{dt^2}\log\gamma(e^t K)\bigg|_{t=0} \leq 0$$

for every symmetric convex K: replacing K by $e^{t_0}K$ gives the inequality at any t_0. We assume $\gamma(K) > 0$ and work first with bounded K, so differentiation under the integral is straightforward. Unbounded sets follow by intersecting with larger and larger centered balls.

Introduce the auxiliary function

$$F(s) := (2\pi)^{n/2}\gamma(sK) = \int_{sK} e^{-|x|^2/2}\,dx = s^n \int_K e^{-s^2|y|^2/2}\,dy, \qquad s > 0.$$

We have

$$F'(s) = ns^{n-1}\int_K e^{-s^2|y|^2/2}\,dy - s^{n+1}\int_K |y|^2 e^{-s^2|y|^2/2}\,dy,$$

and therefore

$$F'(1) = (2\pi)^{n/2}\left(n\gamma(K) - \int_K |y|^2\,d\gamma(y)\right).$$

Differentiating once more gives

$$F''(1) = (2\pi)^{n/2}\left(n(n-1)\gamma(K) - (2n+1)\int_K |y|^2\,d\gamma(y) + \int_K |y|^4\,d\gamma(y)\right).$$

Let Y have the normalized restriction $\gamma(\cdot \cap K)/\gamma(K)$. In particular,

$$\frac{F'(1)}{F(1)} = n - \mathbb{E}|Y|^2, \qquad \frac{F''(1)}{F(1)} = n(n-1) - (2n+1)\mathbb{E}|Y|^2 + \mathbb{E}|Y|^4.$$

Thus

$$\begin{aligned}\frac{d^2}{dt^2}\log\gamma(e^tK)\Big|_{t=0} &= \frac{d}{dt}\frac{e^tF'(e^t)}{F(e^t)}\Big|_{t=0}\\ &= \frac{F'(1)F(1)+F''(1)F(1)-F'(1)^2}{F(1)^2}\\ &= n-\mathbb{E}|Y|^2+n(n-1)-(2n+1)\mathbb{E}|Y|^2+\mathbb{E}|Y|^4-(n-\mathbb{E}|Y|^2)^2\\ &= \mathrm{Var}(|Y|^2)-2\mathbb{E}|Y|^2. \end{aligned} \tag{19}$$

Recall the symmetric Gaussian Poincaré inequality on a symmetric convex set: for every even, locally Lipschitz f,

$$\fint_K f^2\,d\gamma-\left(\fint_K f\,d\gamma\right)^2\le\frac{1}{2}\fint_K|\nabla f|^2\,d\gamma, \tag{20}$$

where $\fint_K f\,d\gamma=\gamma(K)^{-1}\int_K f\,d\gamma$. Applying this to $f(y)=|y|^2$, whose gradient is $2y$, gives

$$\mathrm{Var}(|Y|^2)\le 2\mathbb{E}|Y|^2.$$

Together with (19), this proves the theorem.

The same calculation also gives the matrix version needed below: for any symmetric matrix A,

$$\frac{d^2}{dt^2}\log\gamma(e^{tA}K)\Big|_{t=0}=\mathrm{Var}\left(\langle AY,Y\rangle\right)-2\mathbb{E}|AY|^2\le 0.$$

Here one uses (20) with $f(y)=\langle Ay,y\rangle$. Replacing K by $e^{t_0A}K$ shows that $t\mapsto\gamma(e^{tA}K)$ is log-concave. □

Remark 4.17. *Caffarelli's contraction theorem is a useful tool for transferring Gaussian inequalities to measures $d\mu=e^{-v}\,dx$ with $\nabla^2 v\ge\mathrm{Id}$, or more generally $\nabla^2 v\ge k\,\mathrm{Id}$, with the appropriate scaling. This includes normalized Gaussian restrictions to convex sets, understood by approximation. For inequalities involving even functions, symmetry of the measure ensures that the transport can be chosen odd.*

There are several applications of the B-theorem:

- Klartag and Vershynin [114] applied it to small-ball estimates, which we will discuss later in subsection 5.13.
- Bobkov [24]: let K be a symmetric convex body with $|K|=1$. Among volume-preserving linear transformations T, the maximum of $\gamma(TK)$ is attained at $T=\mathrm{Id}$ if and only if the probability measure

  $$d\gamma_K(x)=\frac{e^{-|x|^2/2}\mathbb{1}_K(x)}{(2\pi)^{n/2}\gamma(K)}\,dx$$

  has covariance matrix $c\,\mathrm{Id}$ for some $c>0$. Its barycenter is already zero by symmetry. This is sometimes called a Gaussian isotropic position; here the scalar c is not required to equal 1. The matrix form of the B-theorem above yields the sufficiency of this condition.

4.14 Some more history on the B-conjecture

The following cases were known at the time of this course:

- The Gaussian measure: theorem 4.14, due to Cordero-Erausquin, Fradelizi and Maurey [52].
- The measure $d\mu = c_n e^{-\|x\|_1}\, dx$: Eskenazis, Nayar and Tkocz [62]. Their result uses the Gaussian B-theorem.
- Dimension 2: Livne Bar-On [136], Böröczky, Lutwak, Yang and Zhang [36], and Saroglou [168].
- Rotation-invariant log-concave measures: Cordero-Erausquin and Rotem [56]. We shall discuss this result later in the semester.
- The unconditional case: Saroglou [169], and Cordero-Erausquin, Fradelizi and Maurey [52]; see subsection 4.15 below.

In fact, the following stronger statement is often referred to as Banaszczyk's original conjecture. An affirmative answer implies the Gaussian B-theorem; see the homework.

Conjecture 4.8 (Banaszczyk)**.** *Let K be a symmetric convex body and let γ be the standard Gaussian measure. For every $z \in \mathbb{R}^n$, the function*

$$t \longmapsto \frac{\gamma(z + tK)}{\gamma(tK)}$$

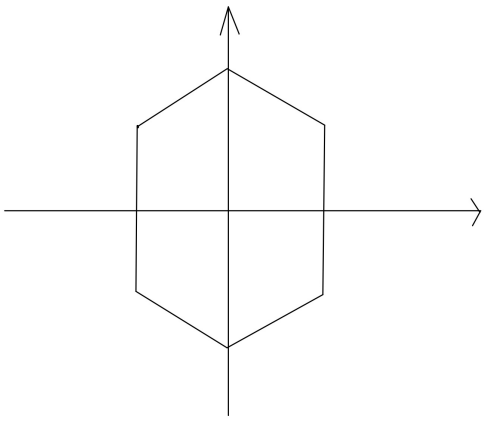
is nondecreasing on $(0, \infty)$.

4.15 B-conjecture in the unconditional case

Definition 4.7. A set $A \subset \mathbb{R}^n$ is called *unconditional* if, whenever $x = (x_1, \ldots, x_n) \in A$, the point $(\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ also belongs to A for every choice of $\epsilon_i \in \{-1, 1\}$. In other words, A is invariant under coordinate reflections. A function is unconditional if

$$f(x_1, \ldots, x_n) = f(\epsilon_1 x_1, \ldots, \epsilon_n x_n)$$

for every $x \in \mathbb{R}^n$ and every choice of signs.



Theorem 4.15 (Multiplicative Prékopa–Leindler inequality on $\mathbb{R}^n_+$)**.** *Let $f, g, h : \mathbb{R}^n_+ \to [0, \infty)$ be measurable functions, where $\mathbb{R}^n_+ = \{x : x_i \geq 0 \text{ for all } i\}$. Suppose*

$$\sqrt{f(x)g(y)} \leq h(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n})$$

for all $x, y \in \mathbb{R}^n_+$. Then

$$\left(\int_{\mathbb{R}^n_+} f\right)\left(\int_{\mathbb{R}^n_+} g\right) \leq \left(\int_{\mathbb{R}^n_+} h\right)^2.$$

Proof. The proof is a change of variables in the Prékopa–Leindler inequality. Define

$$F(s_1, \ldots, s_n) = f(e^{s_1}, \ldots, e^{s_n}) e^{s_1 + \cdots + s_n}$$

on $\mathbb{R}^n$, and define G and H analogously. The assumption gives

$$H\left(\frac{s+t}{2}\right) \geq \sqrt{F(s)G(t)}, \qquad s, t \in \mathbb{R}^n.$$

The substitution $x_i = e^{s_i}$ also gives $\int_{\mathbb{R}^n_+} f = \int_{\mathbb{R}^n} F$, and the corresponding identities for g, G and h, H. The conclusion now follows from theorem 3.5. □

Corollary 4.4 (Cordero-Erausquin, Fradelizi, Maurey)**.** *Let μ be a log-concave unconditional measure and let K be an unconditional convex set. Then $\mu(e^t K)$ is log-concave in $t \in \mathbb{R}$.*

Proof. For a set A write $A_+ = A \cap \mathbb{R}^n_+$. Let $d\mu = e^{-v}\, dx$, and consider

$$f = \mathbb{1}_{(aK)_+} e^{-v}, \qquad g = \mathbb{1}_{(bK)_+} e^{-v}, \qquad h = \mathbb{1}_{(\sqrt{ab}K)_+} e^{-v}.$$

An unconditional convex set is downward closed in the positive orthant: if $0 \leq x_i \leq y_i$ for every i and $y \in K_+$, then $x \in K_+$. Similarly, an unconditional convex function v is nondecreasing in each coordinate on $\mathbb{R}^n_+$.

If $x \in (aK)_+$ and $y \in (bK)_+$, convexity gives $(x/a + y/b)/2 \in K_+$. Coordinatewise AM–GM and downward closure therefore imply

$$\left(\sqrt{\frac{x_1 y_1}{ab}}, \dots, \sqrt{\frac{x_n y_n}{ab}}\right) \in K_+.$$

Moreover,

$$v(\sqrt{x_1 y_1}, \dots, \sqrt{x_n y_n}) \le v\left(\frac{x+y}{2}\right) \le \frac{v(x)+v(y)}{2}.$$

These are precisely the conditions of theorem 4.15. Since each orthant carries the same mass, its conclusion is

$$\mu(\sqrt{ab}K) \ge \sqrt{\mu(aK)\mu(bK)}.$$

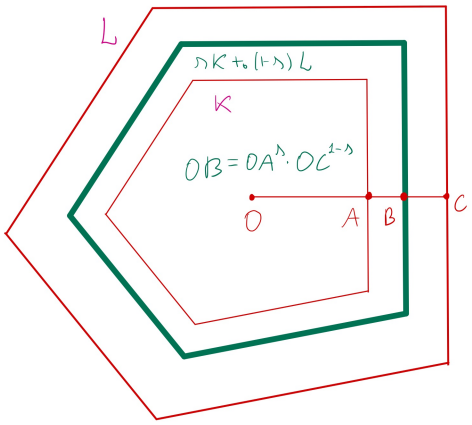
The same proof with weighted geometric means gives all interpolation parameters $\lambda \in [0,1]$; we leave the details as homework. □

4.16 About the Log-Brunn–Minkowski conjecture

Definition 4.8 (Log-addition)**.** Let K, L be symmetric convex bodies. We define their logarithmic midpoint by

$$\frac{K +_0 L}{2} := \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \le \sqrt{h_K(u) h_L(u)} \right\}.$$

Here h_K and h_L are their support functions. The notation denotes the whole operation; in particular, the 2 in the denominator is part of our midpoint notation.



We can also write the usual Minkowski midpoint as

$$\frac{K+L}{2} = \bigcap_{u \in \mathbb{S}^{n-1}} \left\{ x \in \mathbb{R}^n : \langle x, u \rangle \le \frac{h_K(u) + h_L(u)}{2} \right\}.$$

Therefore, by AM–GM,

$$\frac{K +_0 L}{2} \subseteq \frac{K + L}{2}. \tag{21}$$

The following conjecture strengthens the multiplicative Brunn–Minkowski inequality for symmetric convex bodies.

Conjecture 4.9 (Böröczky, Lutwak, Yang, Zhang, 2013 [36])**.** *For symmetric convex bodies $K, L \subset \mathbb{R}^n$,*

$$\left|\frac{K +_0 L}{2}\right| \geq \sqrt{|K||L|}.$$

The validity of this conjecture for Lebesgue measure is equivalent to its validity for all even log-concave measures:

Theorem 4.16 (Saroglou [168])**.** *Conjecture 4.9 in dimension n is equivalent to the following statement: for every even log-concave measure μ on $\mathbb{R}^n$ and every pair of symmetric convex bodies K, L,*

$$\mu\left(\frac{K +_0 L}{2}\right) \geq \sqrt{\mu(K)\mu(L)}. \tag{22}$$

Proof. The forward implication uses the Prékopa–Leindler inequality; see the homework. The reverse implication follows by taking Lebesgue measure. If one restricts the statement to probability measures, apply it to small dilates of K, L under a measure with a positive continuous density at the origin, and pass to the limit. □

Consequently, the Log-Brunn–Minkowski conjecture implies the B-conjecture. Indeed, applying (22) to aK and bK and observing that

$$\frac{aK +_0 bK}{2} = \sqrt{ab}\, K$$

gives $\mu(\sqrt{ab}K) \geq \sqrt{\mu(aK)\mu(bK)}$. Böröczky, Lutwak, Yang and Zhang [36] proved the Log-Brunn–Minkowski conjecture in dimension 2, so this also gives the B-conjecture on the plane.

Let us now return to the Blaschke–Santaló inequality; see theorems 4.1 and 4.8.

4.17 Reverse Log-Sobolev inequality

For a probability density f with respect to Lebesgue measure, recall the notation

$$\mathrm{Ent}(f) = \int_{\mathbb{R}^n} f \log f \, dx.$$

We shall discuss another useful corollary of the Blaschke–Santaló inequality.

Theorem 4.17 (Artstein-Avidan, Klartag, Schütt, Werner [7])**.** *Let v be an even convex function with $\int_{\mathbb{R}^n} e^{-v}\,dx = 1$, and write $d\mu = e^{-v}\,dx$. Then*

$$-\int v\,d\mu = \mathrm{Ent}(e^{-v}) \geq \frac{1}{2}\int \log\det \nabla^2 v\,d\mu - \frac{n}{2}\log(2\pi e). \tag{23}$$

For the calculation below, we assume that v is smooth with positive definite Hessian and that all the displayed integrals and integrations by parts are justified. The general statement uses the almost-everywhere Hessian of a convex function.

Remark 4.18. *Equality holds for every centered nondegenerate Gaussian density, and these are the only equality cases. Thus the equality potentials are*

$$v(x) = \frac{1}{2}\langle Ax, x\rangle + \frac{n}{2}\log(2\pi) - \frac{1}{2}\log\det A, \qquad A = A^\top > 0.$$

In particular, for the standard Gaussian density $\varphi_n(x) = (2\pi)^{-n/2}e^{-|x|^2/2}$,

$$\mathrm{Ent}(\varphi_n) = -\frac{n}{2}\log(2\pi e),$$

since $\int |x|^2\,d\gamma = n$. Hence (23) is equivalently

$$\mathrm{Ent}(e^{-v}) - \mathrm{Ent}(\varphi_n) \geq \frac{1}{2}\int \log\det\nabla^2 v\,d\mu.$$

Remark 4.19. *Recall the generalized Log-Sobolev inequality from theorem 3.10. When $\int e^{-v} = \int e^{-G} = 1$, it gives*

$$\mathrm{Ent}(e^{-v}) \leq \int G^*(\nabla v)\,d\mu - n.$$

Taking $G(x) = |x|^2/2 + (n/2)\log(2\pi)$, so that $G^*(x) = |x|^2/2 - (n/2)\log(2\pi)$, yields

$$\mathrm{Ent}(e^{-v}) \leq \frac{1}{2}\int |\nabla v|^2\,d\mu - n - \frac{n}{2}\log(2\pi).$$

For the operator $Lu = \Delta u - \langle \nabla v, \nabla u\rangle$, integration by parts gives $\int Lu\,d\mu = 0$. With $u = v$, this implies

$$\int \Delta v\,d\mu = \int |\nabla v|^2\,d\mu.$$

Combining the two entropy inequalities therefore gives

$$\frac{1}{2}\int \log\det\nabla^2 v\,d\mu \leq \mathrm{Ent}(e^{-v}) + \frac{n}{2}\log(2\pi e) \leq \frac{1}{2}\int(\Delta v - n)\,d\mu.$$

For $\mu = \gamma$, these become $0 \leq 0 \leq 0$. The lower inequality is sharp for every centered Gaussian, whereas the upper inequality in this fixed normalization is sharp for the standard Gaussian. Optimizing its scale makes the second inequality stronger; the homework asks you to show that

$$\mathrm{Ent}(e^{-v}) \leq \frac{n}{2}\log\left(\frac{\int \Delta v\,d\mu}{n}\right) - \frac{n}{2}\log(2\pi e).$$

Proof of theorem 4.17. We give the calculation in the smooth case described above. The functional Blaschke–Santaló inequality and $\int e^{-v} = 1$ imply

$$\int e^{-v^*}\,dx \le (2\pi)^n.$$

Make the change of variables $x = \nabla v(y)$. Using

$$v^*(\nabla v(y)) = \langle \nabla v(y), y\rangle - v(y),$$

we obtain

$$\begin{aligned}
(2\pi)^n &\ge \int e^{-v^*(\nabla v(y))} \det \nabla^2 v(y)\,dy \\
&= \int e^{-\langle \nabla v(y), y\rangle + v(y)} \det \nabla^2 v(y)\,dy \\
&= \int e^{-\langle \nabla v(y), y\rangle + 2v(y) + \log\det \nabla^2 v(y)}\,d\mu(y) \\
&\ge \exp\left(\int \Big[-\langle \nabla v(y), y\rangle + 2v(y) + \log\det \nabla^2 v(y)\Big]\,d\mu(y)\right).
\end{aligned}$$

The last inequality is Jensen's inequality, theorem 2.3, for the exponential function. Taking logarithms gives

$$n\log(2\pi) \ge \int \Big[-\langle \nabla v(y), y\rangle + 2v(y) + \log\det \nabla^2 v(y)\Big]\,d\mu(y).$$

Finally,

$$L\left(\frac{|x|^2}{2}\right) = n - \langle x, \nabla v(x)\rangle,$$

so integration by parts gives $\int \langle x, \nabla v(x)\rangle\,d\mu = n$. Consequently,

$$n\log(2\pi e) \ge 2\int v\,d\mu + \int \log\det \nabla^2 v\,d\mu,$$

which rearranges to the desired inequality. □

Remark 4.20 (On the reverse Log-Sobolev inequality)**.** *For presentation, we used a smooth potential, a nonsingular gradient map and sufficient decay. These assumptions can be removed by regularizing convex potentials and using the appropriate change-of-variables and integration-by-parts statements for convex functions. A direct treatment of the general case, including equality, is given by Caglar, Fradelizi, Guédon, Lehec, Schütt and Werner [45].*

Evenness is not needed for the reverse Log-Sobolev inequality itself. For a general log-concave probability density, first translate its barycenter to the origin and use the centered functional Blaschke–Santaló inequality [133] in the same argument. Both the entropy and the integrated log-determinant of the Hessian are invariant under translation. This is the version used in the moment-measure proof of Fathi's inequality in the homework; see also [65, Theorem 2.2].

Remark 4.21 (Entropy Power Inequality)**.** *For a random vector X with density f, its differential entropy is*

$$h(X) = -\int f \log f \, dx = -\operatorname{Ent}(f).$$

Let X, Y be independent random vectors with densities and finite differential entropies. Let X', Y' be independent centered Gaussian vectors, each with covariance a scalar multiple of the identity, chosen so that $h(X) = h(X')$ and $h(Y) = h(Y')$. The Entropy Power Inequality is equivalent to

$$h(X + Y) \geq h(X' + Y').$$

Since Gaussians give equality in both Log-Sobolev and reverse Log-Sobolev inequalities, one may ask whether these inequalities, taken together, yield the Entropy Power Inequality. Such a proof would give an interesting connection to information theory.

Remark 4.22. *Suppose F, G are 2-homogeneous convex functions on $\mathbb{R}^n$, meaning $F(tx) = t^2 F(x)$ for $t \geq 0$, and similarly for G. Euler's identity gives*

$$2F(x) = \langle \nabla F(x), x \rangle.$$

Together with proposition 3.2, this yields $F^*(\nabla F) = F$. It leads to the following inequality, whose details are left to the homework:

$$\int e^{-(F+G)/2} \det\left(\frac{\nabla^2 F + \nabla^2 G}{2}\right) dx \geq \left(\int e^{-F} \det \nabla^2 F \, dx\right)^{1/2} \left(\int e^{-G} \det \nabla^2 G \, dx\right)^{1/2}.$$

Indeed, the same change of variables as above identifies $\int e^{-F} \det \nabla^2 F$ with $\int e^{-F^*}$, and similarly for G and $(F + G)/2$. The conclusion is then the concavity of $F \mapsto \log \int e^{-F^*}$, which follows from Prékopa–Leindler. As before, start with smooth, nondegenerate potentials away from the origin and use approximation if needed.

4.18 Fathi's symmetrized Transport-Entropy Inequality

Let μ, ν be probability measures with $\mu \ll \nu$, and write $d\mu = f \, d\nu$. Their relative entropy is

$$\operatorname{Ent}_\nu(\mu) = \int f \log f \, d\nu.$$

We set it equal to $+\infty$ when μ is not absolutely continuous with respect to ν. For a Lebesgue density p we also write $\operatorname{Ent}_{dx}(\mu) = \int p \log p \, dx$; unlike relative entropy between probability measures, this quantity can be negative.

Remark 4.23. *Suppose μ is a probability measure with*

$$d\mu = e^{-v} \, d\gamma = e^{-v - |x|^2/2 - (n/2)\log(2\pi)} \, dx.$$

Then

$$\begin{aligned}\mathrm{Ent}_\gamma(\mu) &= -\int v\,d\mu,\\ \mathrm{Ent}_{dx}(\mu) &= -\int \left(v + \frac{|x|^2}{2} + \frac{n}{2}\log(2\pi)\right) d\mu\\ &= \mathrm{Ent}_\gamma(\mu) - \frac{1}{2}\int |x|^2\,d\mu - \frac{n}{2}\log(2\pi).\end{aligned}$$

Remark 4.24. *For probability measures, Jensen's inequality gives* $\mathrm{Ent}_\nu(\mu) \geq 0$*, with equality exactly when* $\mu = \nu$*. In particular,* $\mathrm{Ent}_\mu(\mu) = 0$*. Relative entropy, also called Kullback–Leibler divergence, measures a discrepancy between measures, but it is not a metric: it is generally asymmetric and does not satisfy the triangle inequality.*

Definition 4.9. Let μ, ν be probability measures on $\mathbb{R}^n$ with finite second moments. Recall that $T_\#\mu = \nu$ means that

$$\int \phi(y)\,d\nu(y) = \int \phi(Tx)\,d\mu(x)$$

for all bounded measurable test functions ϕ. When μ is absolutely continuous, the Wasserstein 2-distance can be written as

$$W_2(\mu,\nu) = \left(\inf_{T:T_\#\mu=\nu} \int |x - Tx|^2\,d\mu(x)\right)^{1/2}.$$

Brenier's theorem, theorem 4.11, says that the infimum is attained by $T = \nabla F$ for a convex function F.

Remark 4.25. *For arbitrary probability measures with finite second moments, use* couplings*: a coupling* π *of* μ, ν *is a probability measure on* $\mathbb{R}^n \times \mathbb{R}^n$ *with these marginals. Denote their set by* $\Pi(\mu,\nu)$*. Thus*

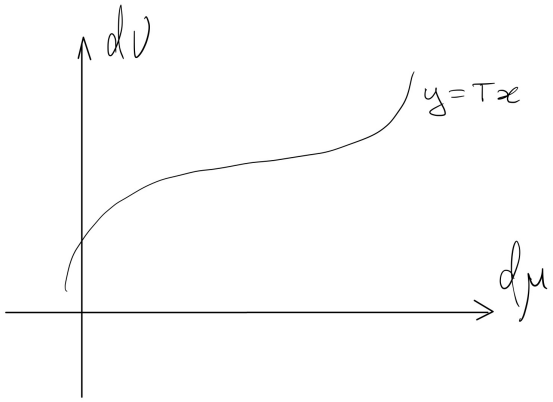
$$\int \phi(x)\,d\pi(x,y) = \int \phi\,d\mu, \qquad \int \psi(y)\,d\pi(x,y) = \int \psi\,d\nu.$$

If π *happens to have a density* $F(x,y)$ *and* $d\mu = f\,dx$*,* $d\nu = g\,dy$*, these conditions become* $\int F(x,y)\,dy = f(x)$ *and* $\int F(x,y)\,dx = g(y)$*. Couplings need not have densities. In particular, the optimal coupling associated with a map is concentrated on its graph.*

The general definition is

$$W_2(\mu,\nu)^2 = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\pi(x,y).$$

When μ *is absolutely continuous, an optimal coupling is* $(\mathrm{Id}, T)_\#\mu$*, where* T *is the Brenier map. This agrees with the preceding definition.*



We now state the main result of this subsection, which we shall derive from the functional Blaschke–Santaló inequality.

Theorem 4.18 (Fathi [65])**.** *For two even probability measures μ, ν on $\mathbb{R}^n$ with finite second moments,*

$$W_2(\mu,\nu)^2 \le 2\,\mathrm{Ent}_\gamma(\mu) + 2\,\mathrm{Ent}_\gamma(\nu).$$

Remark 4.26. *In fact, Fathi's theorem only requires one of the two measures to have barycenter zero; the other need not be centered or even. Our proof below treats the even case, using the even functional Blaschke–Santaló inequality.*

Fathi's full theorem yields the following important result, which was originally proved much earlier by different means:

Corollary 4.5 (Talagrand [171])**.** *For every probability measure μ with finite second moment,*

$$W_2(\mu,\gamma)^2 \le 2\,\mathrm{Ent}_\gamma(\mu).$$

Remark 4.27. *The triangle inequality for W_2 and Talagrand's inequality give*

$$W_2(\mu,\nu)^2 \le 4\,\mathrm{Ent}_\gamma(\mu) + 4\,\mathrm{Ent}_\gamma(\nu).$$

Fathi's theorem improves this factor of 4 to 2 under its centering assumption. This improvement matters in applications to concentration of measure.

As an immediate consequence of Talagrand's inequality and the Gaussian Log-Sobolev inequality, we obtain:

Corollary 4.6 (Talagrand and Gaussian Log-Sobolev)**.** *For a probability measure $d\mu = e^{-v}\,d\gamma$,*

$$W_2(\mu,\gamma)^2 \le \int |\nabla v|^2\,d\mu,$$

whenever the right-hand side is defined. Indeed, apply Gaussian Log-Sobolev to $e^{-v/2}$.

To prove Fathi's result, we need a few lemmas. First, observe:

Lemma 4.3. *Fathi's inequality is equivalent to*

$$\inf_{\pi\in\Pi(\mu,\nu)} \int -\langle x, y\rangle \, d\pi(x, y) \le \mathrm{Ent}_{dx}(\mu) + \mathrm{Ent}_{dx}(\nu) + n \log(2\pi).$$

Proof. Expand $|x - y|^2 = |x|^2 + |y|^2 - 2\langle x, y\rangle$ in the definition of W_2^2. The first two integrals depend only on the marginals, so

$$W_2(\mu, \nu)^2 = \int |x|^2 \, d\mu + \int |y|^2 \, d\nu + 2 \inf_{\pi\in\Pi(\mu,\nu)} \int -\langle x, y\rangle \, d\pi.$$

The conclusion follows from remark 4.23. □

For the remaining calculations, we first work with smooth, positive densities and finite moments and entropies, so that all expressions make sense. Standard truncation and regularization, followed by lower semicontinuity of entropy and transport cost, extend the inequality to the stated setting. Measures with infinite relative entropy give a trivial right-hand side.

We use the following variational formula for entropy; see also Artstein-Avidan, Giannopoulos and Milman [5].

Lemma 4.4 (Variational formula for entropy)**.** *For an absolutely continuous probability measure μ,*

$$\mathrm{Ent}_{dx}(\mu) = \sup_f \left\{ \int f \, d\mu - \log \int e^f \, dx \right\},$$

where the supremum is taken over measurable f with $0 < \int e^f \, dx < \infty$ for which the expression is defined.

Proof. Write $d\mu = p \, dx$. Given an admissible f, put $q = e^f / \int e^f$. Nonnegativity of relative entropy gives

$$0 \le \int p \log \frac{p}{q} \, dx = \mathrm{Ent}_{dx}(\mu) - \int f \, d\mu + \log \int e^f \, dx.$$

Thus the displayed supremum is at most $\mathrm{Ent}_{dx}(\mu)$. Taking $f = \log p$, with $\log 0 = -\infty$, gives equality whenever this expression is directly admissible; truncation gives the general version.

It is also instructive to see the variational calculation. For a candidate f_0 and a bounded test function ϕ, set

$$\mathcal{F}(\epsilon) = \int (f_0 + \epsilon\phi)p \, dx - \log \int e^{f_0+\epsilon\phi} \, dx.$$

Then

$$\mathcal{F}'(\epsilon) = \int \phi p \, dx - \frac{\int \phi e^{f_0+\epsilon\phi} \, dx}{\int e^{f_0+\epsilon\phi} \, dx}.$$

Stationarity for every ϕ means $p = e^{f_0}/\int e^{f_0}$, so $f_0 = \log p$ up to an additive constant. Moreover,

$$\mathcal{F}''(\epsilon) = -\operatorname{Var}_{q_\epsilon}(\phi) \le 0, \qquad q_\epsilon = \frac{e^{f_0+\epsilon\phi}}{\int e^{f_0+\epsilon\phi}}.$$

This explains why the stationary point is a global maximum. No compactness assertion about a class of bounded functions is needed. In fact, globally bounded f cannot be used alone here, since e^f would not be integrable over $\mathbb{R}^n$. □

Lemma 4.5 (Kantorovich duality for the correlation cost)**.** *For probability measures μ, ν with finite second moments,*

$$\inf_{\pi\in\Pi(\mu,\nu)} \int -\langle x, y\rangle \, d\pi(x, y) = \sup_{f(x)+g(y)\le -\langle x,y\rangle} \left(\int f \, d\mu + \int g \, d\nu \right).$$

The supremum is over admissible integrable potentials.

Proof. For any coupling π and any pair satisfying the constraint,

$$\int f \, d\mu + \int g \, d\nu = \int (f(x) + g(y)) \, d\pi(x, y) \le \int -\langle x, y\rangle \, d\pi(x, y).$$

Taking a supremum and then an infimum gives the inequality from right to left.

To see equality in the regular setting, let $T = \nabla\psi$ be a Brenier map from μ to ν. By Fenchel's inequality, $f = -\psi$ and $g = -\psi^*$ satisfy $f(x) + g(y) \le -\langle x, y\rangle$. On the graph $y = \nabla\psi(x)$, however,

$$\psi(x) + \psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x)\rangle.$$

Thus the preceding inequality is an equality for this pair and the optimal coupling. The general statement is the usual Kantorovich duality theorem, or follows by approximation. □

Proof of theorem 4.18 in the even case. The functional Blaschke–Santaló inequality says that, for even functions f, g satisfying

$$f(x) + g(y) \le -\langle x, y\rangle,$$

one has

$$\left(\int e^f \, dx \right) \left(\int e^g \, dy \right) \le (2\pi)^n.$$

For example, the constraint implies $-g \ge (-f)^*$, so this follows from the functional inequality in Legendre-transform form.

When μ, ν are even, the dual supremum in lemma 4.5 may be restricted to even potentials. Indeed, replace an admissible pair by

$$\bar{f}(x) = \frac{f(x) + f(-x)}{2}, \qquad \bar{g}(y) = \frac{g(y) + g(-y)}{2}.$$

Adding the constraints at (x, y) and $(-x, -y)$ preserves admissibility, and evenness of the measures preserves the two integrals in the objective.

For any such even admissible pair, lemma 4.4 and functional Blaschke–Santaló now give

$$\begin{aligned}\mathrm{Ent}_{dx}(\mu) + \mathrm{Ent}_{dx}(\nu) + n\log(2\pi) &\geq \int f\,d\mu + \int g\,d\nu - \log\int e^f\,dx - \log\int e^g\,dy + n\log(2\pi)\\ &\geq \int f\,d\mu + \int g\,d\nu.\end{aligned}$$

In the regular setting the relevant dual potentials can be taken with finite exponential integrals; approximation supplies the general case. Taking the supremum and using lemma 4.5, we obtain

$$\inf_{\pi\in\Pi(\mu,\nu)} \int -\langle x, y\rangle\,d\pi \leq \mathrm{Ent}_{dx}(\mu) + \mathrm{Ent}_{dx}(\nu) + n\log(2\pi).$$

Finally, lemma 4.3 converts this into the desired transport-entropy inequality. □

Remark 4.28. *Fathi's inequality and the* functional *Blaschke–Santaló inequality are equivalent, with corresponding symmetry or centering assumptions. We proved one implication; the reverse is left as homework.*

4.19 Home work

Question 4.1 (1 point)**.** *Let P be the bounded polytope*

$$P = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leq a_i,\ i = 1, \ldots, N\},$$

where u_i are unit vectors and $a_i > 0$. Show that

$$P^\circ = \mathrm{conv}\left\{\frac{u_1}{a_1}, \ldots, \frac{u_N}{a_N}\right\}.$$

Conclude that $(B_1^n)^\circ = B_\infty^n$.

Question 4.2 (1 point)**.** *Throughout this question, K, L are convex bodies with the origin in their interiors.*

a) Prove that $K^{\circ\circ} = K$.

b) For an invertible linear map T, prove that $(T^\top K)^\circ = T^{-1}K^\circ$. Conclude that the polar of an ellipsoid centered at the origin is an ellipsoid.

c) Prove that $(B_p^n)^\circ = B_q^n$ when $p, q > 1$ and $p^{-1} + q^{-1} = 1$.

d) Prove that $(K \cap L)^\circ = \mathrm{conv}(K^\circ \cup L^\circ)$.

e) For a linear subspace H, prove that $(K|H)^\circ \cap H = K^\circ \cap H$. Here the polar on the left is taken in $\mathbb{R}^n$; equivalently, the polar of $K|H$ within H is $K^\circ \cap H$.

f) Prove that $K \subseteq L$ implies $L^\circ \subseteq K^\circ$.

g) Prove that symmetry of K implies symmetry of K°.

h) For any set A, possibly nonconvex, show that $A^\circ = (\operatorname{conv} A)^\circ$. Also verify directly that A° is always convex.

Question 4.3 (1 point)**.** *Let K be a symmetric convex body. Show that $K = K^\circ$ implies $K = B_2^n$.*

Question 4.4 (1 point)**.** *Show that, for every symmetric convex body K and every $\theta \in \mathbb{S}^{n-1}$,*

$$h_K(\theta)\rho_{K^\circ}(\theta) = 1.$$

Question 4.5 (3 points)**.** *Verify Mahler's conjecture in $\mathbb{R}^2$ for symmetric polygons: show that every symmetric polygon P with nonempty interior satisfies*

$$|P||P^\circ| \geq 8.$$

Question 4.6 (1 point)**.** *Given a bounded Borel set $A \subset \mathbb{R}^n$, a bounded function $\alpha : A \to \mathbb{R}$ and $v \in \mathbb{R}^n \setminus \{0\}$, consider the shadow system*

$$K_t = \overline{\operatorname{conv}}\{x + t\alpha(x)v : x \in A\},$$

and the fixed lifted convex set

$$\widetilde{K} = \overline{\operatorname{conv}}\{x + \alpha(x)e_{n+1} : x \in A\} \subset \mathbb{R}^{n+1}.$$

We identify $\mathbb{R}^n$ with $e_{n+1}^\perp$. Show that, for $u \in e_{n+1}^\perp$,

$$h_{K_t}(u) = h_{\widetilde{K}}(u + t\langle u, v\rangle e_{n+1}).$$

Question 4.7 (2 points)**.** *Prove the symmetric Blaschke–Santaló inequality using shadow systems.*

Hint 1. *Express $|K_t^\circ|$ using Questions 2.4, 4.4 and 4.6. Work with a shadow system of origin-symmetric convex bodies containing the origin in their interiors.*

Hint 2. *After choosing coordinates with $v = e_n$, parametrize the two open hemispheres by*

$$\theta = \frac{(z, \pm 1)}{\sqrt{1 + |z|^2}}, \qquad z \in \mathbb{R}^{n-1}.$$

Use the surface Jacobian and the homogeneity of the support function to obtain

$$|K_t^\circ| = \frac{1}{n} \int_{\mathbb{R}^{n-1}} \left[h_{K_t}(z, 1)^{-n} + h_{K_t}(z, -1)^{-n} \right] dz.$$

Hint 3. *Symmetry makes the two integrals equal. By Question 4.6, the function $(z, t) \mapsto h_{\widetilde{K}}(z, 1, t)$ is convex.*

Hint 4. *Apply the Borell–Brascamp–Lieb inequality from Question 3.9, with exponent $-1/n$ in dimension $n-1$, to conclude that $t \mapsto |K_t^\circ|$ is (-1)-concave: its reciprocal is convex.*

Hint 5. *Realize Steiner symmetrization as the midpoint of a shadow system whose endpoints are K and its reflection. The two endpoint polar volumes agree. Convexity of the reciprocal shows that Steiner symmetrization increases polar volume. Iterate symmetrizations and conclude Blaschke–Santaló.*

This proof is due to Campi and Gronchi.

Question 4.8 (1 point)**.** *a) For a proper extended-real-valued function ϕ on $\mathbb{R}^n$, show that ϕ^* is convex.*

b) If ϕ is proper, lower semicontinuous and convex, show that $\phi^{**} = \phi$.

c) Show that $f \geq g$ implies $f^* \leq g^*$.

d) Find the Legendre transform of $x \mapsto |x_1|$ on $\mathbb{R}^n$.

e) For $p, q > 1$, find the Legendre transform of $\|x\|_p^q/q$. Distinguish the exponent conjugate to p from the one conjugate to q.

f) For a convex body K, show that $(\mathbb{1}_K^\infty)^* = h_K$; equivalently, interpret $(-\log \mathbb{1}_K)^*$.

g) For $a > 0$, express $(a\phi)^*$ in terms of ϕ^*.

h) Let $\phi_a(x) = \phi(ax)$ for $a \neq 0$. Find ϕ_a^*.

i) Show that $(\phi + a)^* = \phi^* - a$ for every $a \in \mathbb{R}$.

j) Show that

$$(f^* + g^*)^*(z) = \inf_{x+y=z} \{f(x) + g(y)\}$$

when f, g are proper lower semicontinuous convex functions and their infimal convolution is proper and lower semicontinuous. For presentation, you may assume that f, g are finite convex functions growing faster than linearly at infinity; in general one must take the lower semicontinuous closure on the right.

k) Fix $\alpha > 1$. If f is differentiable, convex and α-homogeneous, meaning $f(tx) = t^\alpha f(x)$ for $t > 0$, show that

$$f^*(\nabla f(x)) = (\alpha - 1) f(x).$$

Hint: *combine Fenchel equality with Euler's identity $\langle \nabla f(x), x \rangle = \alpha f(x)$, and verify the latter by differentiating in t.*

Question 4.9 (1 point)**.** *Find a short alternative proof of the functional Blaschke–Santaló inequality for unconditional functions by passing from $\mathbb{R}^n$ to the positive orthant and making the substitution*

$$(x_1, \ldots, x_n) = (e^{t_1}, \ldots, e^{t_n})$$

in the Prékopa–Leindler inequality. Compare with Question 3.10.

Question 4.10 (1 point). *Show that the Santaló point of a convex body exists and is unique.*

Question 4.11 (4 points). *Find precise statements and proofs of the Blaschke–Santaló inequality for nonsymmetric convex bodies and of its functional version for functions which are not even, including the required translations or centering conditions, as discussed in class.*

Question 4.12 (3 points). *a) Show that the symmetric planar Blaschke–Santaló inequality is equivalent to the following: for every π-periodic C^2 function h with $h > 0$ and $h + \ddot{h} \geq 0$,*

$$\mathcal{F}(h) := \left(\int_{-\pi}^{\pi} h^{-2}\, dt \right) \left(\int_{-\pi}^{\pi} (h^2 - \dot{h}^2)\, dt \right) \leq 4\pi^2.$$

The period π expresses central symmetry of the body. Equivalently, integrating both factors over $[0, \pi]$ gives the bound π^2.

Hint: *use Questions 2.4 and 4.4 to obtain*

$$|K^\circ| = \frac{1}{2} \int_{-\pi}^{\pi} h^{-2}\, dt,$$

and use the area formula from Question 3.23, derived using Question 3.18, for $|K|$.

b) Observe that equality is attained when h is the support function of an ellipse centered at the origin.

c) Find a way to prove this inequality directly.

Option 1: *perhaps use basic harmonic analysis? I would love to see an argument of this kind if it works!*

Option 2: *try a variational approach. Suppose h maximizes $\mathcal{F}$. First work formally in the smooth case with $h > 0$ and $h + \ddot{h} > 0$, so that $h + \epsilon\psi$ remains admissible for sufficiently small signed ϵ and every smooth π-periodic ψ. Use*

$$\frac{d}{d\epsilon} \mathcal{F}(h + \epsilon\psi) \bigg|_{\epsilon=0} = 0$$

to obtain an ODE for h, and identify its positive solutions as support functions of centered ellipses. Justifying the reduction to a smooth, strictly convex maximizer is a separate and potentially difficult step; pursue the calculation even if that step is not yet clear.

Option 3: *try whatever you like! :)*

Question 4.13 (1 point). *Let $K \subset \mathbb{R}^n$ be a convex body with C^2 boundary, $0 \in \operatorname{int} K$, and positive definite second fundamental form. For $x \in \partial K$, put*

$$x^* = \nabla \|x\|_K = \frac{n_x}{\langle x, n_x \rangle} \in \partial K^\circ.$$

Writing $\kappa_K(x)$ for Gauss curvature, show that the centro-affine curvatures *are reciprocal:*

$$\frac{\kappa_K(x)}{\langle x, n_x\rangle^{n+1}} \frac{\kappa_{K^\circ}(x^*)}{\langle x^*, n_{x^*}\rangle^{n+1}} = 1.$$

Equivalently,

$$\kappa_K(x)\kappa_{K^\circ}(x^*) = (|x||x^*|)^{-(n+1)}.$$

Hint: *apply Legendre duality to $F(x) = \frac{1}{2}\|x\|_K^2$ and $F^*(y) = \frac{1}{2}\|y\|_{K^\circ}^2$. Away from the origin these functions have positive definite Hessians, and $\nabla^2 F^*(x^*) = (\nabla^2 F(x))^{-1}$. At a point $x \in \partial K$, verify that*

$$\det \nabla^2 F(x) = \frac{\kappa_K(x)}{\langle x, n_x\rangle^{n+1}}.$$

For this calculation, use a basis consisting of an orthonormal tangent frame followed by x, and Euler's identity $\nabla^2 F(x)x = \nabla F(x)$. Finally, observe that $\langle x, n_x\rangle = |x^*|^{-1}$ and $\langle x^*, n_{x^*}\rangle = |x|^{-1}$.

Question 4.14 (5 points)**.** *a) Find a nonsymmetric convex body whose Santaló point and center of mass do not coincide.*

b) How far apart can they be?

c) If $d(K)$ is their distance, how large can $d(K)/\operatorname{diam}(K)$ be for a convex body in $\mathbb{R}^n$?

Question 4.15 (1 point)**.** *Let H be a Hanner polytope, as defined inductively in class. Show that*

$$|H||H^\circ| = \frac{4^n}{n!}.$$

Question 4.16 (2 points, Saint-Raymond's theorem via Meyer's proof)**.** *Prove the symmetric Mahler conjecture for an unconditional convex body K.*

Hint 1: *the result is true in dimension 1; proceed by induction.*

Hint 2: *write $K_+ = K \cap \mathbb{R}^n_+$. For $x \in K_+$, consider the n cones*

$$K_i = \operatorname{conv}\Big(\{x\} \cup (K_+ \cap e_i^\perp)\Big).$$

Their interiors are disjoint and they lie in K_+, so

$$|K| \geq 2^n \sum_{i=1}^n |K_i|.$$

Use Question 1.3 and unconditionality to deduce that the vector

$$\left(\frac{2|K \cap e_1^\perp|_{n-1}}{n|K|}, \ldots, \frac{2|K \cap e_n^\perp|_{n-1}}{n|K|}\right)$$

belongs to K°.

Hint 3: *apply the same argument to K°. Use polarity and the identity $K \cap e_i^\perp = K|e_i^\perp$, which holds here by unconditionality, to obtain*

$$|K||K^\circ| \geq \frac{4}{n^2} \sum_{i=1}^{n} |K \cap e_i^\perp|_{n-1} \, |(K \cap e_i^\perp)^{\circ_{e_i^\perp}}|_{n-1}.$$

The polar in the last expression is taken within $e_i^\perp$. Finish by induction.

Question 4.17 (5 points)**.** *Iriyeh and Shibata's proof of the symmetric Mahler conjecture in $\mathbb{R}^3$ uses a refinement of the idea in Question 4.16. A symmetric convex body can be put by a linear transformation into a position where the coordinate planes divide it into eight regions of equal volume, and divide each of its three coordinate-plane sections into four regions of equal area.*

a) Assuming this equipartition, prove $|K||K^\circ| \geq 32/3$. The ingredient beyond Meyer's unconditional argument is the following vector-area estimate. For a boundary patch $A \subset \partial K$, oriented by the outer normal, let

$$V(A) = \int_A n_K \, d\mathcal{H}^2, \qquad C(A) = \{tx : x \in A, \ 0 \leq t \leq 1\}.$$

When $|C(A)| > 0$, show that $V(A)/(3|C(A)|) \in K^\circ$. Apply this also to patches of ∂K° to get

$$\langle V(A), V(B) \rangle \leq 9|C(A)||C(B)|, \qquad B \subset \partial K^\circ.$$

Let $A_\omega = \partial K \cap \{x : \omega_j x_j \geq 0, \ j = 1,2,3\}$, where $\omega \in \{-1,1\}^3$. Use the polar correspondence $\Phi(x) = n_x / \langle x, n_x \rangle$ to pair A_ω with $B_\omega = \Phi(A_\omega) \subset \partial K^\circ$, and extend Φ homogeneously along rays. The equipartition and the divergence theorem give

$$|C(A_\omega)| = \frac{|K|}{8}, \qquad V(A_\omega) = \frac{1}{4} \sum_{i=1}^{3} \omega_i |K \cap e_i^\perp|_2 e_i.$$

To sum the vector-area estimates, verify the cancellation identity

$$\sum_{\omega} \omega_i \langle V(B_\omega), e_i \rangle = 2|P_i K^\circ|_2.$$

Here P_i is orthogonal projection onto $e_i^\perp$. One way to see the last identity is to cancel the common oriented interfaces of the cones $\Phi(C(A_\omega))$: only the image of $K \cap e_i^\perp$ remains. Its projection covers $P_i K^\circ$ once, with positive orientation. Indeed, $\Phi = \nabla(\| \cdot \|_K^2/2)$, so $P_i \Phi$ on $e_i^\perp$ is the gradient of the squared gauge of $K \cap e_i^\perp$, divided by 2.

Summing now yields

$$|K||K^\circ| \geq \frac{4}{9} \sum_{i=1}^{3} |K \cap e_i^\perp|_2 \, |P_i K^\circ|_2.$$

The set $P_i K^\circ$ is the polar within $e_i^\perp$ of $K \cap e_i^\perp$. Finish with the planar symmetric Mahler inequality.

b) *Prove the challenging equipartition theorem above. An exposition appears in* Equipartitions and Mahler volumes of symmetric convex bodies*, by Fradelizi, Hubard, Meyer, Roldán-Pensado and Zvavitch. See Theorem 2 and Section 3 for the equipartition and vector-area arguments.*

For the calculation in (a), start with smooth bodies having positive definite second fundamental form; approximate a general symmetric convex body by such bodies, put each in equipartition position, and use continuity and linear invariance of the volume product. Equality of a coordinate projection and section is not required here.

Question 4.18 (3 points)**.** *Verify the nonsymmetric Mahler conjecture in dimension* 2*, using polarity about the Santaló point.*

Question 4.19 (3 points)**.** *Using the ideas of Question 4.16, prove the result of Barthe and Fradelizi: if a convex body $K \subset \mathbb{R}^n$ has all the symmetries of a regular simplex centered at the origin, then*

$$|K||K^\circ| \geq |\Delta_n||\Delta_n^\circ| = \frac{(n+1)^{n+1}}{(n!)^2},$$

where Δ_n is a regular simplex centered at the origin. A regular simplex can be scaled so that its polar is its reflection, $\Delta_n^\circ = -\Delta_n$; it is not literally equal to its polar.

Question 4.20 (10 points)**.** *Can the ideas of Question 4.19 be used to prove the nonsymmetric Mahler conjecture in $\mathbb{R}^3$? In the normalization where the Santaló point of K is at the origin, the desired conclusion is*

$$|K||K^\circ| \geq |\Delta_3||\Delta_3^\circ| = \frac{64}{9}.$$

Perhaps one could prove an appropriate nonsymmetric analogue of the equipartition used by Iriyeh and Shibata?

Question 4.21 (2 points)**.** *Prove the following equivalence due to Fradelizi and Meyer: the symmetric Mahler conjecture in every dimension is equivalent to the following functional statement in every dimension. For every even proper lower semicontinuous convex function ϕ on $\mathbb{R}^n$ with $0 < \int e^{-\phi} < \infty$,*

$$\left(\int e^{-\phi}\,dx\right)\left(\int e^{-\phi^*}\,dx\right) \geq 4^n.$$

Hint: *one direction follows by taking $\phi = \|\cdot\|_K$. For the reverse direction, approximate the function by convex bodies in increasing dimensions. The phrase "in every dimension" matters for this equivalence.*

Question 4.22 (2 points)**.** *Prove the following extension of the functional Blaschke–Santaló inequality due to Fradelizi and Meyer. Let $\rho : [0,\infty) \to [0,\infty)$ be measurable, and suppose that f, g are nonnegative even log-concave functions satisfying*

$$f(x)g(y) \leq \rho(\langle x, y\rangle)^2 \qquad \text{whenever } \langle x, y\rangle \geq 0.$$

Then

$$\left(\int f\,dx\right)\left(\int g\,dx\right) \le \left(\int \rho(|x|^2)\,dx\right)^2.$$

Assume the integrals in the conclusion are finite when proving the nontrivial case.

Question 4.23 (5 points)**.** *We discussed the p-Beckner inequality on the circle,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2\,dt - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^p\,dt\right)^{2/p} \le \frac{2-p}{2\pi}\int_{-\pi}^{\pi} \dot{f}^2\,dt,$$

for 2π-periodic functions. It holds for $p \in [1,2)$, and the corresponding inequality also holds for $p = -2$ when f is strictly positive. Is there a whole range of negative p for which it holds? For negative powers, start with smooth strictly positive periodic f. Perhaps an interpolation or limiting argument, as in Question 3.33, would help.

Question 4.24 (2 points)**.** *Show that Talagrand's transport-entropy inequality implies the Gaussian Poincaré inequality.*

Hint: *linearize.*

Question 4.25 (10 points)**.** *Try to make progress on the question discussed in class: for an even log-concave measure μ and a symmetric convex body K, is it true that*

$$\mu(K)\mu(K^\circ) \le \mu(B_2^n)^2?$$

Perhaps you can prove a partial case: a special class of measures, unconditional measures and bodies, dimension 2, and so on.

Question 4.26 (1 point)**.** *Prove the symmetric Gaussian Poincaré inequality*

$$\mathrm{Var}_\gamma(f) \le \frac{1}{2}\mathbb{E}_\gamma|\nabla f|^2$$

for even $f \in W^{1,2}(\gamma)$, using its expansion in Hermite polynomials instead of the linearization of Blaschke–Santaló used in class.

Question 4.27 (1 point)**.** *Show that the functional Blaschke–Santaló inequality and Fathi's inequality are equivalent, with matching symmetry assumptions. In class, we proved only one implication.*

Question 4.28 (2 points)**.** *Prove Saroglou's result.*

a) Show that the Log-Brunn–Minkowski inequality for Lebesgue measure,

$$\left|\frac{K +_0 L}{2}\right| \ge \sqrt{|K||L|},$$

for all symmetric convex bodies $K, L \subset \mathbb{R}^n$, implies the corresponding inequality

$$\mu\left(\frac{K +_0 L}{2}\right) \geq \sqrt{\mu(K)\mu(L)}$$

for every even log-concave measure μ with full support. Conclude that the Log-Brunn–Minkowski conjecture implies the B-conjecture.

Hint: *use the Prékopa–Leindler inequality.*

b) Show the converse implication.

Hint: *consider small dilates near the origin, where a positive continuous density is approximately constant, and use the homogeneity of the Lebesgue version.*

Question 4.29 (2 points)**.** *Confirm that the validity of the B-conjecture for all rotation-invariant log-concave measures is equivalent to the statement that, for every even log-concave measure μ and every $R > 0$,*

$$\mu(RB_2^n)\mu(R^{-1}B_2^n) \leq \mu(B_2^n)^2.$$

Recall that this is a special case of the conjecture in Question 4.25.

Question 4.30 (2 points)**.** *Prove Klartag's extension of the functional Blaschke–Santaló inequality: for every even log-concave measure μ and every even convex function ϕ for which the integrals make sense,*

$$\left(\int e^{-\phi}\, d\mu\right)\left(\int e^{-\phi^*}\, d\mu\right) \leq \left(\int e^{-|x|^2/2}\, d\mu\right)^2.$$

Hint: *apply Caffarelli's contraction theorem to the probability measure obtained by normalizing $e^{-|x|^2/2}\, d\mu$.*

Question 4.31 (10 points)**.** *Attempt to make progress on the "original B-conjecture": for $z \in \mathbb{R}^n$ and a symmetric convex set K with positive Gaussian measure, is*

$$t \longmapsto \frac{\gamma(tK + z)}{\gamma(tK)}$$

nondecreasing for $t \geq 1$? Rescaling K makes the formulation with $t > 0$ equivalent.

Question 4.32 (2 points)**.** *Show that the B-theorem of Cordero-Erausquin, Fradelizi and Maurey follows from an affirmative answer to Question 4.31.*

Hint: *express the conjecture using the derivative at $t = 1$. Equivalently, compare the ratios at two dilation factors $1 \leq s < t$. Each resulting difference, viewed as a function of z, is nonnegative and vanishes at $z = 0$; hence its Hessian there is nonnegative definite. Take its Laplacian at $z = 0$, using*

$$\Delta_z \gamma(z + sK)\big|_{z=0} = \int_{sK} (|x|^2 - n)\, d\gamma(x).$$

Relate the conclusion to $\frac{d}{d \log s} \log \gamma(sK)$.

Question 4.33 (2 points). *Prove Bobkov's equivalence for a symmetric convex body K of volume 1:*

- *The probability measure*
$$d\gamma_K(x) = \frac{e^{-|x|^2/2}\mathbb{1}_K(x)}{(2\pi)^{n/2}\gamma(K)}\,dx$$
has covariance matrix $c\,\mathrm{Id}$ for some $c > 0$.

- *For every volume-preserving linear transformation T, one has $\gamma(K) \geq \gamma(TK)$.*

Hint: *use the matrix form $t \mapsto \log \gamma(e^{tA}K)$ of the B-theorem, with $A = A^\top$ and $\operatorname{tr} A = 0$, together with polar decomposition of T. Recall that a scalar covariance matrix need not be the identity matrix.*

Question 4.34 (3 points). *Prove the optimized Lebesgue Log-Sobolev inequality: if V is convex, $\int e^{-V}\,dx = 1$ and $d\mu = e^{-V}\,dx$, then*

$$-\int V\,d\mu \leq \frac{n}{2}\log\left(\frac{\int \Delta V\,d\mu}{n}\right) - \frac{n}{2}\log(2\pi e).$$

Work first in the smooth case with finite Fisher information, and optimize a dilation in the usual Log-Sobolev inequality.

Question 4.35 (10 points). *Can the reverse Log-Sobolev inequality and/or the generalized Log-Sobolev inequality be used to deduce the following formulation of the Entropy Power Inequality?*

Let X, Y be independent centered random vectors in $\mathbb{R}^n$ with densities and finite differential entropies. Let X', Y' be independent centered Gaussian vectors with scalar covariance matrices, chosen so that $h(X) = h(X')$ and $h(Y) = h(Y')$. Then

$$h(X+Y) \geq h(X'+Y'),$$

where $h(X) = -\int f \log f$ when f is the density of X.

Question 4.36 (2 points). *Find Fathi's original proof of his inequality using the reverse Log-Sobolev inequality and the following fact, arising from the work of Cordero-Erausquin, Klartag and Santambrogio.*

Let μ be a centered probability measure with finite second moment which is not supported on a hyperplane. There is an essentially continuous convex function ϕ, unique up to translations of its argument, such that $d\rho = e^{-\phi}\,dx$ is a probability measure and $(\nabla\phi)_\# \rho = \mu$. Moreover, ρ minimizes

$$\nu \longmapsto -\frac{1}{2}W_2(\mu,\nu)^2 + \mathrm{Ent}_\gamma(\nu)$$

over probability measures ν with finite second moment for which the expression is defined. The minimizer is understood up to translations.

Clarification: *you are not asked to prove this fact. Derive Fathi's inequality from it and the reverse Log-Sobolev inequality. For a first pass, use smooth densities so that the Monge–Ampère change of variables and the integration by parts are justified; the general proof uses their versions for essentially continuous convex potentials.*

Question 4.37 (1 point)**.** *Suppose u, v are 2-homogeneous convex functions on $\mathbb{R}^n$. Prove that*

$$\int e^{-(u+v)/2} \det\left(\frac{\nabla^2 u + \nabla^2 v}{2}\right) dx \geq \left(\int e^{-u} \det \nabla^2 u \, dx\right)^{1/2} \left(\int e^{-v} \det \nabla^2 v \, dx\right)^{1/2}.$$

Work first with positive definite Hessians away from the origin and finite integrals.

Hint: *use $2u = \langle \nabla u, x \rangle$, the change of variables used in the proof of the reverse Log-Sobolev inequality, and the Prékopa–Leindler inequality.*

Question 4.38 (1 point)**.** *Prove the inequality in Question 4.25 when both K and μ are unconditional.*

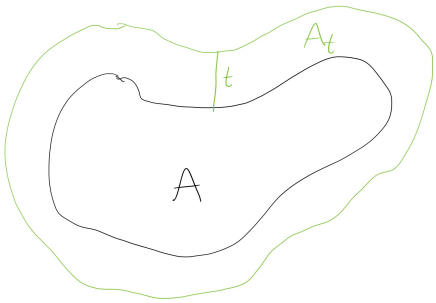
5 Concentration of Measure Phenomenon: the soft approach

We shall now use all the acquired knowledge to study concentration of measure stemming from convexity. Often we will be happy with estimates up to absolute constants. The phenomena we encounter will frequently grow out of the Brunn–Minkowski inequality and its many consequences.

5.1 Introduction to the basic concepts related to concentration of measure

Let μ be a probability measure on a metric space (X, d). For $x \in X$ and a nonempty subset $A \subseteq X$, define

$$d(x, A) := \inf_{y \in A} d(x, y), \qquad A_t := \{x \in X : d(x, A) < t\}, \quad t > 0.$$



We will study lower bounds for

$$\inf_{\mu(A)=\alpha} \mu(A_t \setminus A), \qquad 0 < \alpha < 1.$$

A related question is *isoperimetry*. In Euclidean space, define, as before,

$$\mu^+(\partial A) = \liminf_{\epsilon \downarrow 0} \frac{\mu(A + \epsilon B_2^n) - \mu(A)}{\epsilon}.$$

On a general metric space, use A_ϵ in place of $A + \epsilon B_2^n$. What lower bounds hold for

$$\inf_{\mu(A)=\alpha} \mu^+(\partial A)?$$

Here we prescribe the measure, rather than merely require $\mu(A) \geq \alpha$: the latter class includes X, whose boundary is empty. Infinitesimal enlargement estimates give perimeter bounds, and in many settings perimeter bounds can be integrated to give enlargement estimates.

Example 5.1 (Concentration of measure on the sphere)**.** *Let σ be the uniform probability measure on $\mathbb{S}^{n-1}$, equipped with the geodesic distance $d(x, y) = \angle(x, y)$. For $n \geq 2$, there are absolute constants $c, C > 0$ such that*

$$\sigma(A_t) \geq 1 - Ce^{-cnt^2}, \qquad \sigma(A) \geq \frac{1}{2}.$$

The factor n in the exponent is essential: the concentration scale on the unit sphere is $n^{-1/2}$. A sharper statement follows from spherical isoperimetry; below we give a soft proof with non-sharp constants.

Example 5.2 (Concentration of measure in Gaussian space)**.** *For standard Gaussian measure on $\mathbb{R}^n$ and Euclidean distance,*

$$\gamma(A_t) \geq 1 - \frac{1}{2}e^{-t^2/2}, \qquad \gamma(A) \geq \frac{1}{2}.$$

We will later deduce this from Gaussian isoperimetry.

Example 5.3 (Concentration of measure on the Hamming cube)**.** *Let $Q = \{-1, 1\}^n$, with uniform probability measure $\mu(A) = 2^{-n}\#A$, and normalized Hamming distance*

$$d(x, y) = \frac{1}{n}\#\{i : x_i \neq y_i\}.$$

One has, for example,

$$\mu(A_t) \geq 1 - e^{-nt^2/2}, \qquad \mu(A) \geq \frac{1}{2}.$$

The normalization of the distance matters: replacing d by the unnormalized Hamming distance replaces nt^2 in the exponent by t^2/n. A proof using bounded differences is outlined in the homework.

We now prove a soft version of concentration on the sphere.

Theorem 5.1 (Non-sharp concentration on $\mathbb{S}^{n-1}$)**.** *Let $A \subseteq \mathbb{S}^{n-1}$ be Borel measurable, with $\sigma(A) = 1/2$, and let $0 < t \leq \pi/2$. Then*

$$\sigma(A_t) \geq 1 - c_1 e^{-c_2 n t^2},$$

where $c_1, c_2 > 0$ are absolute constants.

The proof requires the following lemma.

Lemma 5.1. *Let μ be normalized Lebesgue measure on B_2^n. For nonempty Borel sets $A, B \subseteq B_2^n$, set*

$$\rho = \operatorname{dist}(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

If $\mu(A), \mu(B) \geq \alpha > 0$, then

$$\alpha \leq e^{-n\rho^2/8}.$$

Thus, if two subsets of the unit ball are far apart, at least one has small measure.

Proof. By Brunn–Minkowski,

$$\mu\left(\frac{A+B}{2}\right)^{1/n} \geq \frac{1}{2}\mu(A)^{1/n} + \frac{1}{2}\mu(B)^{1/n} \geq \alpha^{1/n}.$$

For $a \in A, b \in B$, the parallelogram identity gives

$$|a + b|^2 = 2|a|^2 + 2|b|^2 - |a - b|^2 \leq 4 - \rho^2.$$

Consequently,

$$\frac{A+B}{2} \subseteq \sqrt{1 - \rho^2/4}\, B_2^n,$$

and therefore

$$\alpha \leq \mu\left(\frac{A+B}{2}\right) \leq (1 - \rho^2/4)^{n/2} \leq e^{-n\rho^2/8}.$$

□

Proof of Theorem 5.1. Let $B = \mathbb{S}^{n-1} \setminus A_t$. If $\sigma(B) = 0$, the conclusion is immediate. Otherwise choose $0 < \lambda < 1$ and consider

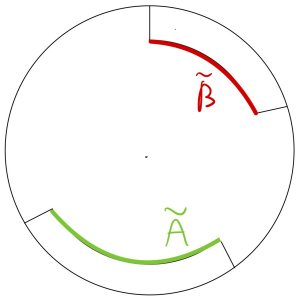
$$\widetilde{A} = \{r\theta : \theta \in A,\ \lambda \le r \le 1\}, \qquad \widetilde{B} = \{s\eta : \eta \in B,\ \lambda \le s \le 1\}.$$

For $\theta \in A, \eta \in B$, their angle is at least t, so

$$|r\theta - s\eta|^2 = (r-s)^2 + 2rs(1 - \cos\angle(\theta,\eta)) \ge 4\lambda^2 \sin^2(t/2).$$

Thus $\operatorname{dist}(\widetilde{A}, \widetilde{B}) \ge 2\lambda \sin(t/2) \ge 2\lambda t/\pi$. By polar coordinates,

$$\mu(\widetilde{A}) = \frac{1-\lambda^n}{2}, \qquad \mu(\widetilde{B}) = (1-\lambda^n)\sigma(B) \le \frac{1-\lambda^n}{2}.$$



The smaller measure in Lemma 5.1 is therefore $\mu(\widetilde{B})$, and

$$1 - \sigma(A_t) = \sigma(B) \le \frac{1}{1-\lambda^n} \exp\left(-\frac{n\lambda^2 t^2}{2\pi^2}\right).$$

Take $\lambda = 1/2$. We may choose $c_1 = 2$ and $c_2 = 1/(8\pi^2)$. □

Proposition 5.1. *Let $A \subseteq \mathbb{R}^n$ be Borel measurable with $\gamma(A) > 0$. Then*

$$\int_{\mathbb{R}^n} e^{d(x,A)^2/4}\, d\gamma(x) \le \frac{1}{\gamma(A)}.$$

Proof. We use Prékopa–Leindler. Write

$$\varphi(x) = (2\pi)^{-n/2} e^{-|x|^2/2}, \quad f(x) = e^{d(x,A)^2/4}\varphi(x), \quad g(x) = \mathbb{1}_A(x)\varphi(x), \quad h(x) = \varphi(x).$$

If $y \notin A$, then $\sqrt{f(x)g(y)} = 0$. If $y \in A$, then $d(x,A) \le |x-y|$, and

$$\frac{d(x,A)^2}{4} - \frac{|x|^2}{2} - \frac{|y|^2}{2} \le -\frac{|x+y|^2}{4}.$$

It follows that

$$f(x)g(y) \le (2\pi)^{-n} e^{-|x+y|^2/4} = h\left(\frac{x+y}{2}\right)^2.$$

Prékopa–Leindler now gives

$$1 = \int h \ge \sqrt{\int f \int g} = \sqrt{\gamma(A) \int e^{d(x,A)^2/4}\, d\gamma(x)},$$

which proves the claim. If the integrability of f has not yet been established, apply the argument to $\min\{f, N\}\mathbb{1}_{NB_2^n}$ and pass to the limit. □

Theorem 5.2 (A non-sharp Gaussian concentration bound)**.** *If A is Borel measurable and $\gamma(A) \ge 1/2$, then*

$$\gamma(A_t) \ge 1 - 2e^{-t^2/4}, \qquad t > 0.$$

Proof. Split the integral in Proposition 5.1 into $\{d(x, A) < t\}$ and its complement. On the complement, $e^{d(x,A)^2/4} \ge e^{t^2/4}$. Hence

$$e^{t^2/4}\gamma(A_t^c) \le \int e^{d(x,A)^2/4}\, d\gamma(x) \le \frac{1}{\gamma(A)} \le 2.$$

Rearranging proves the theorem. □

5.2 Lévy's concentration function and types of estimates

Recall that $A_t = \{x \in X : d(x, A) < t\}$. We have been seeking estimates of the form $\mu(A_t) \ge 1 - g(t)$ for sets with $\mu(A) \ge 1/2$. The following function encodes the best such estimate.

Definition 5.1 (Lévy's concentration function)**.** For a metric probability space (X, d, μ), define

$$\alpha_\mu(t) := \sup_{\mu(A)\ge 1/2} \Big(1 - \mu(A_t)\Big) = 1 - \inf_{\mu(A)\ge 1/2} \mu(A_t), \qquad t > 0,$$

where A ranges over Borel sets.

Thus α_μ is the smallest function for which $\mu(A_t) \ge 1 - \alpha_\mu(t)$ holds for every such A. It is nonincreasing, and $\alpha_\mu(t) \le 1/2$.

Claim 5.1 (Homework)**.** *For every metric probability space, $\alpha_\mu(t) \to 0$ as $t \to \infty$.*

The pointwise assertion $A_t \uparrow X$ is not by itself enough, since we take a supremum over A. To obtain a uniform estimate, fix $x_0 \in X$ and choose R with $\mu(B(x_0, R)) > 1/2$. Every eligible A meets this ball, so, for $t > R$,

$$B(x_0, t - R) \subseteq A_t, \qquad \alpha_\mu(t) \le 1 - \mu(B(x_0, t - R)) \longrightarrow 0.$$

There are two types of concentration that are especially useful.

Definition 5.2. A metric probability space has *normal*, or *sub-Gaussian*, concentration if

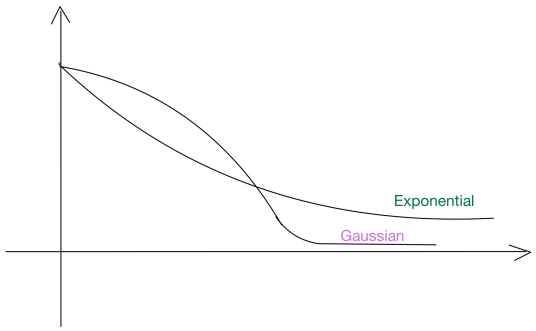
$$\alpha_\mu(t) \le Ce^{-ct^2}.$$

The constants may depend on the space and its normalization.

Gaussian measure, uniform measure on the sphere, and uniform measure on the Hamming cube satisfy such estimates, with the dimension dependence described above.

Definition 5.3. A metric probability space has *exponential concentration* if

$$\alpha_\mu(t) \le Ce^{-ct}.$$



Below we shall see further examples of both phenomena.

5.3 Concentration of measure for Lipschitz functions

Recall that $f : X \to \mathbb{R}$ is L-Lipschitz if $|f(x)-f(y)| \le Ld(x,y)$ for every x, y. For $f \in C^1(\mathbb{R}^n)$, this is equivalent to $|\nabla f| \le L$ everywhere.

Definition 5.4. A *median* of a measurable function $f : X \to \mathbb{R}$ is a number m, denoted $\mathrm{med}(f)$, such that

$$\mu(f \ge m) \ge \frac{1}{2}, \qquad \mu(f \le m) \ge \frac{1}{2}.$$

A median need not be unique; any choice satisfying these inequalities may be used below.

The next result explains why concentration means that Lipschitz functions are almost constant.

Theorem 5.3. *If f is 1-Lipschitz on a metric probability space, then, for $t > 0$,*

$$\mu\Big(|f - \mathrm{med}(f)| \ge t\Big) \le 2\alpha_\mu(t).$$

Proof. Put $m = \mathrm{med}(f)$, $A = \{f \geq m\}$ and $B = \{f \leq m\}$. If $y \in A_t$, choose $x \in A$ with $d(x, y) < t$. Then

$$f(y) \geq f(x) - d(x, y) > m - t.$$

Likewise, $y \in B_t$ implies $f(y) < m + t$. Consequently,

$$\{|f - m| \geq t\} \subseteq A_t^c \cup B_t^c.$$

Both A and B have measure at least $1/2$, so the union bound proves the result. □

Corollary 5.1. *For an L-Lipschitz function with $L > 0$,*

$$\mu\Big(|f - \mathrm{med}(f)| \geq t\Big) \leq 2\alpha_\mu(t/L).$$

Proof. Apply the theorem to f/L. □

Conversely, concentration of all Lipschitz functions controls concentration of sets.

Proposition 5.2. *Fix $t > 0$. Suppose that for every 1-Lipschitz function f and every choice of its median,*

$$\mu\Big(|f - \mathrm{med}(f)| \geq t\Big) \leq \eta.$$

Then $\alpha_\mu(t) \leq \eta$.

Proof. For $\mu(A) \geq 1/2$, the function $f(x) = d(x, A)$ is 1-Lipschitz and has median 0. The assumed inequality gives

$$1 - \mu(A_t) = \mu(d(x, A) \geq t) \leq \eta.$$

Take the supremum over A. □

Thus the correspondence is sharp up to the factor 2.

Corollary 5.2. *There are absolute constants $c, C > 0$ such that*

1. *if $f : \mathbb{S}^{n-1} \to \mathbb{R}$ is 1-Lipschitz, then $\sigma(|f - \mathrm{med}(f)| \geq t) \leq Ce^{-cnt^2}$;*
2. *if $f : \mathbb{R}^n \to \mathbb{R}$ is 1-Lipschitz, then $\gamma(|f - \mathrm{med}(f)| \geq t) \leq Ce^{-ct^2}$.*

The mean is often easier to compute than a median. We next transfer the estimate to deviations from the mean.

Theorem 5.4. *Let $f : \mathbb{S}^{n-1} \to \mathbb{R}$ have Lipschitz constant at most $b > 0$. Then*

$$\sigma\Big(|f - \mathbb{E}_\sigma f| \geq bt\Big) \leq 4e^{-cnt^2}, \qquad t > 0,$$

where $c > 0$ is absolute.

Proof. By rescaling, take $b = 1$. Let θ, η be independent and uniform on the sphere. By the preceding median estimate and the triangle inequality, after changing the absolute constant,

$$\mathbb{P}\Big(|f(\theta) - f(\eta)| \geq t\Big) \leq 4e^{-ant^2}$$

for some $a > 0$. For $0 < \Lambda^2 < an$, the layer-cake formula yields

$$\begin{aligned}\mathbb{E}e^{\Lambda^2|f(\theta)-f(\eta)|^2} &= 1 + 2\Lambda^2 \int_0^\infty te^{\Lambda^2 t^2}\mathbb{P}\Big(|f(\theta) - f(\eta)| > t\Big)\, dt \\ &\leq 1 + 8\Lambda^2 \int_0^\infty te^{-(an-\Lambda^2)t^2}\, dt = 1 + \frac{4\Lambda^2}{an - \Lambda^2}.\end{aligned}$$

Choose $\Lambda^2 = an/4$. Jensen's inequality (Theorem 2.3) in η gives

$$\mathbb{E}_\sigma e^{(an/4)|f-\mathbb{E}_\sigma f|^2} \leq 4. \tag{24}$$

Markov's inequality now implies

$$\sigma(|f - \mathbb{E}_\sigma f| \geq t) \leq e^{-ant^2/4}\mathbb{E}_\sigma e^{(an/4)|f-\mathbb{E}_\sigma f|^2} \leq 4e^{-ant^2/4}.$$

□

5.4 Log-concave measures, Borell's lemma and the reverse Hölder inequality

Lemma 5.2 (Borell's lemma)**.** *Let μ be a log-concave probability measure and let $A \subset \mathbb{R}^n$ be symmetric and convex, with $\mu(A) = \alpha \in [1/2, 1)$. For $t > 1$,*

$$1 - \mu(tA) \leq \alpha\left(\frac{1-\alpha}{\alpha}\right)^{(t+1)/2}.$$

For $\alpha > 1/2$ this is an exponential bound in t. At $\alpha = 1/2$, the displayed inequality remains valid but gives no decay.

Proof. We claim that

$$\frac{2}{t+1}(tA)^c + \frac{t-1}{t+1}A \subseteq A^c.$$

Indeed, if $x \notin tA$, $y \in A$, and $z = (2x + (t-1)y)/(t+1)$ were in A, then

$$\frac{x}{t} = \frac{t+1}{2t}z + \frac{t-1}{2t}(-y) \in A$$

by symmetry and convexity, a contradiction. Log-concavity therefore gives

$$1 - \alpha \geq (1 - \mu(tA))^{2/(t+1)}\alpha^{(t-1)/(t+1)}.$$

Rearrange to obtain the claim. □

Theorem 5.5 (Reverse Hölder inequality for seminorms of log-concave measures)**.** *Let μ be a nondegenerate log-concave probability measure on $\mathbb{R}^n$, and let f be a seminorm, that is, a finite even, convex, positively 1-homogeneous function. For $q \geq p \geq 1$,*

$$\left(\int f^p \, d\mu \right)^{1/p} \leq \left(\int f^q \, d\mu \right)^{1/q} \leq \frac{Cq}{p} \left(\int f^p \, d\mu \right)^{1/p},$$

where $C > 0$ is absolute.

Proof. The first inequality is Hölder's inequality. For the other direction, set

$$A = \{x : f(x) \leq 3\|f\|_p\}, \qquad \|f\|_p = \left(\int f^p \, d\mu \right)^{1/p}.$$

The case $f = 0$ is immediate. Markov's inequality gives

$$\alpha := \mu(A) \geq 1 - 3^{-p}.$$

In particular, $(1-\alpha)/\alpha \leq e^{-cp}$ for an absolute $c > 0$. By homogeneity, $tA = \{f \leq 3t\|f\|_p\}$, so Borell's lemma gives

$$\mu(f > 3t\|f\|_p) \leq e^{-c_1 pt}, \qquad t \geq 1.$$

The layer-cake formula now yields

$$\begin{aligned} \int f^q \, d\mu &= \int_0^\infty q s^{q-1} \mu(f > s) \, ds \\ &\leq (3\|f\|_p)^q \left(1 + q \int_1^\infty t^{q-1} e^{-c_1 pt} \, dt \right) \\ &\leq (3\|f\|_p)^q \left(1 + \frac{\Gamma(q+1)}{(c_1 p)^q} \right) \leq \left(\frac{Cq}{p} \right)^q \|f\|_p^q. \end{aligned}$$

Here we used

$$\int_0^\infty t^{q-1} e^{-c_1 pt} \, dt = (c_1 p)^{-q} \Gamma(q)$$

and $\Gamma(q+1)^{1/q} \leq Cq$. Taking qth roots completes the proof. □

Here is an application.

Claim 5.2. *Let X be an isotropic log-concave random vector, so $\mathbb{E}X = 0$ and $\mathbb{E}X_i X_j = \delta_{ij}$. Then, for $t \geq 1$,*

$$\mathbb{P}(|X| \geq Ct\sqrt{n}) \leq e^{-c_1 t},$$

with absolute constants.

Proof. Chernoff's trick gives, for $\Lambda > 0$,

$$\mathbb{P}(|X| \geq Ct\sqrt{n}) \leq e^{-Ct\Lambda\sqrt{n}}\mathbb{E}e^{\Lambda|X|}.$$

By the reverse Hölder inequality,

$$\mathbb{E}e^{\Lambda|X|} = 1 + \sum_{k=1}^{\infty} \frac{\Lambda^k \mathbb{E}|X|^k}{k!} \leq 1 + \sum_{k=1}^{\infty} \frac{(C_0 \Lambda k \mathbb{E}|X|)^k}{k!}.$$

Since $\mathbb{E}|X| \leq \sqrt{\mathbb{E}|X|^2} = \sqrt{n}$, choosing $\Lambda = c_0/\sqrt{n}$ with c_0 small makes the series bounded by an absolute constant. Substitute this in the tail estimate and enlarge C to absorb the prefactor for $t \geq 1$. □

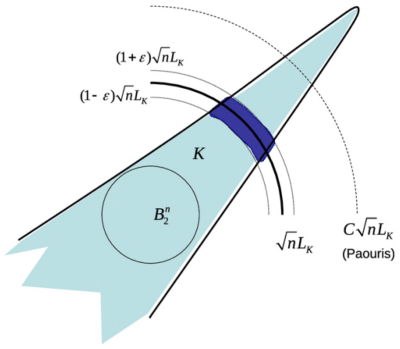
5.5 Paouris's inequality (statement)

The exponent in Claim 5.2 misses an additional factor $\sqrt{n}$.

Theorem 5.6 (Paouris [157])**.** *If X is an isotropic log-concave random vector, then, for $t \geq 1$,*

$$\mathbb{P}(|X| \geq Ct\sqrt{n}) \leq e^{-t\sqrt{n}},$$

where $C > 0$ is absolute.



Picture (c) Klartag, Milman

Guédon and E. Milman [81] later gave another proof and refined moment and deviation estimates.

5.6 Sub-exponential concentration of linear functions for log-concave measures, and of other seminorms

Repeating the preceding moment-generating-function argument gives the following.

Claim 5.3. *Let X have a log-concave distribution, and let f be a nonzero seminorm with $m = \mathbb{E}f(X) > 0$. Then*

$$\mathbb{P}(f(X) \geq m + t) \leq Ce^{-ct/m}, \qquad t > 0.$$

For $f(x) = |\langle x, \theta\rangle|$, the exponential order is sharp.

Corollary 5.3. *Let X have a nondegenerate log-concave distribution. For $\theta \in S^{n-1}$ and $t > 0$,*

$$\mathbb{P}\left(|\langle X, \theta\rangle| \geq t\mathbb{E}|\langle X, \theta\rangle|\right) \leq 2e^{-ct}.$$

Proof. Put $m = \mathbb{E}|\langle X, \theta\rangle|$. Reverse Hölder and the power series for the exponential show that

$$\mathbb{E}\exp\left(\frac{c|\langle X, \theta\rangle|}{m}\right) \leq 1 + \sum_{k=1}^{\infty} \frac{(cCk)^k}{k!} \leq 2$$

when $c > 0$ is small enough. Markov's inequality gives the result. □

To see sharpness of the order, use the normalized density $2^{-n}e^{-\|x\|_1}$ and a coordinate linear functional.

There is also a spherical version. Write a standard Gaussian vector as $G = R\Theta$, with $R = |G|$ independent of the uniform direction Θ. Combining homogeneity, Gaussian radial moments, and reverse Hölder gives:

Corollary 5.4 (Homework)**.** *Let f be a seminorm on $\mathbb{R}^n$, restricted to the sphere. For $q \geq p \geq 1$,*

$$\left(\int_{S^{n-1}} f^q \, d\sigma\right)^{1/q} \leq \frac{Cq}{p}\sqrt{\frac{n+p}{n+q}}\left(\int_{S^{n-1}} f^p \, d\sigma\right)^{1/p}.$$

In particular, if $m = \mathbb{E}_\sigma f > 0$, then

$$\sigma(f \geq m + t) \leq Ce^{-ct/m}.$$

5.7 The Thin Shell Conjecture and the KLS conjecture

Conjecture 5.1 (Anttila, Ball, Perissinaki [3]; Bobkov, Koldobsky [25])**.** *If X is an isotropic log-concave random vector, then*

$$\mathrm{Var}(|X|) \leq C$$

for an absolute constant C.

Remark 5.1. *For a centered probability measure with positive definite covariance matrix Σ, the map $T = \Sigma^{-1/2}$ makes the covariance the identity. If the map is required to preserve volume, take instead*

$$T = (\det \Sigma)^{1/(2n)}\Sigma^{-1/2}, \qquad T\Sigma T^\top = (\det \Sigma)^{1/n}\,\mathrm{Id}\,.$$

Thus a determinant-one map makes the covariance a scalar multiple of the identity, but cannot change its determinant. The associated variational characterization concerns the second moment*:*

$$\inf_{\det T = 1} \mathbb{E}|TX|^2 = n(\det \Sigma)^{1/n}.$$

Indeed, $\mathbb{E}|TX|^2 = \mathrm{tr}(T\Sigma T^\top)$, and the arithmetic–geometric mean inequality applied to its eigenvalues proves the claim. The same assertion with $\mathrm{Var}(|TX|)$ in place of $\mathbb{E}|TX|^2$ is false in general.

Remark 5.2. *The thin-shell conjecture implies Bourgain's slicing conjecture (Conjecture 4.4), as shown by Eldan and Klartag [61].*

Recall the Kannan–Lovász–Simonovits conjecture [92] (1995), stated above as Conjecture 4.5: for an isotropic log-concave probability measure μ,

$$\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 \, d\mu$$

with an absolute C, for every locally Lipschitz f.

Remark 5.3. *KLS implies the thin-shell conjecture: take $f(x) = |x|$, whose gradient has length 1 almost everywhere. Then*

$$\mathrm{Var}_\mu(|x|) \le C \int 1 \, d\mu = C.$$

The following was the logarithmic bound available at the time of the course.

Theorem 5.7 (Klartag, 2023 [107])**.** *For an isotropic log-concave probability measure on $\mathbb{R}^n$,*

$$\mathrm{Var}_\mu(f) \le C \log(en) \int |\nabla f|^2 \, d\mu.$$

Remark 5.4. *Here the Poincaré constant denotes the coefficient multiplying $\int |\nabla f|^2$. The corresponding isoperimetric length scale is of order $\sqrt{\log(en)}$. Confusing these two conventions would incorrectly put $\sqrt{\log n}$ in the displayed variance estimate.*

5.8 Klartag's bound on the thin shell in the unconditional case

We outline a relatively uncomplicated proof, due to Klartag [105], in the case of measures invariant under coordinate reflections. See also [103].

Theorem 5.8 (Klartag, 2008 [105])**.** *If X is an unconditional isotropic log-concave random vector on $\mathbb{R}^n$, then $\mathrm{Var}(|X|) \le C$, with C independent of the dimension.*

Lemma 5.3. *Let μ be the probability measure on $\mathbb{R}^n_+$ obtained by restricting and normalizing an unconditional log-concave measure. For $f \in W^{1,2}(\mu)$,*

$$\mathrm{Var}_\mu(f) \le 4 \int_{\mathbb{R}^n_+} \sum_{i=1}^n x_i^2 |\partial_i f(x)|^2 \, d\mu(x).$$

Proof. Write $d\mu = e^{-\psi(x)} dx$ on $\mathbb{R}^n_+$, with the normalization included in ψ. An unconditional convex potential is nondecreasing in each coordinate on the positive orthant. Thus, for

$$\pi(y) = (y_1^2, \dots, y_n^2),$$

$\psi \circ \pi$ is convex. Define

$$\varphi(y) = \psi(\pi(y)) - \sum_{i=1}^{n} \log(2y_i), \qquad d\nu(y) = e^{-\varphi(y)} dy,$$

on the open positive orthant. The change of variables $x = \pi(y)$ gives $\pi_{\#}\nu = \mu$. Moreover,

$$\nabla^2 \varphi(y) \succeq \operatorname{diag}(y_1^{-2}, \dots, y_n^{-2}), \qquad (\nabla^2 \varphi(y))^{-1} \preceq \operatorname{diag}(y_1^2, \dots, y_n^2).$$

Apply Brascamp–Lieb (Theorem 3.15) to $g = f \circ \pi$:

$$\operatorname{Var}_\nu(g) \le \int \sum_i y_i^2 |\partial_i g(y)|^2 \, d\nu(y). \tag{25}$$

The push-forward identity and the chain rule give

$$\operatorname{Var}_\nu(g) = \operatorname{Var}_\mu(f), \tag{26}$$

and

$$\int \sum_i y_i^2 |\partial_i g|^2 \, d\nu = 4 \int \sum_i y_i^4 |\partial_i f(\pi(y))|^2 \, d\nu(y) = 4 \int \sum_i x_i^2 |\partial_i f(x)|^2 \, d\mu(x). \tag{27}$$

Combine these identities. For presentation, we used a smooth potential and smooth test functions. Restricting away from the coordinate hyperplanes, regularizing the convex potential, and then passing to the limit justifies the general case. □

Proof of Theorem 5.8. The radial functions involved are unconditional, so we may work on the positive orthant. Since $\mathbb{E}|X|^2 = n$,

$$\begin{aligned} \operatorname{Var}(|X|) &\le \mathbb{E}(|X| - \sqrt{n})^2 = \mathbb{E} \frac{(|X|^2 - n)^2}{(|X| + \sqrt{n})^2} \\ &\le \frac{1}{n} \operatorname{Var}(|X|^2) \le \frac{4}{n} \sum_i \mathbb{E}\big[X_i^2 (2X_i)^2\big] = \frac{16}{n} \sum_i \mathbb{E} X_i^4 \le C. \end{aligned}$$

The lemma gives the penultimate inequality. For the last one, reverse Hölder and isotropy give $\mathbb{E}X_i^4 \le C(\mathbb{E}X_i^2)^2 = C$. □

5.9 Exponential concentration via Poincaré inequality

Recall that a probability measure μ satisfies a Poincaré inequality with constant β if

$$\operatorname{Var}_\mu(f) \le \beta \int |\nabla f|^2 \, d\mu$$

for every locally Lipschitz function in the appropriate Sobolev space. Throughout these notes, the Poincaré constant is the coefficient multiplying the squared gradient integral. We have the following examples:

- For Gaussian measure, $\beta = 1$.
- For Gaussian measure and *even* functions, the improved coefficient is $1/2$. This is an inequality on a restricted class of functions; by itself it does not give concentration for arbitrary sets.
- For isotropic log-concave probability measures, the KLS conjecture predicts an absolute constant. The bound available in 2023 from Klartag's work is $\beta \le C \log(en)$.
- By Payne–Weinberger (Theorem 3.16), if μ is uniform on a convex body K, then $\beta \le \operatorname{diam}(K)^2/\pi^2$.
- Other examples include uniform measure on the sphere and discrete product spaces, with their respective gradients or difference operators; see the homework.
- If $d\mu = e^{-V} dx$ and $\nabla^2 V \succeq k \operatorname{Id}$ for $k > 0$, then Brascamp–Lieb (Theorem 3.15) gives $\beta \le k^{-1}$.

Establishing a Poincaré inequality allows us to obtain exponential concentration; see also [5]. Recall that

$$\alpha_\mu(t) = \sup_{\mu(A) \ge 1/2} (1 - \mu(A_t)).$$

Theorem 5.9 (Gromov–Milman [77])**.** *Suppose a metric probability space satisfies the Poincaré inequality with constant $\beta > 0$, with a gradient satisfying the usual chain rule and $|\nabla f| \le 1$ for 1-Lipschitz f. Then*

$$\alpha_\mu(t) \le e^{-ct/\sqrt{\beta}}, \qquad t > 0,$$

for an absolute constant $c > 0$. The proof below gives $c = 1/4$.

We immediately obtain some corollaries.

Corollary 5.5. *If $d\mu = e^{-V} dx$ and $\nabla^2 V \succeq k \operatorname{Id}$, then*

$$\alpha_\mu(t) \le e^{-c\sqrt{k}\, t}.$$

Corollary 5.6. *For an isotropic log-concave probability measure, the 2023 bound on the Poincaré constant gives*

$$\alpha_\mu(t) \le \exp\left(-\frac{ct}{\sqrt{\log(en)}}\right).$$

Remark 5.5. *Payne–Weinberger gives, for uniform measure on a convex body K,*

$$\alpha_\mu(t) \le \exp\left(-\frac{ct}{\operatorname{diam} K}\right).$$

Before proving the theorem, let us relate concentration to deviations from the mean. Earlier we did this for deviations from a median.

Proposition 5.3 (Homework)**.** *Suppose that every bounded* 1*-Lipschitz function satisfies*

$$\mu(f \geq \mathbb{E}_\mu f + t) \leq a(t), \qquad t > 0.$$

Then, for every Borel set A *with* $\mu(A) > 0$*,*

$$1 - \mu(A_t) \leq a(\mu(A)t).$$

In particular, $\alpha_\mu(t) \leq a(t/2)$*.*

Proof. Use the bounded 1-Lipschitz function $f(x) = \min\{d(x, A), t\}$. It vanishes on A, is at most t, and equals t outside A_t. Hence

$$\mathbb{E}_\mu f \leq t(1 - \mu(A)), \qquad A_t^c \subseteq \{f \geq \mathbb{E}_\mu f + t\mu(A)\}.$$

Apply the assumed deviation bound. If $\mu(A) \geq 1/2$, the same inclusion gives $A_t^c \subseteq \{f \geq \mathbb{E}_\mu f + t/2\}$, proving the last assertion without a monotonicity assumption on a. □

Proof of Theorem 5.9. Let f be bounded, 1-Lipschitz, and mean zero. Set

$$S(\lambda) = \int e^{\lambda f} \, d\mu.$$

Apply Poincaré to $g_\lambda = e^{\lambda f/2}$:

$$S(\lambda) - S(\lambda/2)^2 \leq \frac{\beta\lambda^2}{4} \int e^{\lambda f} |\nabla f|^2 \, d\mu \leq \frac{\beta\lambda^2}{4} S(\lambda).$$

Thus, for $0 \leq \lambda < 2/\sqrt{\beta}$,

$$S(\lambda) \leq \frac{S(\lambda/2)^2}{1 - \beta\lambda^2/4}.$$

The square in this recurrence is important. Iterating it gives

$$\log S(\lambda) \leq \sum_{j=0}^{N-1} 2^j \left[-\log\left(1 - \frac{\beta\lambda^2}{4^{j+1}}\right)\right] + 2^N \log S(\lambda/2^N).$$

Since $\mathbb{E}f = 0$ and f is bounded, $\log S(s) = O(s^2)$ as $s \to 0$, so the last term tends to zero. At $\lambda = 1/\sqrt{\beta}$, use $-\log(1 - u) \leq 4u/3$ for $0 \leq u \leq 1/4$ to get

$$\log S(1/\sqrt{\beta}) \leq \frac{4}{3} \sum_{j=0}^{\infty} \frac{2^j}{4^{j+1}} = \frac{2}{3}.$$

Chernoff's trick now gives

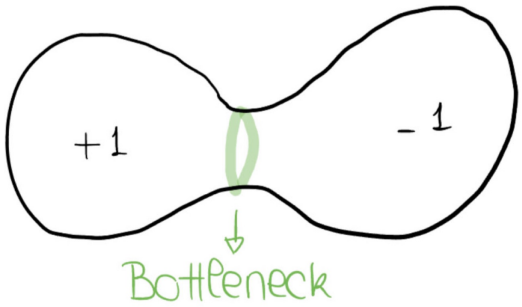
$$\mu(f \geq s) \leq e^{2/3 - s/\sqrt{\beta}}.$$

By Proposition 5.3,

$$\alpha_\mu(t) \le \min\left\{\frac{1}{2}, e^{2/3 - t/(2\sqrt{\beta})}\right\} \le e^{-t/(4\sqrt{\beta})}.$$

For the last step, split into $t/\sqrt{\beta} \le 8/3$ and $t/\sqrt{\beta} \ge 8/3$. All the functions needed to estimate the concentration function can be taken bounded, so no exponential-integrability assumption was required beforehand. □

Remark 5.6. *Disconnected domains need not satisfy a Poincaré inequality. Connecting two large pieces by a narrow neck restores an inequality in many standard examples, but its constant becomes large as the neck becomes thinner. The picture below illustrates this bottleneck phenomenon (courtesy of Klartag).*



Picture (c) Klartag

5.10 Marton's argument: almost-optimal concentration for the Gaussian via Fathi

Recall Talagrand's inequality,

$$W_2(\mu, \gamma)^2 \le 2\,\mathrm{Ent}_\gamma(\mu),$$

and the one-centered version of Fathi's theorem (Theorem 4.18),

$$W_2(\mu, \nu)^2 \le 2\,\mathrm{Ent}_\gamma(\mu) + 2\,\mathrm{Ent}_\gamma(\nu).$$

Here μ, ν are probability measures and at least one has barycenter zero. As a corollary, we obtain the following concentration estimate.

Corollary 5.7 (Fathi)**.** *Let $A \subseteq \mathbb{R}^n$ be Borel measurable with $\gamma(A) > 0$ and Gaussian barycenter zero, meaning*

$$\int_A x\, d\gamma(x) = 0.$$

Then

$$1 - \gamma(A_r) \le \frac{1}{\gamma(A)} e^{-r^2/2}, \qquad r > 0.$$

Proof. The conclusion is immediate if $\gamma(A_r^c) = 0$. Otherwise, let

$$d\mu = \frac{\mathbb{1}_A}{\gamma(A)}\, d\gamma, \qquad d\nu = \frac{\mathbb{1}_{A_r^c}}{\gamma(A_r^c)}\, d\gamma.$$

The first measure is centered, and

$$\mathrm{Ent}_\gamma(\mu) = -\log \gamma(A), \qquad \mathrm{Ent}_\gamma(\nu) = -\log \gamma(A_r^c).$$

Every coupling π of μ, ν is supported on pairs at distance at least r. Consequently,

$$W_2(\mu,\nu)^2 = \inf_\pi \int |x-y|^2\, d\pi(x,y) \geq r^2.$$

Fathi's inequality gives

$$r^2 \leq -2\log \gamma(A) - 2\log \gamma(A_r^c),$$

which is the desired result. □

Remark 5.7. *This is not the sharp Gaussian concentration estimate, but it illustrates a useful general technique: transport between the restrictions to a set and the complement of its enlargement. This is usually called Marton's argument; see [145].*

5.11 Concentration and the Laplace functional

We follow the presentation in [5]. Let (X, d, μ) be a metric probability space. For $\lambda \geq 0$, define its *Laplace functional* by

$$E_\mu(\lambda) = \sup\left\{ \int e^{\lambda f}\, d\mu : f \text{ is bounded and 1-Lipschitz, } \mathbb{E}_\mu f = 0 \right\}.$$

One may equivalently allow integrable 1-Lipschitz functions, interpreting exponential moments as extended real numbers; centered truncations justify this extension. We have the following relation to concentration.

Proposition 5.4. *For every $t > 0$,*

$$\alpha_\mu(t) \leq \inf_{\lambda \geq 0} \left(e^{-\lambda t/2} E_\mu(\lambda) \right).$$

Proof. For bounded 1-Lipschitz f, put $g = f - \mathbb{E}_\mu f$. Markov's inequality gives, for $s > 0$ and $\lambda \geq 0$,

$$\mu(g \geq s) \leq e^{-\lambda s} \int e^{\lambda g}\, d\mu \leq e^{-\lambda s} E_\mu(\lambda).$$

Apply Proposition 5.3 and take the infimum over λ. □

In particular, control of the Laplace functional can give sub-Gaussian concentration.

Corollary 5.8. *If $E_\mu(\lambda) \le e^{C_0\lambda^2}$ for all $\lambda \ge 0$, where $C_0 > 0$, then*

$$\alpha_\mu(t) \le e^{-t^2/(16C_0)}.$$

Proof. Minimize $C_0\lambda^2 - \lambda t/2$ at $\lambda = t/(4C_0)$. □

For example, if $\operatorname{diam} X \le D$, Hoeffding's elementary bounded-variable estimate gives

$$E_\mu(\lambda) \le e^{D^2\lambda^2/8}.$$

Applying such bounds successively to independent coordinates gives nearly optimal concentration on the Hamming cube. The product structure is essential for the factor n in the exponent: the diameter bound alone does not produce that factor. See the homework.

5.12 The Herbst argument

This classical argument due to Herbst [80] obtains sub-Gaussian concentration from a log-Sobolev inequality.

Theorem 5.10 (Herbst)**.** *Suppose a metric probability space, with the usual gradient chain rule, satisfies*

$$\operatorname{Ent}_\mu(f^2) \le 2\beta \int |\nabla f|^2 \, d\mu$$

for every locally Lipschitz function in the relevant Sobolev space, where $\beta > 0$. Then

$$\alpha_\mu(t) \le e^{-t^2/(8\beta)}, \qquad t > 0.$$

For example, if $d\mu = e^{-V} dx$ and $\nabla^2 V \succeq k \operatorname{Id}$, then $\beta \le k^{-1}$. This follows from Gaussian log-Sobolev and Caffarelli's contraction theorem; see the homework. Therefore:

Corollary 5.9. *Under this Hessian bound, every Borel set A with $\mu(A) \ge 1/2$ satisfies*

$$\mu(A_t) \ge 1 - e^{-kt^2/8}, \qquad t > 0.$$

Remark 5.8. *Log-Sobolev with constant β implies Poincaré with the same constant: substitute $f = 1 + \epsilon g$ and compare the terms of order ϵ^2. Thus Herbst's theorem has a stronger assumption than Gromov–Milman [77] and gives a stronger concentration estimate.*

To prove the theorem, we introduce some notation.

Definition 5.5 (Centered log-moment generating function)**.** For an integrable function $f : X \to \mathbb{R}$, set

$$\psi_f(\lambda) = \log \mathbb{E}_\mu e^{\lambda(f - \mathbb{E}_\mu f)}.$$

This is defined as an extended real-valued function of $\lambda \in \mathbb{R}$.

Lemma 5.4. *Let $\lambda > 0$. Suppose that the relevant exponential moments are finite and, for every $0 < s \le \lambda$,*

$$\mathrm{Ent}_\mu(e^{sf}) \le \frac{s^2\sigma^2}{2}\mathbb{E}_\mu e^{sf}.$$

Then $\psi_f(\lambda) \le \sigma^2\lambda^2/2$.

Proof. For presentation, take f bounded so differentiation under the integral is immediate. Direct differentiation gives

$$\frac{d}{ds}\frac{\psi_f(s)}{s} = \frac{\mathrm{Ent}_\mu(e^{sf})}{s^2\mathbb{E}_\mu e^{sf}} \le \frac{\sigma^2}{2}.$$

Also $\psi_f(0) = \psi_f'(0) = 0$, so $\lim_{s\to 0}\psi_f(s)/s = 0$. Integration from 0 to λ proves the claim. The same argument applies under suitable exponential integrability; bounded truncations supply the version needed below. □

If f is a centered Gaussian random variable of variance σ^2, then $\psi_f(\lambda) = \sigma^2\lambda^2/2$. This motivates the following definition.

Definition 5.6. An integrable random variable f is *sub-Gaussian with parameter σ* if

$$\psi_f(\lambda) \le \frac{\sigma^2\lambda^2}{2} \qquad \text{for every } \lambda \in \mathbb{R}.$$

Chernoff's bound can be expressed using a one-sided Legendre transform.

Lemma 5.5. *For $t > 0$, put*

$$\psi_{f,+}^*(t) = \sup_{\lambda\ge 0}\{\lambda t - \psi_f(\lambda)\}, \qquad \psi_{f,-}^*(t) = \sup_{\lambda\ge 0}\{\lambda t - \psi_f(-\lambda)\}.$$

Then

$$\mu(f - \mathbb{E}_\mu f \ge t) \le e^{-\psi_{f,+}^*(t)}, \qquad \mu(f - \mathbb{E}_\mu f \le -t) \le e^{-\psi_{f,-}^*(t)}.$$

Consequently,

$$\mu(|f - \mathbb{E}_\mu f| \ge t) \le e^{-\psi_{f,+}^*(t)} + e^{-\psi_{f,-}^*(t)}.$$

Proof. For $\lambda \ge 0$, Markov's inequality gives

$$\mu(f - \mathbb{E}_\mu f \ge t) \le e^{-\lambda t}\mathbb{E}e^{\lambda(f-\mathbb{E}f)} = e^{-(\lambda t - \psi_f(\lambda))}.$$

Take the infimum over λ. Apply the same argument to $-f$ for the lower tail. In general the two rate functions need not coincide. □

Corollary 5.10. *For sub-Gaussian f with parameter σ,*

$$\mu(|f - \mathbb{E}_\mu f| \ge t) \le 2e^{-t^2/(2\sigma^2)}.$$

Proof of Theorem 5.10. Let g be a bounded 1-Lipschitz function and substitute $f = e^{\lambda g/2}$ in log-Sobolev. The chain rule gives

$$\mathrm{Ent}_\mu(e^{\lambda g}) \le \frac{\beta\lambda^2}{2}\int |\nabla g|^2 e^{\lambda g}\, d\mu \le \frac{\beta\lambda^2}{2}\mathbb{E}_\mu e^{\lambda g}.$$

By Lemma 5.4, applied also to $-g$, we have

$$\psi_g(\lambda) \le \frac{\beta\lambda^2}{2}.$$

The one-sided Chernoff bound therefore yields

$$\mu(g - \mathbb{E}_\mu g \ge t) \le e^{-t^2/(2\beta)}.$$

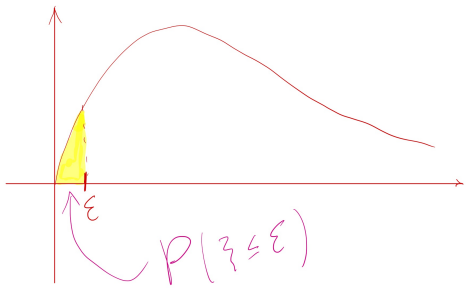
Finally, Proposition 5.3 gives $\alpha_\mu(t) \le e^{-t^2/(8\beta)}$. Smooth approximation and bounded truncation extend the intermediate calculations to the usual Sobolev setting. For the concentration bound itself, bounded truncated distance functions already suffice. □

5.13 Small-ball estimates for norms on the sphere by Klartag and Vershynin

Recall that for a non-negative random variable ξ, a *small-ball estimate* is an inequality of the form

$$\mathbb{P}(\xi \le \epsilon) \le \delta(\epsilon),$$

where ideally $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, the faster the better.



More generally, for a random vector X in $\mathbb{R}^n$, one may seek upper bounds for $\mathbb{P}(|X - z| \le \epsilon)$ that hold uniformly in the center z. Such estimates say that the random vector rarely falls in a prescribed small ball.

From Corollary 5.4, we know that a seminorm f has an exponential upper-tail estimate on the sphere:

$$\sigma(f \ge \mathbb{E}_\sigma f + t) \le C\exp\left(-\frac{ct}{\mathbb{E}_\sigma f}\right).$$

This does not provide a useful estimate for the probability that f is much smaller than its mean. Even a bound of the same shape for the lower tail would have a right-hand side of constant order when $0 \le t \le \mathbb{E}f$.

This leads to a natural question: can one bound

$$\sigma(f \le \epsilon \mathbb{E}_\sigma f)$$

by a quantity tending to zero with ϵ? Klartag and Vershynin answered this question for norms in [114]; see also later work of Paouris, Tikhomirov and Valettas [159]. We outline their argument.

We begin with the following standard notation introduced by Milman.

Definition 5.7. For a norm $\|\cdot\|$ on $\mathbb{R}^n$, let

$$M = \int_{\mathbb{S}^{n-1}} \|\theta\| \, d\sigma(\theta).$$

Definition 5.8. The Klartag–Vershynin dimension is

$$d(\|\cdot\|) = \min\left\{ n, -\log \sigma\left(\|\theta\| \le \frac{M}{2} \right) \right\},$$

where $-\log 0 = +\infty$.

For the Euclidean norm, $M = 1$ and the probability in the definition is zero, so $d = n$. For various families of norms, this dimension tends to infinity with n; see [114]. The homework asks you to verify this for $\|\cdot\|_\infty$. This is not asserted for every possible sequence of norms.

Theorem 5.11 (Klartag–Vershynin [114])**.** *There are absolute constants $c, C, c' > 0$ such that every norm on $\mathbb{R}^n$ satisfies*

$$\sigma(\|\theta\| \le \epsilon M) \le (C\epsilon)^{c\, d(\|\cdot\|)}, \qquad 0 < \epsilon < 1.$$

Consequently,

$$\left(\int_{\mathbb{S}^{n-1}} \|\theta\|^{-\ell} \, d\sigma(\theta) \right)^{-1/\ell} \ge c' M, \qquad 0 < \ell \le c' d(\|\cdot\|).$$

Remark 5.9. *Hölder's inequality, or monotonicity of power means, gives the reverse bound*

$$\left(\int_{\mathbb{S}^{n-1}} \|\theta\|^{-\ell} \, d\sigma(\theta) \right)^{-1/\ell} \le M.$$

Thus the negative moments in this range are comparable to the ordinary mean. The exponent outside the integral is $-1/\ell$.

We first compare spherical and Gaussian measures.

Lemma 5.6. *For every origin-symmetric convex body K,*

$$\frac{1}{2}\sigma\left(\mathbb{S}^{n-1}\cap\frac{1}{2}K\right)\le\gamma(\sqrt{n}K)\le\sigma(\mathbb{S}^{n-1}\cap 2K)+e^{-c_0 n},$$

where $c_0>0$ is absolute.

Proof. Write a standard Gaussian vector as $G=R\Theta$, where $R=|G|$ and Θ is uniform on the sphere, independently of R. Two elementary radial estimates are

$$\mathbb{P}(R\le 2\sqrt{n})\ge\frac{3}{4},\qquad \mathbb{P}(R<\sqrt{n}/2)\le e^{-c_0 n}.$$

The first is Markov's inequality, since $\mathbb{E}R^2=n$. The second follows either from the radial Gaussian integral or from the exponential Markov inequality applied to e^{-R^2}.

If $R\le 2\sqrt{n}$ and $\Theta\in K/2$, then $R\Theta\in\sqrt{n}K$. This proves the lower bound. Conversely, if $R\Theta\in\sqrt{n}K$ and $R\ge\sqrt{n}/2$, then $\Theta\in 2K$. This proves the upper bound. □

Remark 5.10. *The comparison only requires K to be star-shaped with respect to the origin; convexity and symmetry are not needed here.*

Proof of Theorem 5.11. Let K be the unit ball of the norm, write $d=d(\|\cdot\|)$, and set

$$L=M\sqrt{n}K,\qquad q(s)=\gamma(sL),\quad s>0.$$

A median m of the norm on the sphere also provides a natural normalization: $m\le 2M$, since $M\ge m\,\sigma(\|\theta\|\ge m)\ge m/2$. The B-theorem of Cordero-Erausquin, Fradelizi and Maurey (Theorem 4.14) says that $t\mapsto\log q(e^t)$ is concave. In particular, if $0<\epsilon<a<b$, concavity gives the extrapolation estimate

$$\log q(\epsilon)\le\log q(a)-\frac{\log(a/\epsilon)}{\log(b/a)}\Big(\log q(b)-\log q(a)\Big).\tag{28}$$

The point is that a substantial loss of Gaussian mass between two fixed scales forces a power bound at all smaller scales.

Let $F=\|G\|$. Since $G=R\Theta$,

$$\mathbb{E}F=(\mathbb{E}R)M\le\sqrt{n}M,$$

so Markov gives $q(4)\ge 3/4$. The comparison lemma and the definition of d give

$$q(1/8)\le\sigma(\|\theta\|\le M/4)+e^{-c_0 n}\le e^{-d}+e^{-c_0 n}\le 2e^{-c_1 d},$$

where $c_1=\min\{1,c_0\}$. Thus, whenever d exceeds a fixed absolute constant,

$$\log q(4)-\log q(1/8)\ge\frac{c_1 d}{2}.$$

Use (28) with $a = 1/8$, $b = 4$ to obtain

$$q(\epsilon) \le (8\epsilon)^{c_2 d}, \qquad 0 < \epsilon \le 1/8.$$

Increasing the constant inside the parentheses makes the assertion automatic at larger ϵ.

For completeness, the bounded range of d can be handled by the same argument at two other absolute scales. Reverse Hölder gives $\mathbb{E}F^2 \le C(\mathbb{E}F)^2$, and Paley–Zygmund therefore gives

$$\mathbb{P}(F > \mathbb{E}F/2) \ge c_3 > 0.$$

Also $\mathbb{E}R \ge c_4\sqrt{n}$. Hence there are absolute $a, b > 0$ with

$$q(a) \le 1 - c_3, \qquad q(b) \ge 1 - c_3/2.$$

For the second inequality, take b sufficiently large and use Markov. Extrapolation now gives $q(\epsilon) \le (C\epsilon)^{c_5}$, which implies the desired bound when d is bounded. The same reverse Hölder and Paley–Zygmund argument on the sphere shows $d \ge c_6 > 0$ for every norm.

Finally, the comparison lemma yields

$$\sigma(\|\theta\| \le \epsilon M) \le 2q(2\epsilon) \le (C\epsilon)^{cd},$$

after adjusting the absolute constants; the uniform lower bound on d allows us to absorb the factor 2.

To deduce the negative-moment statement, put $Y = \|\theta\|/M$, $k = cd$, and choose a fixed $\delta > 0$ so small that $C\delta \le 1/2$. By the layer-cake formula, with $r = k/2$,

$$\begin{aligned}
\mathbb{E}Y^{-r} &= r\int_0^\infty t^{-r-1}\mathbb{P}(Y < t)\,dt \\
&\le rC^k \int_0^\delta t^{k-r-1}\,dt + r\int_\delta^\infty t^{-r-1}\,dt \\
&\le \delta^{-r}(1 + 2^{-k}).
\end{aligned}$$

Since k is bounded below by an absolute positive constant, this gives $(\mathbb{E}Y^{-r})^{-1/r} \ge c' > 0$. Monotonicity of power means gives the same bound for $0 < \ell \le r$. Multiplying by M completes the proof. □

5.14 Homework

Question 5.1 (3 points)**.** *Recall that a spherical cap is a nonempty set of the form*

$$\mathbb{S}^{n-1} \cap \{x : \langle x, v\rangle \ge t\},$$

for $v \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$. Prove the isoperimetric inequality on $\mathbb{S}^{n-1}$: among Borel sets of prescribed spherical probability $\alpha \in (0,1)$, spherical caps minimize perimeter. Here perimeter is defined using infinitesimal geodesic enlargements and normalized spherical measure.

Hint: use a spherical analogue of Steiner symmetrization, or another approach.

Question 5.2 (1 point). *Using Question 5.1, deduce sharp concentration on the sphere: a set of spherical measure at least* $1/2$ *has enlargements at least as large as the corresponding enlargements of a hemisphere.*

Hint: integrate the isoperimetric inequality, as we will later do to obtain sharp Gaussian concentration from Gaussian isoperimetry.

Question 5.3 (2 points). *a) Prove the Efron–Stein inequality. If* $X_1, \ldots, X_n$ *are independent and* $f(X) \in L^2$*, then*

$$\operatorname{Var}(f(X)) \leq \sum_{i=1}^{n} \mathbb{E} \operatorname{Var}_{X_i}(f(X)),$$

where Var_{X_i} *means that the other coordinates are held fixed and variance is taken only in* X_i*.*

b) Prove tensorization of the Poincaré inequality: if $\mu_1, \ldots, \mu_m$ *are probability measures on* $\mathbb{R}^{k_1}, \ldots, \mathbb{R}^{k_m}$*, then*

$$C_{\mathrm{Poin}}(\mu_1 \otimes \cdots \otimes \mu_m) = \max_i C_{\mathrm{Poin}}(\mu_i),$$

using the Euclidean product gradient and the convention $\operatorname{Var}_\mu f \leq C_{\mathrm{Poin}}(\mu) \int |\nabla f|^2 d\mu$*.*

Question 5.4 (1 point). *a) Let* $X_1, \ldots, X_n$ *be independent and let* $f \geq 0$ *be measurable, with finite entropy. Prove*

$$\operatorname{Ent}(f(X)) \leq \sum_{i=1}^{n} \mathbb{E} \operatorname{Ent}_{X_i}(f(X)).$$

As in the preceding exercise, Ent_{X_i} *integrates only in coordinate* i*, holding the others fixed. Recall that*

$$\operatorname{Ent}_\mu(f) = \int f \log f \, d\mu - \left(\int f \, d\mu \right) \log \left(\int f \, d\mu \right),$$

with $0 \log 0 = 0$*.*

Hint: use the variational characterization of entropy proved earlier.

b) Define the log-Sobolev constant to be the smallest β *such that* $\operatorname{Ent}_\mu(g^2) \leq 2\beta \int |\nabla g|^2 d\mu$*. Prove that the log-Sobolev constant of a product of probability measures is the maximum of the constants of its factors.*

Question 5.5 (5 points). *This question is intentionally vague, allowing some freedom: find an interesting extension or generalization of Herbst's argument, for example using a generalized log-Sobolev inequality, or in another setting.*

Question 5.6 (1 point). *Show that if* ψ *is convex and coordinatewise non-decreasing on the positive orthant, then*

$$x \longmapsto \psi(x_1^2, \ldots, x_n^2)$$

is convex wherever it is defined. Explain why the restriction of an unconditional convex function to the positive orthant is coordinatewise non-decreasing. This supplies the composition step in the unconditional thin-shell proof. The monotonicity assumption is essential.

Question 5.7 (1 point). *Show that for every Borel probability measure on a nonempty metric space,*

$$\alpha_\mu(t) \longrightarrow 0 \qquad (t \to \infty).$$

Your argument should give a bound uniform over the sets A with $\mu(A) \geq 1/2$.

Question 5.8 (1 point). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be p-Lipschitz, that is, $|f(x) - f(y)| \leq p|x - y|$. Show that $|\nabla f| \leq p$ at every point of differentiability, and hence almost everywhere by Rademacher's theorem.*

Question 5.9 (3 points). *a) Let f be an integrable log-concave density with $0 < f(0) < \infty$ and 0 in the interior of its support. For $p \geq 1$, define Ball's body $K_p(f)$ by its radial function*

$$\rho_{K_p(f)}(\theta) = \left(\frac{p}{f(0)} \int_0^\infty f(t\theta) t^{p-1}\, dt \right)^{1/p}.$$

Prove that

$$\|x\|_{K_p(f)} = |x| \left(\frac{p}{f(0)} \int_0^\infty f\left(t \frac{x}{|x|} \right) t^{p-1}\, dt \right)^{-1/p}, \qquad x \neq 0,$$

extended by zero at $x = 0$, is convex, and hence is the Minkowski functional of a convex body. If f is even, this body is origin-symmetric. These bodies were introduced by Keith Ball.

b) Suppose f is an even *isotropic probability density. For which p does $K_p(f)$ have scalar covariance? After dilating to volume one, this is isotropic position in the convex-body convention (the covariance is $L_K^2 \,\mathrm{Id}$, not necessarily Id).*

Hint: for a homogeneous function H of degree $k \geq 0$, polar coordinates give

$$\int_{K_{n+k}(f)} H(x)\, dx = \frac{1}{f(0)} \int_{\mathbb{R}^n} H(x) f(x)\, dx,$$

provided the integrals are finite. The cases $k = 1$ and $k = 2$ involve different bodies. *For a general centered density, the first identifies the barycenter of $K_{n+1}(f)$, whereas the second identifies the raw second moments of $K_{n+2}(f)$. Evenness ensures that both bodies are centered.*

c) Prove that a dimension-independent thin-shell bound for all isotropic log-concave probability measures is equivalent to such a bound for uniform probability measures on convex bodies.

Hint for (c): use the equivalent squared-radius formulation $\mathrm{Var}(|X|^2) \leq Cn$; the pointwise comparison with the radial formulation is given in [2, Proposition 2.3]. Symmetrize by taking $(X - X')/\sqrt{2}$, where X' is an independent copy. For an even isotropic density $f = f(0)e^{-V}$, couple X to uniform measure on a suitable dilate of $K_n(f)$ by matching the conditional radial distribution functions. In high dimensions the useful dilation is $e^{\mathbb{E}V(X)/n}$. Show that the quadratic transport cost is bounded by an absolute constant, and then normalize the body's covariance. The needed estimate on $V(X) - \mathbb{E}V(X)$ follows by applying Prékopa to the convex perspective $pV(x/p)$.

d) Prove the analogous equivalence for the KLS conjecture.

Hint for (d): after subtracting a constant from the convex potential, assume $V \geq 0$ and approximate $e^{-V(x)}$ by $(1-V(x)/m)_+^m$. This is proportional to the marginal density of uniform measure on the convex body

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : V(x) \leq m, \quad |y| \leq 1 - V(x)/m\}.$$

Center and normalize the covariance in the two coordinate blocks, and test the resulting Poincaré inequality on functions depending only on x. Pass to the limit using bounded smooth test functions first.

The reductions in (c) and (d) require separate arguments; the moment calculation in (b) alone does not establish them.

Question 5.10 (3 points)**.** *Show that the KLS conjecture is equivalent to the following statement: for every convex body whose uniform measure is isotropic, some affine hyperplane divides the body into two parts of equal volume with relative perimeter at most an absolute constant times the least relative perimeter among all equal-volume divisions. Here relative perimeter counts the interface* inside *the body, not the outside boundary of the body. For one point, prove the implication from KLS.*

Hint: relate isoperimetry to the Poincaré inequality for log-concave measures. The appropriate rigorous version of "the gradient of an indicator measures its boundary" is the total variation identity $|D\mathbb{1}_M| = \mathcal{H}^{n-1} \lfloor_{\partial M}$ for a smooth region M; alternatively use smooth approximations to $\mathbb{1}_M$.

Question 5.11 (1 point)**.** *Let f be the restriction to $\mathbb{S}^{n-1}$ of a seminorm on $\mathbb{R}^n$. Show that, for $1 \leq p \leq q$,*

$$\left(\int_{\mathbb{S}^{n-1}} |f(\theta)|^q \, d\sigma(\theta) \right)^{1/q} \leq \frac{Cq}{p} \sqrt{\frac{n+p}{n+q}} \left(\int_{\mathbb{S}^{n-1}} |f(\theta)|^p \, d\sigma(\theta) \right)^{1/p}.$$

Hint: use reverse Hölder for Gaussian measure, polar coordinates, and the radial Gaussian moments, which can be estimated using integration by parts or the Gamma function.

Question 5.12 (1 point)**.** *Let μ be any even log-concave measure on $\mathbb{R}^n$. Show that for any symmetric measurable set A in $\mathbb{R}^n$ and any $t > 0$ one has*

$$\mu\left(x \in \mathbb{R}^n : \exists y \in A : \langle x, y \rangle > -t\right) \geq 1 - \frac{1}{\mu(A)} e^{-t}.$$

Hint: Use Klartag's extension of the functional Blaschke-Santalo inequality.

Question 5.13 (2 points)**.** *Let μ be a probability measure on a metric space of diameter at most $D < \infty$. Prove that its Laplace functional satisfies*

$$E_\mu(\lambda) \leq e^{D^2\lambda^2/2}, \qquad \lambda \geq 0.$$

The sharper coefficient $1/8$ also holds.

Question 5.14 (1 point). *Show that if $E_\mu(\lambda) \le e^{\lambda^2/(2c)}$ for every $\lambda \ge 0$, where $c > 0$, then*

$$\alpha_\mu(t) \le e^{-ct^2/8}, \qquad t > 0.$$

Hint: use Proposition 5.4.

Remark: even on a bounded convex body, a diameter-based Laplace estimate gives a quadratic exponent, whereas using only Payne–Weinberger and Gromov–Milman gives a linear exponent.

Question 5.15 (1 point). *Prove the following sub-Gaussian bound for the uniform measure on $\{-1,1\}^n$, equipped with the* normalized *Hamming distance used above:*

$$\alpha_\mu(t) \le e^{-nt^2/8}.$$

Hint: expose the independent coordinates one at a time. A 1-Lipschitz function changes by at most $1/n$ when one coordinate changes; the corresponding martingale increments have conditional range of length at most $1/n$. Apply the bounded-variable exponential estimate from Question 5.13 conditionally, and then use Question 5.14.

Question 5.16 (2 points). *Show that Paouris's inequality follows from the following moment estimates of Guédon and Milman. For an isotropic log-concave random vector X in $\mathbb{R}^n$,*

$$1 - C\frac{|p-2|}{n^{1/3}} \le n^{-1/2}(\mathbb{E}|X|^p)^{1/p} \le 1 + C\frac{|p-2|}{n^{1/3}}$$

when $1 \le |p-2| \le c_1 n^{1/6}$, and

$$1 - C\frac{|p-2|^{1/2}}{n^{1/4}} \le n^{-1/2}(\mathbb{E}|X|^p)^{1/p} \le 1 + C\frac{|p-2|^{1/2}}{n^{1/4}}$$

when $c_1 n^{1/6} \le |p-2| \le c_2 n^{1/2}$. The constants are absolute, and the assertions concern the range in which the indicated moments are finite. At $p = 0$, interpret the power mean by its limit $\exp(\mathbb{E}\log|X|)$.

Hint: first use a positive moment of order $c\sqrt{n}$, then combine it with reverse Hölder and Markov's inequality to obtain the full upper-tail bound.

Remark: these estimates also include negative moments, unlike the reverse Hölder inequality proved above.

Question 5.17 (1 point). *Give a non-Lipschitz function on the sphere that violates the concentration-around-a-median bound stated for 1-Lipschitz functions. For example, exhibit a sequence of discontinuous functions on $\mathbb{S}^{n-1}$ for which a fixed-size deviation has probability bounded away from zero as $n \to \infty$.*

Question 5.18 (1 point). *Suppose that every bounded 1-Lipschitz function on (X, d, μ) satisfies*

$$\mu(f \ge \mathbb{E}f + t) \le a(t), \qquad t > 0.$$

Show that $\alpha_\mu(t) \le a(t/2)$. Use a truncated distance function so no integrability assumption on the metric is needed.

Question 5.19 (2 points). • *If* $\operatorname{diam}(X, d) \le D$*, prove that every probability measure on* X *satisfies* $E_\mu(\lambda) \le e^{CD^2\lambda^2}$*.*

- *By applying the estimate conditionally in each independent coordinate, deduce nearly sharp sub-Gaussian concentration on the Hamming cube.*

Question 5.20 (2 points). *Prove that* $d(\|\cdot\|_\infty) \to \infty$ *as the dimension tends to infinity, where* d *is the Klartag–Vershynin dimension defined above.*

Question 5.21 (1 point). *Estimate from below the Poincaré constant of uniform measure on a domain consisting of two unit balls in* $\mathbb{R}^n$*,* $n \ge 2$*, joined by a cylindrical neck of radius* ϵ *and fixed positive length. Track the dependence on* ϵ *as* $\epsilon \downarrow 0$*.*

Question 5.22 (1 point). *Prove that for a log-concave probability measure on* $\mathbb{R}$*, and each prescribed mass in* $(0, 1)$*, an isoperimetric minimizer can be chosen to be a ray. The choice between a left ray and a right ray may depend on the measure and on the prescribed mass.*

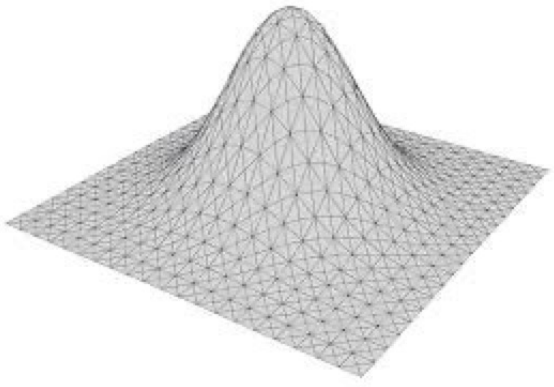
6 Gaussian Measure and its special properties

6.1 A general discussion

Recall that the standard Gaussian measure on $\mathbb{R}^n$ is given by

$$d\gamma(x) = (2\pi)^{-n/2} e^{-|x|^2/2}\, dx.$$

When the dimension needs to be specified, we write γ_n.



We have already seen that it has many wonderful properties:

- It is a log-concave isotropic probability measure. In particular,

$$\int x\, d\gamma = 0, \qquad \int \langle x, \theta\rangle^2\, d\gamma = 1 \quad (\theta \in S^{n-1}).$$

- In dimension at least two, a nondegenerate rotation-invariant product probability measure is a centered Gaussian measure, up to a common variance factor.
- Centered Gaussian random vectors, including linear images of a standard Gaussian vector, are determined by their covariance matrices. For a general Gaussian vector one also specifies its mean.
- Gaussian measure plays the main role in the Central Limit Theorem, and the class of Gaussian distributions is closed under convolution.
- Gaussian functions are extremizers in the classical log-Sobolev and reverse log-Sobolev inequalities.
- They give equality in the functional Blaschke–Santaló inequality.
- Gaussian distributions give equality in the Entropy Power Inequality, with the appropriate proportionality condition on the covariance matrices.
- There is a nice "Gaussian Fourier system" given by Hermite polynomials.
- The standard Gaussian measure satisfies the Poincaré inequality with constant 1.
- It satisfies the B-theorem and an improved Poincaré inequality for even functions with constant $1/2$.

It is worthwhile mentioning also:

Theorem 6.1 (Gaussian Correlation Inequality, Royen [164])**.** *If A, B are symmetric convex sets in $\mathbb{R}^n$, then*

$$\gamma(A \cap B) \geq \gamma(A)\gamma(B).$$

In addition to Royen [164], see the exposition by Latała and Matlak [126] of this breakthrough result.

The following proposition quantifies the idea that *the Gaussian measure is a "role model" for isotropic probability measures, especially for log-concave ones.* Throughout this section, the Poincaré constant $C_{\mathrm{Poin}}(\mu)$ is the smallest coefficient in

$$\mathrm{Var}_\mu(f) \leq C_{\mathrm{Poin}}(\mu) \int |\nabla f|^2 \, d\mu.$$

Proposition 6.1. *If μ is an isotropic probability measure, then*

$$C_{\mathrm{Poin}}(\mu) \geq 1 = C_{\mathrm{Poin}}(\gamma).$$

Proof. Isotropy gives $\int x_i \, d\mu = 0$ and $\int x_i^2 \, d\mu = 1$. Apply the Poincaré inequality to $f(x) = x_i$:

$$1 = \mathrm{Var}_\mu(x_i) \leq C_{\mathrm{Poin}}(\mu) \int 1 \, d\mu = C_{\mathrm{Poin}}(\mu).$$

□

In this section, we shall see several strong isoperimetric properties and phenomena special to Gaussian measure.

Remark 6.1 (A convention about regularity)**.** *For the sake of presentation, differential calculations below are first carried out for smooth functions with sufficient decay and for sets with sufficiently regular boundaries. In the general statements, perimeter may be understood in the usual weighted finite-perimeter sense; smooth approximation and lower semicontinuity then give the corresponding inequalities. For enlargement arguments, the outer Minkowski perimeter defined below is convenient. We will mention additional assumptions when they affect the mathematical content, rather than interrupt every calculation with an approximation argument.*

6.2 The isoperimetric profile

Recall the isoperimetric problem for a probability measure μ: for a prescribed $a \in [0,1]$, find

$$\inf_{\mu(A)=a} \mu^+(\partial A),$$

where the outer Minkowski perimeter is

$$\mu^+(\partial A) = \liminf_{\varepsilon \downarrow 0} \frac{\mu(A + \varepsilon B_2^n) - \mu(A)}{\varepsilon}.$$

For regular sets and a measure with continuous density, this is the integral of the density over the boundary.

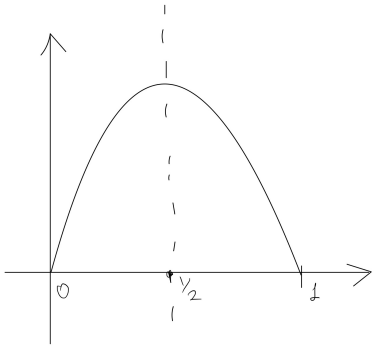
Definition 6.1. The isoperimetric profile of μ is

$$I_\mu(a) = \inf_{\mu(A)=a} \mu^+(\partial A).$$

For the full-dimensional log-concave probability measures considered here, the profile has the following properties (with the standard perimeter convention):

- $I_\mu(a) \geq 0$, and $I_\mu(0) = I_\mu(1) = 0$. For Gaussian measure the profile also tends to zero at the endpoints. Such continuity is not a consequence of non-atomicity alone: for example, the uniform measure on an interval has a positive constant profile on $(0,1)$.
- $I_\mu(a) = I_\mu(1-a)$, so $s \mapsto I_\mu(1/2+s)$ is even. The reason is that a set and its complement have the same perimeter.
- I_μ is *concave* for log-concave measures; see E. Milman [147] and the references there.

The complement argument uses measure-theoretic perimeter, or regular representatives of sets; arbitrary topological boundaries need not behave this way.



Remark 6.2. *For Lebesgue measure, the isoperimetric inequality reads*

$$|\partial A| \geq n|B_2^n|^{1/n}|A|^{(n-1)/n}.$$

Thus its profile is also concave. Recall that we obtained this bound from the Brunn–Minkowski inequality.

For a log-concave measure, one might similarly try to use Prékopa–Leindler. For $0 < t < 1$,

$$\begin{aligned}\mu(K + \varepsilon B_2^n) &= \mu\left((1-t)\frac{K}{1-t} + t\frac{\varepsilon B_2^n}{t}\right) \\ &\geq \mu\left(\frac{K}{1-t}\right)^{1-t} \mu\left(\frac{\varepsilon B_2^n}{t}\right)^t.\end{aligned}$$

Subtracting $\mu(K)$*, dividing by* ε*, and choosing* t *as a function of* ε *can give useful estimates. However, this log-concavity bound does not by itself identify the sharp Gaussian isoperimetric profile. For that, we need a stronger inequality.*

6.3 The Ehrhard Inequality

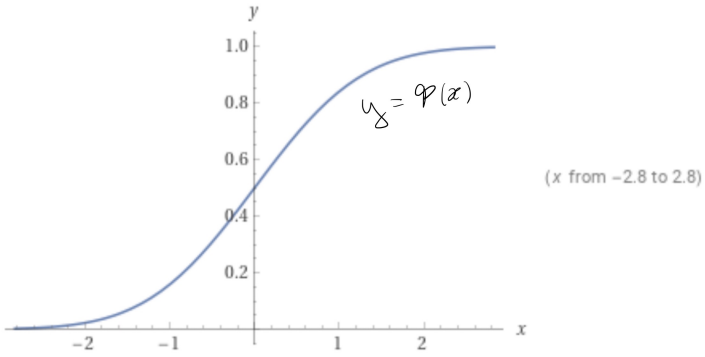
Ehrhard's inequality is a sharper Gaussian analogue of the Prékopa–Leindler inequality.

Definition 6.2. The standard Gaussian cumulative distribution function is

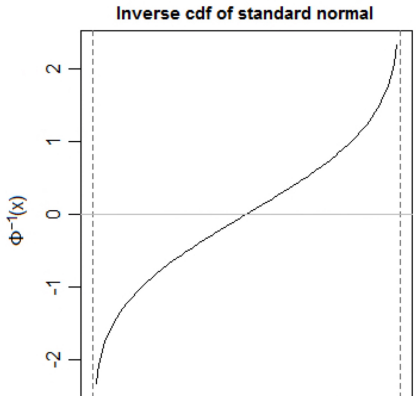
$$\Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

We write

$$\varphi(s) = \Phi'(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}.$$



We shall use the inverse $\Phi^{-1} : (0,1) \to \mathbb{R}$, extended by $\Phi^{-1}(0) = -\infty$ and $\Phi^{-1}(1) = +\infty$ when taking limits.



We are ready to state arguably the most important Gaussian Brunn–Minkowski type inequality. It was proved by Ehrhard for convex sets [58, 59], by Latała when one set is convex [124], and by Borell for arbitrary Borel sets [34, 33].

Theorem 6.2 (Ehrhard–Borell)**.** *For Borel sets $A, B \subset \mathbb{R}^n$ of Gaussian measures in $(0,1)$ and $\lambda \in (0,1)$,*

$$\Phi^{-1}\Big(\gamma(\lambda A + (1-\lambda)B)\Big) \geq \lambda \Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)).$$

The endpoint cases are interpreted by limits whenever the right-hand side is well-defined.

Other proofs and extensions are given by Neeman and Paouris [154], van Handel [82], and Ivanisvili [91], among others.

Gaussian log-concavity gives

$$\log \gamma(\lambda A + (1-\lambda)B) \geq \lambda \log \gamma(A) + (1-\lambda)\log \gamma(B).$$

For convex A, B, put $m(t) = \gamma((1-t)A + tB)$. Ehrhard's inequality says that $\Phi^{-1} \circ m$ is concave, whereas Prékopa–Leindler gives concavity of $\log m$. Convexity of A, B ensures that Minkowski interpolation between two members of this family is again the corresponding member of the family.

Claim 6.1. *Ehrhard's inequality is stronger than Gaussian log-concavity: if m takes values in $(0,1)$ and $\Phi^{-1} \circ m$ is concave, then $\log m$ is concave.*

Proof. It is enough to show that $\log \Phi$ is increasing and concave, since

$$\log m = (\log \Phi) \circ (\Phi^{-1} \circ m).$$

Monotonicity is immediate. For concavity, compute

$$(\log \Phi)''(s) = -\frac{\varphi(s)}{\Phi(s)^2}\Big(s\Phi(s) + \varphi(s)\Big).$$

The expression in parentheses is nonnegative for $s \geq 0$. If $s < 0$, put $b = -s > 0$ and use the elementary tail estimate

$$\int_b^\infty e^{-t^2/2}\,dt \leq \frac{1}{b}\int_b^\infty te^{-t^2/2}\,dt = \frac{1}{b}e^{-b^2/2}.$$

Thus $\varphi(s) + s\Phi(s) \geq 0$ for every s.

Here is the same comparison in the local notation used earlier. For $a \in (0,1)$ and $q = \Phi^{-1}(a)$,

$$(\Phi^{-1})'(a) = \frac{1}{\varphi(q)}, \qquad (\Phi^{-1})''(a) = \frac{q}{\varphi(q)^2}.$$

Consequently,

$$-\frac{(\Phi^{-1})''(a)}{(\Phi^{-1})'(a)} = -\frac{q}{\varphi(q)} \leq \frac{1}{a} = -\frac{(\log)''(a)}{(\log)'(a)}, \tag{29}$$

which is precisely

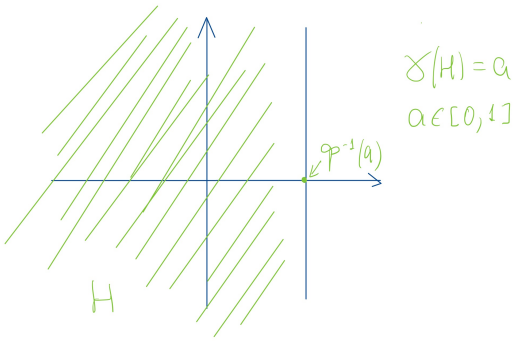
$$-q\Phi(q) \leq \varphi(q). \tag{30}$$

This is the inequality we just proved, so (29) follows. The composition argument has the advantage that it also covers points where $m' = 0$ and does not require differentiating m. □

Remark 6.3. *What is the geometric meaning of $\Phi^{-1}(a)$? The half-space*

$$H_a = \{x \in \mathbb{R}^n : x_1 \leq \Phi^{-1}(a)\}$$

has Gaussian measure a. Thus $\Phi^{-1}(a)$ is the signed distance from the origin to its boundary.



Compare this with $a^{1/n}$, which, up to a dimensional constant, is the radius of a ball of Lebesgue volume a.

Remark 6.4. *Shenfeld and van Handel [83] characterized equality in Ehrhard's inequality. In particular, equality holds for parallel half-spaces with the same orientation. Indeed, if*

$$A = \{x_1 \le \Phi^{-1}(a)\}, \qquad B = \{x_1 \le \Phi^{-1}(b)\},$$

then

$$\lambda A + (1-\lambda)B = \{x_1 \le \lambda\Phi^{-1}(a) + (1-\lambda)\Phi^{-1}(b)\}.$$

Taking Gaussian measure and applying Φ^{-1} gives equality. The midpoint case has the especially simple geometric interpretation of averaging the two boundary locations.

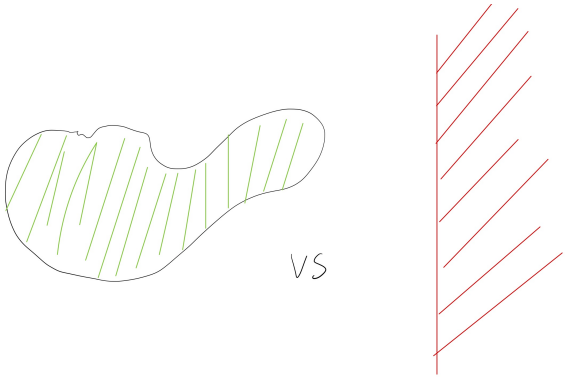
6.4 Gaussian isoperimetric inequality

Ehrhard's inequality implies the following classical result.

Theorem 6.3 (Gaussian Isoperimetric Inequality, Sudakov–Tsirelson [170], Borell [33])**.** *If $A \subset \mathbb{R}^n$ is Borel and $\gamma(A) = a$, then*

$$\gamma^+(\partial A) \ge \gamma^+(\partial H_a) = \varphi(\Phi^{-1}(a)),$$

where H_a is a half-space of Gaussian measure a. At $a = 0, 1$, the last expression is understood as zero.



In other words, the Gaussian profile is independent of the dimension:

$$I_\gamma(a) = I(a) := \varphi(\Phi^{-1}(a)).$$

For $0 < a < 1$,

$$I(a) = \frac{1}{(\Phi^{-1})'(a)}, \qquad I'(a) = -\Phi^{-1}(a), \qquad I(a)I''(a) = -1.$$

Proof from Ehrhard's inequality. Let $a = \gamma(K) \in (0,1)$, $q = \Phi^{-1}(a)$, and fix a large $R > 0$. Set $\lambda = \varepsilon/R$. Ehrhard's inequality gives

$$\gamma(K + \varepsilon B_2^n) \geq \Phi\left((1-\lambda)\Phi^{-1}\Big(\gamma\left(\frac{K}{1-\lambda}\right)\Big) + \lambda\Phi^{-1}(\gamma(RB_2^n))\right).$$

The dilation derivative, obtained by a change of variables, is

$$D(K) := \frac{d}{ds}\gamma(sK)\bigg|_{s=1} = \int_K (n - |x|^2)\, d\gamma.$$

For fixed R, expansion as $\varepsilon \downarrow 0$ therefore yields

$$\gamma^+(\partial K) \geq \frac{D(K)}{R} + \frac{I(a)}{R}\Big(\Phi^{-1}(\gamma(RB_2^n)) - q\Big).$$

Now let $R \to \infty$. The Gaussian tail asymptotics imply

$$\lim_{R\to\infty} \frac{\Phi^{-1}(\gamma(RB_2^n))}{R} = 1;$$

we leave its verification as homework. The other two terms divided by R vanish, and the desired bound follows. □

One may also ask for an anisotropic version, replacing B_2^n by a convex body L. This is left as homework.

6.5 Gaussian concentration inequality

Theorem 6.4 (Gaussian concentration inequality)**.** *Let $A_t = \{x : \mathrm{dist}(x, A) < t\}$ for $t > 0$. If $\gamma(A) = a \in (0,1)$, then*

$$\gamma(A_t) \geq \Phi(\Phi^{-1}(a) + t) = \gamma((H_a)_t).$$

In particular, if $\gamma(A) \geq 1/2$, then

$$\gamma(A_t) \geq \Phi(t) \geq 1 - \frac{1}{2}e^{-t^2/2}.$$

Proof. For the differential argument, write

$$h(t) = \Phi^{-1}(\gamma(A_t)).$$

At points of differentiability, Gaussian isoperimetry gives

$$h'(t) = \frac{\frac{d}{dt}\gamma(A_t)}{I(\gamma(A_t))} \geq \frac{\gamma^+(\partial A_t)}{I(\gamma(A_t))} \geq 1.$$

Integrating, with the usual approximation for arbitrary Borel A, gives

$$h(t) \geq \Phi^{-1}(\gamma(A)) + t.$$

Applying Φ proves the first assertion. If $\gamma(A) \geq 1/2$, then $\Phi^{-1}(\gamma(A)) \geq 0$. Finally,

$$\begin{aligned} 1 - \Phi(t) &= \int_0^\infty \varphi(t+s)\, ds \\ &= e^{-t^2/2} \int_0^\infty e^{-ts} \varphi(s)\, ds \leq \frac{1}{2} e^{-t^2/2}. \end{aligned}$$

□

Remark 6.5. *This is the enlargement form of Gaussian isoperimetry. Borell's noise-stability theorem is a related but different result: it compares the joint probability that two correlated Gaussian vectors lie in given sets. The enlargement statement above does not require introducing noise stability.*

6.6 Isoperimetry on the cube

We now discuss another application of Gaussian isoperimetry. For the uniform probability measure on a convex body K, the isoperimetric problem asks for the smallest *relative perimeter* of a set occupying a prescribed fraction of K. For regular A, this perimeter is proportional to $|\partial A \cap \operatorname{int} K|_{n-1}$: the part of the boundary lying on ∂K is not counted.

For $K = B_2^n$, the minimizers are spherical caps, with boundaries meeting the sphere orthogonally, and limiting half-ball configurations. How about $K = [0,1]^n$? The following seemingly simple problem is open in general in the setting of these lectures.

Conjecture 6.1 (Isoperimetry on the cube)**.** *For $a \in (0,1)$, the smallest relative perimeter in $[0,1]^n$ among sets of volume a is attained by a suitable coordinate round cylinder centered at a vertex, or by the complement of such a cylinder.*

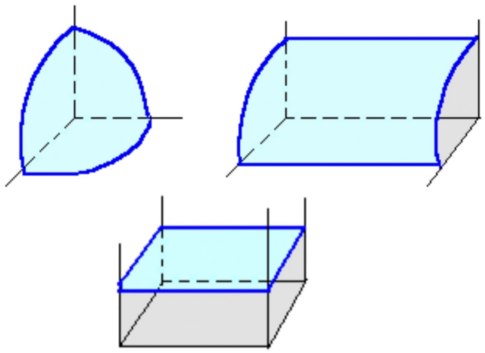
More explicitly, consider sets

$$C_k(a) = rB_2^k \times \mathbb{R}^{n-k} + x_0,$$

where the active coordinates are coordinate directions, x_0 is a cube vertex, and r is chosen so that $|C_k(a) \cap [0,1]^n| = a$. Then the proposed bound is

$$|\partial A \cap (0,1)^n|_{n-1} \geq \min_{1 \leq k \leq n} \left\{ |\partial C_k(a) \cap (0,1)^n|_{n-1}, |\partial C_k(1-a) \cap (0,1)^n|_{n-1} \right\}.$$

The second candidate corresponds to the complement of a cylinder of volume $1-a$.



Although the exact problem is difficult beyond dimension two, a useful dimension-free lower bound is available.

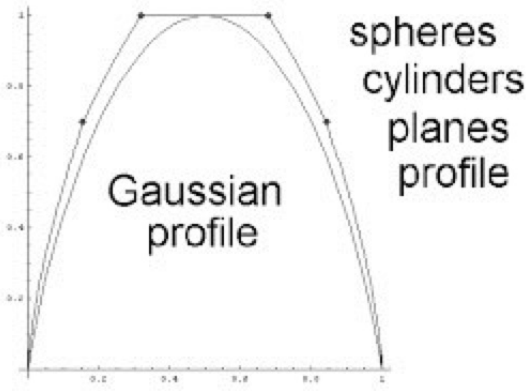
Theorem 6.5 (Barthe–Maurey [20], Hadwiger)**.** *For the uniform probability measure μ on $[0,1]^n$,*

$$I_\mu(a) \geq \sqrt{2\pi}\, I_\gamma(a).$$

Thus, if $|A \cap [0,1]^n| = a$, then

$$|\partial A \cap (0,1)^n|_{n-1} \geq e^{-\Phi^{-1}(a)^2/2},$$

with relative perimeter understood as above.



See also Glaudo [74].

Remark 6.6 (Transport of concentration and isoperimetry)**.** *Suppose $T_\#\mu = \nu$ and T is L-Lipschitz, with $L > 0$. Then*

$$\alpha_\nu(t) \leq \alpha_\mu(t/L), \qquad I_\nu(a) \geq \frac{1}{L} I_\mu(a).$$

Indeed, if B is a set in the target space and $A = T^{-1}(B)$, then $\mu(A) = \nu(B)$ and

$$A_{t/L} \subset T^{-1}(B_t).$$

This inclusion gives the concentration comparison and, after dividing by t and taking a lower limit, the perimeter comparison. The scaling acts on the argument *of the concentration function.*

Corollary 6.1. *Let $d\mu = e^{-V}\,dx$ be a probability measure with $\nabla^2 V \geq k\,\mathrm{Id}$, $k > 0$. Then*

$$\alpha_\mu(t) \leq \alpha_\gamma(\sqrt{k}\,t) \leq \frac{1}{2}e^{-kt^2/2}, \qquad I_\mu(a) \geq \sqrt{k}\,I_\gamma(a).$$

Proof. Caffarelli's contraction theorem 4.12, after rescaling, provides a transport from γ to μ with Lipschitz constant at most $1/\sqrt{k}$. Apply the preceding remark. □

After the curvature application in Corollary 6.1, we return to the uniform measure on the unit cube and use an explicit transport map.

Lemma 6.1. *There is a $1/\sqrt{2\pi}$-Lipschitz map $\pi : \mathbb{R}^n \to [0,1]^n$ such that*

$$\pi_\# \gamma = \mathrm{unif}([0,1]^n).$$

Proof. Recall from Example 4.2 that $\Phi_\# \gamma_1 = \mathrm{unif}[0,1]$: for a test function f,

$$\int_0^1 f(u)\,du = \int_{-\infty}^{\infty} f(\Phi(s))\Phi'(s)\,ds = \int f(\Phi(s))\,d\gamma_1(s).$$

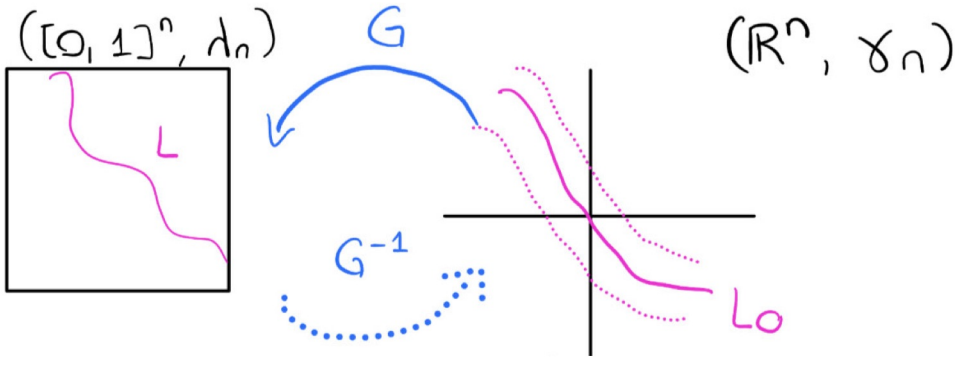
Define

$$\pi(x_1, \ldots, x_n) = (\Phi(x_1), \ldots, \Phi(x_n)).$$

The product structure gives the pushforward identity. Moreover,

$$|\pi(x) - \pi(y)|^2 = \sum_{i=1}^n |\Phi(x_i) - \Phi(y_i)|^2 \leq \frac{1}{2\pi}|x - y|^2,$$

because $\sup_s \Phi'(s) = 1/\sqrt{2\pi}$. □



Picture (c) Klartag

Proof of the Barthe–Maurey bound. Combine Lemma 6.1 with the transport comparison for isoperimetric profiles. □

Corollary 6.2. *If $|A \cap [0,1]^n| = 1/2$, then*

$$|\partial A \cap (0,1)^n|_{n-1} \geq 1.$$

Equality is attained for $A = \{x_1 \leq 1/2\}$.

In particular, we obtain the following nontrivial hyperplane-section bound.

Corollary 6.3 (Vaaler [173], the hyperplane case)**.** *For every $\theta \in S^{n-1}$,*

$$\left|\theta^\perp \cap [-1/2, 1/2]^n\right|_{n-1} \geq 1,$$

with equality for $\theta = e_1$.

Remark 6.7. *In the other direction, Ball [8] showed that, for $n \geq 2$,*

$$\left|\theta^\perp \cap [-1/2, 1/2]^n\right|_{n-1} \leq \sqrt{2}.$$

Equality holds for $\theta = (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots, 0)$.

6.7 Gaussian symmetrization and the proof of the Ehrhard inequality

The following concept was introduced by Ehrhard [58, 59]; see also Borell [34] and Bogachev [28]. It is the Gaussian counterpart of Steiner symmetrization, but now a fiber is replaced by a half-space of the same *Gaussian* measure.

Definition 6.3 (Gaussian symmetrization)**.** Fix $1 \leq k \leq n$, a linear subspace $L \subset \mathbb{R}^n$ with $\dim L = n - k$, and a unit vector $e \in L^\perp$. For a Borel set A, write

$$m_A(x) = \gamma_k\Big(\{y \in L^\perp : x + y \in A\}\Big), \qquad x \in L,$$

where γ_k is the centered standard Gaussian measure on $L^\perp$. Its Gaussian symmetrization is

$$S(L,e)(A) = \{x + y : x \in L,\ y \in L^\perp,\ \langle y, e\rangle \leq \Phi^{-1}(m_A(x))\}.$$

Zero-measure fibers are taken to be empty, and fibers of measure one are taken to be all of $L^\perp$. Boundary choices do not affect Gaussian measure.

Thus we consistently use lower half-spaces in the direction e. Here are some examples:

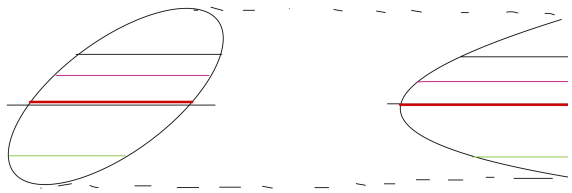
- If $L = e_1^\perp$ and $e = e_1$, each line parallel to e_1 is replaced by a lower ray whose endpoint depends on the base point in $e_1^\perp$.

- If $k = n$, so that $L = \{0\}$, then

$$S(\{0\}, e)(A) = \{x : \langle x, e\rangle \le \Phi^{-1}(\gamma(A))\}.$$

  This is a half-space of the same Gaussian measure as A.

- If $k = n - 1$, the result is a cylinder over a two-dimensional set: one coordinate records the base point in L, and the other records the endpoint of the symmetrized fiber.



Some elementary properties are left as homework.

Lemma 6.2. *For Borel sets $A, B \subset \mathbb{R}^n$,*

- $\gamma(S(L, e)(A)) = \gamma(A)$*;*
- $A \subset B$ *implies* $S(L, e)(A) \subset S(L, e)(B)$*;*
- *if $v \in L$, then* $S(L, e)(A + v) = S(L, e)(A) + v$.

Remark 6.8. *The restriction $v \in L$ matters. Translation in a fiber changes its Gaussian measure, so Gaussian symmetrization does not commute with arbitrary translations.*

Lemma 6.3. *Suppose $e \in L_1^\perp \cap L_2^\perp$ is a unit vector, and*

$$(L_1 \cap L_2)^\perp \cap L_1 \quad \text{and} \quad (L_1 \cap L_2)^\perp \cap L_2$$

are orthogonal. Then, with consistent choices on null fibers,

$$S(L_1, e) \circ S(L_2, e) = S(L_2, e) \circ S(L_1, e) = S(L_1 \cap L_2, e).$$

The identity follows by integrating over the orthogonal fiber coordinates. Consequently:

Corollary 6.4. *If $n \ge 3$ and $2 \le k \le n$, every k-symmetrization $S(L, e)$ is a composition of $k - 1$ two-dimensional symmetrizations, all using the same direction e.*

Lemma 6.4 (The limiting step in the symmetrization argument)**.** *In dimension two, a suitable sequence of one-dimensional Gaussian symmetrizations can be chosen so that the successive symmetrals of a closed set A, with $0 < \gamma_2(A) < 1$, converge to a half-space H of the same Gaussian measure. In particular,*

$$\gamma_2(A_j \triangle H) \longrightarrow 0.$$

In fact, Ehrhard proves local Hausdorff convergence after intersecting with sufficiently large balls [58]. The relevant topology is local, rather than the global Hausdorff distance on arbitrary unbounded sets.

Combining this planar limiting construction with Corollary 6.4 gives the corresponding approximation in higher dimensions.

Corollary 6.5. *The analogous limiting symmetrization construction applies in every dimension.*

Remark 6.9. *Gaussian symmetrization sends half-spaces to half-spaces, up to null sets, but does not leave every half-space fixed. Write $u = u_L + u_{L^\perp}$. The symmetrization of $\{\langle x, u\rangle \le a\}$ has normal $u_L + |u_{L^\perp}|e$. Thus it is fixed precisely when its fiber component is already a nonnegative multiple of e (including zero).*

The substantive geometric property used in the isoperimetric argument is:

Theorem 6.6. *For a closed set $A \subset \mathbb{R}^n$, $r > 0$, and a Gaussian symmetrization $S = S(L, e)$,*

$$S(A) + rB_2^n \subset S(A + rB_2^n),$$

with harmless closure choices if needed.

Proof. The proof is left as homework. In the one-dimensional fibers, it rests on the fact that rays minimize Gaussian enlargements; one then combines the fiber comparisons with the orthogonal decomposition of Euclidean distance. □

Remark 6.10. *This inclusion and measure preservation imply that Gaussian symmetrization decreases Gaussian perimeter. Together with the limiting construction in Corollary 6.5, this gives a symmetrization proof of Gaussian isoperimetry. To make this an independent proof, the fiber comparisons and limiting step must be established first; using the full Gaussian isoperimetric theorem to prove them would be circular. We are outlining the classical proof scheme here, not treating those substantial steps as consequences of Fubini alone.*

Finally, the following convexity property is crucial for Ehrhard's inequality.

Theorem 6.7. *If A is an open convex set, then $S(L, e)(A)$ is convex, with the open-fiber convention.*

Proof. This is a central property of Ehrhard symmetrization; see [58, 59]. Its proof is left as homework. When it is used to prove Ehrhard's inequality, it must be obtained independently, rather than deduced from the inequality one is trying to prove. □

We now explain how this property proves Ehrhard's inequality for convex sets A, B:

$$\Phi^{-1}(\gamma(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)). \tag{31}$$

The idea is to realize A, B as parallel sections of a convex set in one higher dimension. Gaussian symmetrization turns their section measures into the heights of a concave function.

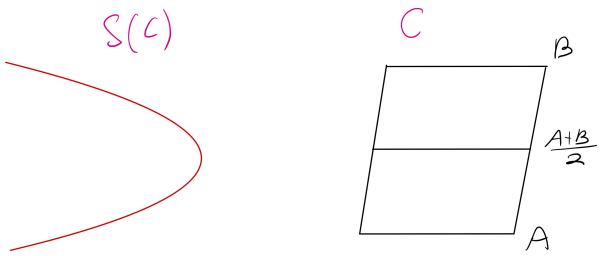
Proof for convex sets, using the symmetrization property. In $\mathbb{R}^{n+1}$, set

$$\widetilde{A} = A \times \{0\}, \qquad \widetilde{B} = B \times \{1\}, \qquad C = \mathrm{conv}(\widetilde{A} \cup \widetilde{B}).$$

Then, after identifying $\mathbb{R}^n \times \{t\}$ with $\mathbb{R}^n$,

$$C_t := \{x : (x,t) \in C\} = (1-t)A + tB \qquad (0 \leq t \leq 1).$$

In particular, the middle section is $(A+B)/2$.



Apply the n-symmetrization with base $L = \mathbb{R}e_{n+1}$ and a fixed unit direction $e \in \mathbb{R}^n \times \{0\}$. Its section at height t is

$$\{x : \langle x, e\rangle \leq f(t)\}, \qquad f(t) = \Phi^{-1}(\gamma_n(C_t)).$$

The symmetrized set is convex by Theorem 6.7; equivalently, its upper endpoint function f is concave. Thus

$$f(t) \geq (1-t)f(0) + tf(1),$$

which is (31) after exchanging t and $1-\lambda$. Open approximations handle the boundary slices and general convex sets. □

6.8 Latała's Functional Ehrhard inequality

We now present the functional version of Ehrhard's inequality observed by Latała [129].

Theorem 6.8 (Functional Ehrhard inequality, Latała [129])**.** *Let $\lambda \in (0,1)$, and let $F, G, H : \mathbb{R}^n \to [0,1]$ be Borel functions such that, for all x, y,*

$$\Phi^{-1}(H(\lambda x + (1-\lambda)y)) \geq \lambda \Phi^{-1}(F(x)) + (1-\lambda)\Phi^{-1}(G(y)). \tag{32}$$

Then

$$\Phi^{-1}\left(\int H\, d\gamma\right) \geq \lambda \Phi^{-1}\left(\int F\, d\gamma\right) + (1-\lambda)\Phi^{-1}\left(\int G\, d\gamma\right).$$

As usual, functions taking values 0 or 1 are treated by the extended-value convention and approximation when necessary; $\lambda = 0, 1$ are immediate endpoint cases.

In particular, for proper convex functions f, g for which the quantities are defined,

$$\Phi^{-1}\left(\int \Phi\big(-(\lambda f + (1-\lambda)g)^*\big)\, d\gamma\right) \geq \lambda \Phi^{-1}\left(\int \Phi(-f^*)\, d\gamma\right) + (1-\lambda)\Phi^{-1}\left(\int \Phi(-g^*)\, d\gamma\right).$$

Thus the functional

$$f \longmapsto \Phi^{-1}\left(\int \Phi(-f^*)\, d\gamma\right)$$

is concave on its natural domain.

Remark 6.11. *The weighted infimal convolution explains this formulation. In the convention*

$$u \square_t v(z) = \inf_{(1-t)x + ty = z} \{(1-t)u(x) + tv(y)\},$$

one has, under the usual closedness assumptions,

$$((1-t)f + tg)^* = f^* \square_t g^*.$$

The inequality defining this infimum gives exactly the hypothesis of functional Ehrhard after applying Φ to its negative.

Proof of the functional theorem. Consider the subgraphs in $\mathbb{R}^{n+1}$

$$A = \{(x,s) : s \leq \Phi^{-1}(F(x))\}, \qquad B = \{(x,s) : s \leq \Phi^{-1}(G(x))\}.$$

The hypothesis (32) implies

$$\lambda A + (1-\lambda)B \subset \{(x,s) : s \leq \Phi^{-1}(H(x))\}. \tag{33}$$

Ehrhard's inequality in dimension $n+1$ gives

$$\Phi^{-1}(\gamma_{n+1}(\lambda A + (1-\lambda)B)) \geq \lambda \Phi^{-1}(\gamma_{n+1}(A)) + (1-\lambda)\Phi^{-1}(\gamma_{n+1}(B)). \tag{34}$$

By Fubini,

$$\gamma_{n+1}(A) = \int F \, d\gamma_n,$$

and similarly for B and the subgraph of $\Phi^{-1} \circ H$. Combine this with (33) and (34). $\square$

Remark 6.12. *Functional Ehrhard tensorizes; this is left as homework. Its one-dimensional case, however, is already substantial.*

6.9 Generalized Bobkov's inequality via functional Ehrhard

We now apply the same procedure to Ehrhard's inequality that led from Prékopa–Leindler to the generalized log-Sobolev inequality, following Barthe, Cordero-Erausquin, Ivanisvili, and Livshyts [16]. Kolesnikov and Milman [119] obtained geometric consequences by linearizing the set version directly. The relation between the geometric formulations in [119] and the functional ones below is an interesting question.

For suitable smooth convex f, g, consider

$$\begin{aligned}\alpha(t) = \Phi^{-1}\left(\int \Phi\Big(-((1-t)f + tg)^*\Big)\, d\gamma\right) \\ - (1-t)\Phi^{-1}\left(\int \Phi(-f^*)\, d\gamma\right) - t\Phi^{-1}\left(\int \Phi(-g^*)\, d\gamma\right).\end{aligned}$$

Functional Ehrhard (Theorem 6.8) gives $\alpha(t) \geq 0$ and $\alpha(0) = 0$, hence $\alpha'(0) \geq 0$.

Recall

$$I(a) = \varphi(\Phi^{-1}(a)), \qquad (\Phi^{-1})'(a) = \frac{1}{I(a)}.$$

By Lemma 3.3,

$$\frac{d}{dt}((1-t)f + tg)^*\Big|_{t=0} = (f-g)(\nabla f^*).$$

It follows that

$$\begin{aligned}0 \leq \alpha'(0) = \frac{\int \varphi(f^*)\,[g(\nabla f^*) - f(\nabla f^*)]\, d\gamma}{I\left(\int \Phi(-f^*)\, d\gamma\right)} \\ + \Phi^{-1}\left(\int \Phi(-f^*)\, d\gamma\right) - \Phi^{-1}\left(\int \Phi(-g^*)\, d\gamma\right).\end{aligned}$$

Set $G = g^*$ and $h = \Phi(-f^*)$. Then

$$f^* = -\Phi^{-1}(h), \qquad \nabla f^* = -\frac{\nabla h}{I(h)}, \qquad \varphi(f^*) = I(h).$$

Collecting the terms independent of G into $\spadesuit(h)$, we obtain

$$\spadesuit(h) + \int G^*\left(-\frac{\nabla h}{I(h)}\right) I(h)\, d\gamma \geq I\left(\int h\, d\gamma\right) \Phi^{-1}\left(\int \Phi(-G)\, d\gamma\right).$$

For example, writing $F = -\Phi^{-1}(h)$, the term being suppressed is

$$\spadesuit(h) = I\left(\int h\, d\gamma\right) \Phi^{-1}\left(\int h\, d\gamma\right) - \int I(h)\left[\langle x, \nabla F\rangle - F\right] d\gamma.$$

Remark 6.13 (Removing the convexity restriction on $-\Phi^{-1}(h)$)**.** *The same first variation can be obtained directly from the weighted infimal convolution $F \square_t G$, without assuming F convex. For the sake of presentation, first take F smooth and bounded with bounded derivatives, and restrict G to a compact convex domain. Writing the constraint as $y = (x - tz)/(1-t)$ and differentiating at $t = 0$ gives*

$$\left.\frac{d}{dt}(F \square_t G)(x)\right|_{t=0} = \langle x, \nabla F(x)\rangle - F(x) - G^*(\nabla F(x)).$$

Applying functional Ehrhard to this interpolation gives the displayed inequality for smooth h taking values strictly between 0 and 1, without a convexity assumption on $-\Phi^{-1}(h)$. To remove the compact-domain restriction, apply the inequality to $G + \mathbb{1}^{\infty}_{RB_2^n}$ and let $R \to \infty$ at each fixed scaling parameter; the conjugates increase to G^*. Truncation of h away from $0, 1$, cutoff, and smoothing give the usual locally Lipschitz versions whenever the displayed integrals converge. These approximation steps are taken before the large-scaling limit below.

Now use the rescaling

$$G_\lambda(x) = \lambda G(x/\lambda), \qquad G^*_\lambda(z) = \lambda G^*(z), \qquad \lambda > 0.$$

Indeed, substituting $y = \lambda u$ in the definition of the conjugate proves the second identity. The preceding inequality becomes

$$\spadesuit(h) + \lambda \int G^*\left(-\frac{\nabla h}{I(h)}\right) I(h)\, d\gamma \geq I\left(\int h\, d\gamma\right) \Phi^{-1}\left(\int \Phi\left(-\lambda G(x/\lambda)\right)\, d\gamma(x)\right).$$

Divide by λ and let $\lambda \to \infty$.

Theorem 6.9 (Generalized Bobkov inequality [16])**.** *Let G be a proper lower-semicontinuous convex function and $h : \mathbb{R}^n \to [0,1]$ a locally Lipschitz function for which the quantities below are defined. Then*

$$\int G^*\left(-\frac{\nabla h}{I(h)}\right) I(h)\, d\gamma \geq I\left(\int h\, d\gamma\right) \limsup_{\lambda\to\infty} \frac{1}{\lambda} \Phi^{-1}\left(\int \Phi\left(-\lambda G(x/\lambda)\right)\, d\gamma(x)\right).$$

Where h takes the values 0 or 1, the integrand is interpreted by approximation. In the examples below the limit exists, so lim sup *may be replaced by* lim*.*

Remark 6.14. *If $G(0) < \infty$ and G is ray-increasing, meaning $t \mapsto G(t\theta)$ is nondecreasing on $[0, \infty)$ for every θ, then $G(x/\lambda) \geq G(0)$. Consequently,*

$$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \Phi^{-1} \left(\int \Phi \left(-\lambda G(x/\lambda) \right) \, d\gamma \right) \leq -G(0).$$

In useful examples equality holds, but it must be checked; the upper bound alone does not supply the lower bound needed in the theorem.

Consider the example

$$G(x) = \begin{cases} -\sqrt{1 - |x|^2}, & |x| \leq 1, \\ +\infty, & |x| > 1. \end{cases}$$

Here

$$p_\lambda := \int \Phi(-\lambda G(x/\lambda)) \, d\gamma_n(x) = \int_{\lambda B_2^n} \Phi \left(\sqrt{\lambda^2 - |x|^2} \right) \, d\gamma_n(x).$$

The full ball λB_2^{n+1} lies inside the subgraph defining this integral, while its height is at most λ. Therefore

$$\gamma_{n+1}(\lambda B_2^{n+1}) \leq p_\lambda \leq \Phi(\lambda).$$

The Gaussian ball-tail limit used earlier now shows that

$$\frac{\Phi^{-1}(p_\lambda)}{\lambda} \longrightarrow 1.$$

Recall from Example 3.3, part 5, that $G^*(z) = \sqrt{1 + |z|^2}$. Hence

$$I(h) G^* \left(-\frac{\nabla h}{I(h)} \right) = \sqrt{I(h)^2 + |\nabla h|^2}.$$

We obtain the celebrated inequality originally proved by Bobkov by different means.

Theorem 6.10 (Bobkov [22])**.** *For locally Lipschitz $h : \mathbb{R}^n \to [0, 1]$,*

$$\int_{\mathbb{R}^n} \sqrt{I(h)^2 + |\nabla h|^2} \, d\gamma \geq I \left(\int_{\mathbb{R}^n} h \, d\gamma \right).$$

This inequality tensorizes. One proof uses two-point inequalities and approximation of Gaussian measure by discrete product measures, as in [22]. Alternative proofs and related results are due to Barthe and Ivanisvili [18], Carlen and Kerce [44], and Neeman and Paouris [154], among others. The linearization proof outlined here follows [16].

Remark 6.15. *Bobkov's inequality implies Gaussian isoperimetry. Approximate $\mathbf{1}_K$ by functions that change from 1 to 0 across a thin boundary layer. In the limit, the gradient term gives $\gamma^+(\partial K)$, the interior contribution $I(\mathbf{1}_K)$ is zero, and the right-hand side tends to $I(\gamma(K))$. Thus*

$$\gamma^+(\partial K) \geq I(\gamma(K)).$$

In fact, the two statements are equivalent; see, for example, Neeman [153]. The gradient of an indicator is understood through this approximation, or as a boundary measure, not as an ordinary pointwise gradient.

The preceding discussion exhibits the following parallel chains:

Ehrhard	Prékopa–Leindler
↓	↓
Generalized Bobkov	Generalized log-Sobolev
↓	↓
Bobkov	Log-Sobolev
↕	
Gaussian isoperimetry	Classical isoperimetry and Sobolev inequalities

The columns describe related geometric and functional phenomena. In particular, the displayed parallel does not assert that the log-Sobolev inequality and the sharp classical isoperimetric inequality are interchangeable without further arguments.

For another example, take a convex body K containing the origin in its interior, and set

$$G(x) = \begin{cases} -1, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

Then $G^*(z) = 1 + h_K(z)$; if K is symmetric, $h_K(z) = \|z\|_{K^\circ}$. We get:

Corollary 6.6. *With the preceding assumptions,*

$$I\left(\int h\,d\gamma\right) \limsup_{\lambda\to\infty} \frac{\Phi^{-1}(\Phi(\lambda)\gamma(\lambda K))}{\lambda} \le \int h_K(-\nabla h)\,d\gamma + \int I(h)\,d\gamma.$$

In particular, for $K = B_2^n$, the limit is 1 and

$$I\left(\int h\,d\gamma\right) - \int I(h)\,d\gamma \le \int |\nabla h|\,d\gamma.$$

Remark 6.16. *The last inequality is weaker than Bobkov's inequality, since*

$$\sqrt{I(h)^2 + |\nabla h|^2} \le I(h) + |\nabla h|.$$

Remark 6.17. *More generally, for $p > 1$ and a symmetric convex body K, one may use*

$$G(x) = \begin{cases} -(1 - \|x\|_K^p)^{1/p}, & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

If $q = p/(p-1)$, then $G^*(z) = (1 + h_K(z)^q)^{1/q}$. Substitution gives corresponding p-type Bobkov inequalities, with the asymptotic constant in Theorem 6.9 determined by the chosen body and scaling.

6.10 An Ehrhard–Brascamp–Lieb type inequality

We now differentiate functional Ehrhard twice. The perturbation g need not be convex: it suffices that $f_t = f + tg$ remain convex for t in a neighborhood of zero. This allows general smooth test functions after the change of variables below. Concavity gives

$$\frac{d^2}{dt^2}\Phi^{-1}\left(\int \Phi(-f_t^*)\,d\gamma\right) \le 0. \tag{35}$$

Write $W = f^*$ and $h = g(\nabla W)$. At $t = 0$, the Legendre-transform differentiation formulas give

$$\dot{W} = -h, \qquad \ddot{W} = \left\langle (\nabla^2 W)^{-1}\nabla h, \nabla h\right\rangle =: Q.$$

The second derivative has a *positive* quadratic term. For

$$a(t) = \int \Phi(-f_t^*)\,d\gamma,$$

we therefore have

$$a'(0) = \int \varphi(W)h\,d\gamma, \qquad a''(0) = \int \varphi(W)(Wh^2 - Q)\,d\gamma.$$

Since $(\Phi^{-1})''/(\Phi^{-1})' = \Phi^{-1}/I$, inequality (35) becomes

$$a''(0) + \frac{\Phi^{-1}(a(0))}{I(a(0))}\,a'(0)^2 \le 0.$$

Normalizing the weight $e^{-W^2/2}$ gives:

Theorem 6.11 (Barthe, Cordero-Erausquin, Ivanisvili, Livshyts [16])**.** *Let $W \ge 0$ be a smooth convex function with positive-definite Hessian, and set*

$$Z = \int e^{-W^2/2}\,d\gamma, \qquad d\mu = Z^{-1}e^{-W^2/2}\,d\gamma, \qquad a = \int \Phi(-W)\,d\gamma.$$

Assume $0 < a \le 1/2$, and define

$$A = Z\,\Phi^{-1}(a)e^{\Phi^{-1}(a)^2/2}.$$

Then, for locally Lipschitz h for which the integrals are finite,

$$\int Wh^2\,d\mu + A\left(\int h\,d\mu\right)^2 \le \int \left\langle (\nabla^2 W)^{-1}\nabla h, \nabla h\right\rangle\,d\mu.$$

Here $A \le 0$. Degenerate Hessians can be handled by regularization whenever the limiting right-hand side is meaningful.

Remark 6.18. *One may equivalently obtain Theorem 6.11 by linearizing the* pre-limit first-variation inequality *in the preceding subsection at its equality case $f = g$, just as we derived Brascamp–Lieb from generalized log-Sobolev. This is left as homework. For this argument, retain the terms denoted by ♠(h): the subsequent large-scaling limit used in Theorem 6.9 discards them. Likewise, Brascamp–Lieb itself can be obtained by taking the second derivative of Prékopa–Leindler.*

6.11 Minimizing centered in-radius of a convex set of fixed Gaussian measure

We now discuss a few other Gaussian extremal inequalities.

Definition 6.4. For a convex set K containing the origin, its *centered in-radius* is

$$r(K) = \sup\{r \geq 0 : rB_2^n \subset K\}.$$

Proposition 6.2. *Let K be convex with $\gamma(K) = a > 1/2$. Then $0 \in \operatorname{int} K$ and*

$$r(K) \geq r(H_a) = \Phi^{-1}(a),$$

where $H_a = \{x_1 \leq \Phi^{-1}(a)\}$.

Proof. If the origin is not in the interior of K, separation puts K in a half-space of Gaussian measure at most $1/2$, a contradiction. A supporting hyperplane at minimal distance from the origin gives, after taking a limit if necessary,

$$K \subset \{x : \langle x, \theta \rangle \leq r(K)\}.$$

Thus $a \leq \Phi(r(K))$, and applying Φ^{-1} proves the assertion. □

Proposition 6.3. *Let K be a symmetric convex set with $\gamma(K) = a \in (0,1)$. If S_K is a symmetric strip with the same Gaussian measure, then*

$$r(K) \geq r(S_K) = \Phi^{-1}\left(\frac{1+a}{2}\right).$$

Proof. Symmetry gives $K \subset \{x : |\langle x, \theta \rangle| \leq r(K)\}$ in a direction of minimal support distance. Compare the measures of these two sets. □

6.12 Gaussian barycenter inequality and some extensions

Proposition 6.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing, and let $K \subset \mathbb{R}^n$ be a Borel set. For $\theta \in S^{n-1}$, define*

$$H_K(\theta) = \{x : \langle x, \theta \rangle \leq \Phi^{-1}(\gamma(K))\}.$$

Then, whenever the integrals are defined,

$$\int_K F(\langle x, \theta \rangle)\, d\gamma \geq \int_{H_K(\theta)} F(\langle x, \theta \rangle)\, d\gamma.$$

The proposition has a simple interpretation: among sets of a prescribed measure, a lower half-space selects the smallest values of an increasing function of one coordinate. The following lemma also provides a useful convexity proof in the case of convex K.

Lemma 6.5. *If F is increasing and $G'(s) = F(s)e^{-s^2/2}$, then $G \circ \Phi^{-1}$ is convex on $(0,1)$.*

Proof. For smooth F, put $s = \Phi^{-1}(a)$. Then

$$(G \circ \Phi^{-1})'(a) = \sqrt{2\pi}\, F(s), \qquad (G \circ \Phi^{-1})''(a) = 2\pi F'(s) e^{s^2/2} \geq 0.$$

Approximation gives the assertion for arbitrary increasing F. □

Proof of Proposition 6.4. Let $q = \Phi^{-1}(\gamma(K))$ and $H = H_K(\theta)$. The sets $K \setminus H$ and $H \setminus K$ have the same Gaussian measure. On the first set $F(\langle x, \theta \rangle) \geq F(q)$, and on the second it is at most $F(q)$. Subtracting the two integrals proves the result, without a convexity assumption on K.

Here is the fiber argument when K is convex, which explains the preceding lemma. Rotate so that $\theta = e_1$. For $x \in e_1^{\perp}$, the fiber is an interval

$$I_x = [\Phi^{-1}(\alpha_x), \Phi^{-1}(\alpha_x + a_x)], \qquad a_x = \gamma_1(I_x).$$

With $J = G \circ \Phi^{-1}$, Fubini gives

$$\int_K F(x_1)\, d\gamma = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^{n-1}} \Big[J(\alpha_x + a_x) - J(\alpha_x) \Big]\, d\gamma_{n-1}(x). \tag{36}$$

Convexity of J implies

$$J(\alpha_x + a_x) - J(\alpha_x) \geq J(a_x) - J(0).$$

Apply this bound in (36). Jensen's inequality and $\int a_x\, d\gamma_{n-1} = \gamma(K)$ then give

$$\int_K F(x_1)\, d\gamma \geq \frac{J(\gamma(K)) - J(0)}{\sqrt{2\pi}} = \int_{H_K(e_1)} F(x_1)\, d\gamma.$$

Endpoint values are understood as limits; truncation deals with unbounded F. □

Theorem 6.12 (Gaussian barycenter inequality, Bobkov [23])**.** *Among Borel sets of prescribed Gaussian measure, half-spaces maximize the Euclidean norm of the unnormalized Gaussian barycenter:*

$$\left| \int_K x\, d\gamma \right| \leq \left| \int_{H_K} x\, d\gamma \right| = I(\gamma(K)),$$

where H_K is any half-space of the same Gaussian measure.

Proof. Rotate so that $\int_K x\, d\gamma = -te_1$, $t \geq 0$. Proposition 6.4 with $F(s) = s$ gives

$$t = -\int_K x_1\, d\gamma \leq -\int_{H_K(e_1)} x_1\, d\gamma = \varphi(\Phi^{-1}(\gamma(K))).$$

□

6.13 Gaussian measure of dilates of convex sets in a direction, and an improved Gaussian Poincaré inequality for linear functions on convex sets

The second moment requires more care than the increasing-function comparison above: $s \mapsto s^2$ is not increasing on all of $\mathbb{R}$.

Proposition 6.5 (The one-dimensional comparison)**.** *If $J \subset \mathbb{R}$ is an interval and $H = (-\infty, \Phi^{-1}(\gamma_1(J))]$, then*

$$\int_J s^2 \, d\gamma_1(s) \le \int_H s^2 \, d\gamma_1(s).$$

We shall prove this as part of a rearrangement lemma below.

Remark 6.19. *The corresponding uncentered directional comparison is false for general convex sets in higher dimensions. For example, take $K = \{x_2 \le q\}$ in $\mathbb{R}^2$, with $q > 0$, $a = \Phi(q)$, and $\theta = e_1$. Independence gives*

$$\int_K x_1^2 \, d\gamma = a, \qquad \int_{\{x_1 \le q\}} x_1^2 \, d\gamma = a - q\varphi(q) < a.$$

The correct general comparison will include the barycenter term.

Consider the directional dilation

$$T_t = \operatorname{diag}(t, 1, \dots, 1), \qquad t > 0. \tag{37}$$

Corollary 6.7. *If K is symmetric and convex, then $t \mapsto \gamma(T_t K)$, with T_t defined in* (37)*, is nondecreasing on $(0, \infty)$.*

Proof. A change of variables gives

$$\gamma(T_t K) = (2\pi)^{-n/2} \int_K t \exp\left(-\frac{t^2 y_1^2}{2} - \frac{1}{2}\sum_{i=2}^n y_i^2\right) dy.$$

Thus

$$\frac{d}{dt}\gamma(T_t K)\bigg|_{t=1} = \gamma(K) - \int_K x_1^2 \, d\gamma.$$

The Gaussian Poincaré inequality restricted to the convex set K gives

$$⨍_K x_1^2 \, d\gamma - \left(⨍_K x_1 \, d\gamma\right)^2 \le 1.$$

The mean is zero by symmetry, so the derivative is nonnegative. Apply the same argument to $T_t K$, which remains symmetric and convex, to obtain

$$\frac{d}{dt}\gamma(T_t K) = \frac{1}{t}\left(\gamma(T_t K) - \int_{T_t K} x_1^2 \, d\gamma\right) \ge 0.$$

□

Remark 6.20. *This property is special to the Gaussian setting and uses symmetry. A general convex set need not have zero barycenter, so the uncentered second moment is not bounded by its Gaussian measure.*

Definition 6.5. For a nonnegative measurable function f on $\mathbb{R}$, its decreasing Gaussian rearrangement f^* is the nonincreasing function equimeasurable with f:

$$\gamma_1(\{f > s\}) = \gamma_1(\{f^* > s\}) \qquad (s \geq 0).$$

Here the star denotes rearrangement, not the Legendre transform.

Lemma 6.6. *Suppose $f \geq 0$ is integrable with respect to γ_1 and every superlevel set $\{f > s\}$ is an interval. Then*

$$\int_{\mathbb{R}} x^2 f(x)\, d\gamma_1(x) \leq \int_{\mathbb{R}} x^2 f^*(x)\, d\gamma_1(x),$$

allowing the value $+\infty$.

Proof. First let J be an interval of Gaussian measure a, and let $J^* = (-\infty, \Phi^{-1}(a)]$. Under $u = \Phi(x)$, J becomes an interval of length a in $(0,1)$, and $[-r, r]$ becomes the centered interval $[\Phi(-r), \Phi(r)]$. Among intervals of length a, placing one at an endpoint of $(0,1)$ minimizes its overlap with this centered interval. Therefore

$$\gamma_1(J \cap [-r, r]) \geq \gamma_1(J^* \cap [-r, r]).$$

Equivalently, $\gamma_1(J \cap \{|x| > r\}) \leq \gamma_1(J^* \cap \{|x| > r\})$. Applying layer cake proves the preceding proposition and, on the superlevel intervals of f, gives

$$\begin{aligned} \int x^2 f(x)\, d\gamma_1(x) &= \int_0^\infty \int_{\{f>s\}} x^2\, d\gamma_1(x)\, ds \\ &\leq \int_0^\infty \int_{\{f^*>s\}} x^2\, d\gamma_1(x)\, ds = \int x^2 f^*(x)\, d\gamma_1(x). \end{aligned} \tag{38}$$

□

Remark 6.21. *For a convex set $K \subset \mathbb{R}^n$, the slice function*

$$f(t) = \gamma_{n-1}(\{y \in e_1^\perp : te_1 + y \in K\})$$

is log-concave, so the rearrangement inequality (38) *applies. However, f^* is generally a decreasing* function*, not the indicator of a single ray. Thus $\int x^2 f^*\, d\gamma_1$ cannot be identified with the directional second moment of the half-space of measure $\gamma(K)$. This is precisely where an attempted higher-dimensional uncentered comparison breaks down.*

The correct strengthened linear Poincaré inequality, discussed in [141], has a particularly short proof from Ehrhard's inequality.

Theorem 6.13 ([141])**.** *Let K be convex with $a = \gamma(K) \in (0,1)$, and set*

$$\eta(a) = \frac{a\Phi^{-1}(a)}{I(a)} = \sqrt{2\pi}\, a\Phi^{-1}(a)e^{\Phi^{-1}(a)^2/2}. \tag{39}$$

For every $\theta \in S^{n-1}$,

$$⨍_K \langle x,\theta\rangle^2\, d\gamma + \eta(a)\left(⨍_K \langle x,\theta\rangle\, d\gamma\right)^2 \le 1.$$

One has $\eta(a) > -1$; it is negative for $a < 1/2$, zero at $1/2$, and positive for $a > 1/2$.

Proof. Put $b(t) = \gamma(K + t\theta)$. Since K is convex, Ehrhard's inequality makes $\Phi^{-1}(b(t))$ concave. A change of variables gives

$$b(t) = \int_K e^{-t\langle x,\theta\rangle - t^2/2}\, d\gamma(x),$$

and hence

$$b'(0) = -\int_K \langle x,\theta\rangle\, d\gamma, \qquad b''(0) = \int_K (\langle x,\theta\rangle^2 - 1)\, d\gamma.$$

Twice differentiating the concavity statement yields

$$b''(0) + \frac{\Phi^{-1}(a)}{I(a)} b'(0)^2 \le 0.$$

Divide by a to obtain the claimed inequality. The bound $\eta(a) > -1$ follows from the strict Gaussian tail estimate used in (30); its limiting value at $a \downarrow 0$ is -1. □

Remark 6.22. *Writing $m = ⨍_K \langle x,\theta\rangle\, d\gamma$, this says*

$$\mathrm{Var}_{\gamma|K}(\langle x,\theta\rangle) \le 1 - (1+\eta(a))m^2 \le 1,$$

where $\gamma|K$ denotes the normalized restriction. Thus it strengthens the usual restricted Gaussian Poincaré inequality for linear functions. A half-space with normal θ gives equality in the strengthened formula.

6.14 Gaussian measure of dilates of convex sets

For fixed $\gamma(K)$ and $t \ge 1$, which sets minimize or maximize $\gamma(tK)$? Let us denote a prospective minimizing model by ♠ and a maximizing model by ♣:

$$\gamma(tK) \ge \gamma(t♠), \qquad \gamma(K) = \gamma(♠), \tag{40}$$

$$\gamma(tK) \le \gamma(t♣), \qquad \gamma(K) = \gamma(♣). \tag{41}$$

Lemma 6.7 (The infinitesimal comparison)**.** *For a Borel set K,*

$$\frac{d}{dt}\gamma(tK)\bigg|_{t=1} = n\gamma(K) - \int_K |x|^2 \, d\gamma.$$

Consequently, at equal Gaussian measure, the derivative comparison corresponding to (40) *is*

$$\int_K |x|^2 \, d\gamma \le \int_{\spadesuit} |x|^2 \, d\gamma,$$

and the comparison corresponding to (41) *has the reverse inequality.*

Proof. Differentiate

$$\gamma(tK) = (2\pi)^{-n/2} \int_K t^n e^{-t^2|y|^2/2} \, dy$$

at $t = 1$. □

Recall that the shorthand x^2 used elsewhere in these notes means $\langle x, x\rangle = |x|^2$.

Remark 6.23. *A derivative comparison at a single set and a single parameter is a necessary local condition, not by itself an equivalence with the full dilation comparison. To integrate it, one applies the local inequality at every dilated set, comparing to the model with the* same current measure*, and uses the resulting differential inequality. We will do this explicitly below.*

Proposition 6.6. *If K is measurable and $\gamma(K) = \gamma(RB_2^n)$, then*

$$\int_K |x|^2 \, d\gamma \ge \int_{RB_2^n} |x|^2 \, d\gamma.$$

Proof. The two difference sets have equal Gaussian measure. Since $|x|^2 \ge R^2$ on $K \setminus RB_2^n$ and $|x|^2 \le R^2$ on $RB_2^n \setminus K$,

$$\begin{aligned}\int_K |x|^2 \, d\gamma - \int_{RB_2^n} |x|^2 \, d\gamma &= \int_{K\setminus RB_2^n} |x|^2 \, d\gamma - \int_{RB_2^n\setminus K} |x|^2 \, d\gamma \\ &\ge R^2\Big(\gamma(K) - \gamma(RB_2^n)\Big) = 0.\end{aligned}$$

This argument works for arbitrary measurable K; a radial-function calculation would require K to be star-shaped. □

Corollary 6.8. *Let $\gamma(K) = a \in (0,1)$, and choose R, r so that*

$$\gamma(RB_2^n) = a, \qquad \gamma(\mathbb{R}^n \setminus rB_2^n) = a.$$

For every $t \ge 1$,

$$\gamma\Big(t(\mathbb{R}^n \setminus rB_2^n)\Big) \le \gamma(tK) \le \gamma(tRB_2^n).$$

Proof. There is also a direct rearrangement proof. The change of variables gives

$$\gamma(tK) = t^n \int_K e^{-(t^2-1)|x|^2/2}\, d\gamma(x).$$

For $t \geq 1$, this radial weight is nonincreasing in $|x|$. At a fixed Gaussian measure, its integral is therefore largest on a centered ball and smallest on the complement of a centered ball, by the same difference-set argument. □

Next, which *convex* set minimizes the Gaussian measure of its dilation? Without convexity, the answer was the complement of a centered ball. Under convexity, half-spaces are the relevant model.

Theorem 6.14. *Let K be convex with $a = \gamma(K) \geq 1/2$, and let H_a be a half-space of measure a. For $t \geq 1$,*

$$\gamma(tK) \geq \gamma(tH_a) = \Phi(t\Phi^{-1}(a)).$$

Proof. For a smooth convex set containing the origin, the dilation derivative also has the boundary expression

$$D(K) = \int_{\partial K} \langle x, n_x \rangle \, d\gamma_{\partial K}(x) = \int_{\partial K} h_K(n_x)\, d\gamma_{\partial K}(x),$$

where $d\gamma_{\partial K} = (2\pi)^{-n/2} e^{-|x|^2/2}\, d\mathcal{H}^{n-1}$. Thus

$$D(K) \geq r(K)\gamma^+(\partial K) \geq \Phi^{-1}(a) I(a).$$

The second inequality uses Proposition 6.2 and Gaussian isoperimetry, and is valid because $\Phi^{-1}(a) \geq 0$. Approximation handles nonsmooth sets and $a = 1/2$.

Apply this local bound to tK, and let $b(t) = \gamma(tK)$ and $q(t) = \Phi^{-1}(b(t))$. Since $0 \in K$, $b(t) \geq a \geq 1/2$ for $t \geq 1$. We obtain

$$b'(t) = \frac{D(tK)}{t} \geq \frac{q(t) I(b(t))}{t}, \qquad q'(t) \geq \frac{q(t)}{t}.$$

Therefore $q(t)/t$ is nondecreasing, so $q(t) \geq tq(1)$. Applying Φ proves the result. □

Remark 6.24. *We have obtained two expressions for the same derivative:*

$$\left.\frac{d}{dt}\gamma(tK)\right|_{t=1} = n\gamma(K) - \int_K |x|^2\, d\gamma = \int_{\partial K} \langle x, n_x \rangle \, d\gamma_{\partial K}.$$

They can also be equated by the divergence theorem. The dilation calculation is useful when one wishes to avoid a direct integration by parts with boundary.

Remark 6.25. *The conclusion of Theorem 6.14 is also true when $\gamma(K) < 1/2$. A convenient way to see this uses the broader Ehrhard–Borell inequality: for $\lambda, \mu \geq 0$ with $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$,*

$$\Phi^{-1}(\gamma(\lambda A + \mu B)) \geq \lambda \Phi^{-1}(\gamma(A)) + \mu \Phi^{-1}(\gamma(B)).$$

See [83] for this formulation and its equality cases. Take $A = B = K$ and $\lambda = \mu = t/2$. Convexity gives $(t/2)K + (t/2)K = tK$, proving

$$\Phi^{-1}(\gamma(tK)) \geq t \Phi^{-1}(\gamma(K))$$

for every $t \geq 1$, without a restriction on $\gamma(K)$.

6.15 Gaussian measure of dilates of symmetric convex sets: the S-inequality of R. Latała and K. Oleszkiewicz

We ask the same question, now restricting to *symmetric convex sets*: which such set minimizes $\gamma(tK)$ for $t \geq 1$, when $\gamma(K)$ is fixed? The answer is a symmetric strip, by the theorem of Latała and Oleszkiewicz [127].

Theorem 6.15 (Latała–Oleszkiewicz, S-inequality)**.** *Let K be symmetric and convex with $0 < \gamma(K) < 1$, and let*

$$S_K = \{x : |\langle x, \theta \rangle| \leq p\}$$

be a symmetric strip with $\gamma(S_K) = \gamma(K)$. Then

$$\gamma(tK) \geq \gamma(tS_K) \quad (t \geq 1), \qquad \gamma(tK) \leq \gamma(tS_K) \quad (0 < t \leq 1).$$

Corollary 6.9. *Under the same assumptions,*

$$\int_K |x|^2 \, d\gamma \leq \int_{S_K} |x|^2 \, d\gamma.$$

Proof. Differentiate the dilation comparison at $t = 1$ and use the formula for $D(K)$. □

The key estimate proved in [127] is:

Theorem 6.16 (Latała–Oleszkiewicz)**.** *For a symmetric convex set K and a symmetric strip S_K of the same Gaussian measure,*

$$r(K)\gamma^+(\partial K) \geq r(S_K)\gamma^+(\partial S_K).$$

Remark 6.26. *This statement is substantially stronger than the elementary in-radius comparison. Although $r(K) \geq r(S_K)$, one does* not *in general have*

$$\gamma^+(\partial K) \geq \gamma^+(\partial S_K).$$

The product of radius and perimeter is the quantity that admits the sharp comparison.

Let us mention a related conjecture.

Conjecture 6.2 (Morgan; see Heilman [85])**.** *For an origin-symmetric measurable set K with $\gamma(K) = a$,*

$$\gamma^+(\partial K) \geq \min_{1\leq k\leq n} \left\{\gamma^+(\partial C_k(a)), \gamma^+(\partial C_k(1-a))\right\},$$

where $C_k(b) = R_k(b)B_2^k \times \mathbb{R}^{n-k}$ is a centered round k-cylinder of Gaussian measure b. The second candidate is the complement of $C_k(1-a)$, whose measure is a and whose perimeter is the same as that of $C_k(1-a)$.

For a sufficiently close to 1, symmetric strips are known to minimize perimeter among symmetric sets; see Barchiesi and Julin [14].

Proof of the S-inequality from Theorem 6.16. As above,

$$D(K) \geq r(K)\gamma^+(\partial K) \geq r(S_K)\gamma^+(\partial S_K) = D(S_K).$$

To integrate this comparison correctly, put

$$\Psi(p) = \gamma_1([-p,p]) = 2\Phi(p) - 1, \qquad p(t) = \Psi^{-1}(\gamma(tK)).$$

At each $t > 0$, apply the local comparison to tK and the strip of measure $\gamma(tK)$. Since the strip derivative is $p\Psi'(p)$, we get

$$\frac{d}{dt}\gamma(tK) \geq \frac{p(t)\Psi'(p(t))}{t}, \qquad p'(t) \geq \frac{p(t)}{t}.$$

Thus $p(t)/t$ is nondecreasing. Comparing with $t = 1$ gives both directions of the S-inequality. □

The key estimate reduces to a precise two-dimensional statement. The parameter in this reduction is a *strip half-width*; it is not the centered in-radius of the symmetrized planar set.

Lemma 6.8 (The planar estimate of Latała–Oleszkiewicz)**.** *Let $w > 0$ and*

$$A = \{(x,y) : |x| < w,\ y \leq f(x)\},$$

where $f : (-w, w) \to \mathbb{R}$ is smooth, even, and concave, with $f(x) \to -\infty$ as $x \uparrow w$. If S is a symmetric strip of the same Gaussian measure, with half-width p, then

$$w\,\gamma_2^+(\partial A) \geq p\,\gamma_2^+(\partial S) = 2p\varphi(p).$$

The limiting versions needed for nonsmooth concave f follow by approximation.

Reduction of Theorem 6.16 to Lemma 6.8. In dimension one the assertion is immediate, so assume $n \geq 2$. Let $w = r(K)$, and rotate so that $K \subset \{|x_1| \leq w\}$. For $|s| < w$, let

$$K_s = \{y \in \mathbb{R}^{n-1} : (s, y) \in K\}, \qquad f(s) = \Phi^{-1}(\gamma_{n-1}(K_s)).$$

Convexity of K and Ehrhard's inequality make f concave. Symmetry of K makes f even. The planar symmetral

$$A = \{(s, y) : |s| < w,\ y \leq f(s)\}$$

has $\gamma_2(A) = \gamma_n(K)$. Gaussian symmetrization decreases perimeter, so

$$\gamma_2^+(\partial A) \leq \gamma_n^+(\partial K).$$

After the regular approximation allowed in Lemma 6.8, that lemma gives

$$r(K)\gamma_n^+(\partial K) \geq w\gamma_2^+(\partial A) \geq r(S_K)\gamma_n^+(\partial S_K).$$

Notice that no comparison with the centered in-radius of A is needed; A may not even contain the origin. □

Once the proof scheme is in place, the S-inequality follows from this planar lemma. The lemma itself is the difficult step: the original proof contains a substantial sequence of analytic estimates and explicit numerical bounds. We do not reproduce those estimates here, and refer the interested reader to [127].

6.16 Home work

Question 6.1 (3 points)**.** *Find an alternative proof of Bobkov's inequality by approximating Gaussian measure with normalized sums of coordinates on the uniform Hamming cube. Specify the discrete inequality and the normalization used in the limiting argument.*

Question 6.2 (1 point)**.** *Verify that*

$$\frac{\Phi^{-1}(\gamma_n(tB_2^n))}{t} \longrightarrow 1 \qquad (t \to \infty).$$

Recall that we used this fact to deduce Gaussian isoperimetry from Ehrhard's inequality.

Question 6.3 (1 point)**.** *Deduce Gaussian isoperimetry directly from Bobkov's inequality, using smooth approximations of indicators.*

Question 6.4 (2 points)**.** *Prove the sharp Gaussian Kahane inequality. Let $g_1, \ldots, g_m$ be independent $N(0,1)$ random variables, let $z_1, \ldots, z_m \in \mathbb{R}^n$, and let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. For $q \geq p > 0$, show that*

$$\left(\mathbb{E}\left\|\sum_{i=1}^m g_i z_i\right\|^q\right)^{1/q} \leq \frac{\alpha_q}{\alpha_p}\left(\mathbb{E}\left\|\sum_{i=1}^m g_i z_i\right\|^p\right)^{1/p}, \qquad \alpha_p = (\mathbb{E}|g_1|^p)^{1/p}.$$

Question 6.5 (2 points)**.** *Let $S = S(L, e)$ be a Gaussian symmetrization. Prove the following properties, with consistent boundary conventions:*

- *$\gamma(S(A)) = \gamma(A)$ for Borel A;*
- *$A \subset B$ implies $S(A) \subset S(B)$;*
- *$S(A + v) = S(A) + v$ for $v \in L$;*
- *if $A_1 \subset A_2 \subset \cdots$ and $A = \bigcup_{i=1}^{\infty} A_i$, then $S(A) = \bigcup_{i=1}^{\infty} S(A_i)$ up to fiber boundary sets, and exactly with the open-ray convention.*

Explain why the translation identity cannot be claimed for arbitrary $v \in \mathbb{R}^n$.

Question 6.6 (1 point)**.** *Let L_1, L_2 be linear subspaces such that $(L_1 \cap L_2)^{\perp} \cap L_1$ and $(L_1 \cap L_2)^{\perp} \cap L_2$ are orthogonal. For a unit vector $e \in L_1^{\perp} \cap L_2^{\perp}$, prove*

$$S(L_1, e) \circ S(L_2, e) = S(L_2, e) \circ S(L_1, e) = S(L_1 \cap L_2, e).$$

Question 6.7 (1 point)**.** *Let $n \geq 3$ and $2 \leq k \leq n$. Show that every k-symmetrization can be written as a composition of $k - 1$ two-dimensional symmetrizations:*

$$S = S_1 \circ \cdots \circ S_{k-1}.$$

Hint: use Question 6.6 and an orthonormal basis of the fiber containing the distinguished direction e.

Question 6.8 (1 point)**.** *Establish the planar limiting symmetrization step described in Lemma 6.4 for closed sets: choose suitable directions so that successive one-dimensional Gaussian symmetrals converge locally, and hence in Gaussian measure, to a half-space of the same measure. Explain the regular approximation needed when passing to arbitrary measurable sets and perimeter. Global Hausdorff convergence is not asserted for arbitrary sets.*

Question 6.9 (2 points)**.** *Prove, for $\varepsilon > 0$ and a Gaussian symmetrization S,*

$$S(A) + \varepsilon B_2^n \subset S(A + \varepsilon B_2^n),$$

first for regular sets and then with the appropriate closure convention. Conclude that Gaussian symmetrization decreases Gaussian perimeter. Together with Questions 6.8 and 6.7, explain the symmetrization proof of Gaussian isoperimetry, without using the full Gaussian isoperimetric theorem as an input to this proof.

Question 6.10 (2 points)**.** *Prove that Gaussian symmetrization preserves convexity. Recall that this is the substantive geometric step in the symmetrization proof of Ehrhard's inequality; organize the argument so that it does not assume the very inequality it is intended to prove.*

Question 6.11 (2 points)**.** *Find lower estimates for the isoperimetric profiles of some product probability measures of your choice, beyond the uniform measure on the cube and Gaussian measure.*

Question 6.12 (4 points)**.** *Solve the relative isoperimetric problem in the square. For $|A \cap [0,1]^2| = a \in (0,1)$, prove that the least relative perimeter in $(0,1)^2$ is attained by an appropriate quarter-disk centered at a vertex, a strip cut off by a line parallel to an edge, or the complement of a quarter-disk. Identify the ranges of a in which the candidates minimize the perimeter.*

Question 6.13 (3 points)**.** *Let L be a convex body. Find a lower estimate for the anisotropic Gaussian perimeter of a set A with $\gamma(A) = a$:*

$$\liminf_{\varepsilon \downarrow 0} \frac{\gamma(A + \varepsilon L) - \gamma(A)}{\varepsilon}.$$

Begin with $0 \in \operatorname{int} L$, so that these are genuine enlargements, and discuss what changes for general L. For which L and A is your estimate sharp?

Question 6.14 (2 points)**.** *Prove the following special case of Gaussian correlation, called Šidák's lemma: if K, L are symmetric strips, then*

$$\gamma(K \cap L) \geq \gamma(K)\gamma(L).$$

Hint: use Prékopa–Leindler to analyze the Gaussian measures of the relevant sections.

Question 6.15 (1 point)**.** *Prove the Gaussian log-Sobolev inequality by a suitable limiting linearization of Bobkov's inequality. Specify the substitution and limiting regime; a perturbation around a fixed constant gives Poincaré rather than log-Sobolev.*

Question 6.16 (1 point)**.** *Show that functional Ehrhard tensorizes: knowing it in dimensions k and m, deduce it in dimension $k + m$.*

Question 6.17 (5 points)**.** *Find a proof of the one-dimensional functional Ehrhard inequality that does not use the geometric Ehrhard inequality.*

Question 6.18 (1 point)**.** *For η defined in (39), prove that $\eta(a) > -1$ for $0 < a < 1$, and determine its limits as $a \downarrow 0$ and $a \uparrow 1$. Explain why η is not confined to $[-1, 0]$ when $a > 1/2$.*

Question 6.19 (3 points)**.** *For a convex set K with $a = \gamma(K) \in (0,1)$, prove again that*

$$\frac{1}{a} \int_K \langle x, \theta \rangle^2 \, d\gamma + \frac{\eta(a)}{a^2} \left(\int_K \langle x, \theta \rangle \, d\gamma \right)^2 \leq 1.$$

Compare the proof by differentiating Ehrhard along translations with a proof using one of its functional consequences, such as generalized Bobkov or the Ehrhard–Brascamp–Lieb inequality.

Question 6.20 (1 point)**.** *Extend the conclusion of Theorem 6.14 to $\gamma(K) \leq 1/2$:*

$$\gamma(tK) \geq \gamma(tH) \qquad (t \geq 1),$$

where K is convex and H is a half-space of the same Gaussian measure. Hint: use the broader coefficient form of Ehrhard–Borell stated after that theorem, taking the same convex set in both slots.

7 Hörmander's L^2 method

7.1 The Bochner formula and its extension

Let $d\mu = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^n$, where V is convex, and let

$$Lu = \Delta u - \langle \nabla u, \nabla V \rangle.$$

Recall the integration by parts formula

$$\int f Lu \, d\mu = -\int \langle \nabla f, \nabla u \rangle \, d\mu.$$

We use V for the potential and f for a test function, so that these two roles are not confused.

Remark 7.1 (A convention about regularity)**.** *For the sake of presentation, we first perform the computations with smooth potentials and smooth functions for which all integrations by parts and differentiations are justified; compact support is a convenient sufficient condition. Cutoffs, mollification, and, where an inverse Hessian occurs, a small strictly convex perturbation give the corresponding extensions when the relevant energies are finite. On a convex domain one must either use the appropriate Neumann boundary condition or approximate the restriction by smooth convex potentials on the whole space. We will mention a boundary term when it is important to the argument.*

The following classical "double integration by parts" formula is particularly useful.

Theorem 7.1 (Bochner–Lichnerowicz)**.** *Under the preceding regularity convention,*

$$\int (Lu)^2 \, d\mu = \int \left(\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u \rangle \right) d\mu.$$

Here $\nabla^2 u = (\partial_i \partial_j u)_{ij}$ is the Hessian and $\|A\|_{HS}^2 = \sum_{i,j} a_{ij}^2$ is the squared Hilbert–Schmidt norm.

Standard proof. The commutation formula is

$$\nabla(Lu) = L\nabla u - \nabla^2 V \nabla u, \qquad L\nabla u = (L\partial_1 u, \dots, L\partial_n u)^T.$$

Consequently,

$$\begin{aligned} \int (Lu)^2 \, d\mu &= -\int \langle \nabla u, \nabla Lu \rangle \, d\mu \\ &= \int \langle \nabla^2 V \nabla u, \nabla u \rangle \, d\mu - \sum_{i=1}^n \int \partial_i u \, L \partial_i u \, d\mu \\ &= \int \langle \nabla^2 V \nabla u, \nabla u \rangle \, d\mu + \sum_{i=1}^n \int |\nabla \partial_i u|^2 \, d\mu, \end{aligned}$$

which is the desired identity. □

There is also a useful weighted version.

Theorem 7.2. *Under the same convention,*

$$\int f\left((Lu)^2 - \|\nabla^2 u\|_{HS}^2 - \langle \nabla^2 V \nabla u, \nabla u\rangle\right) d\mu$$
$$= \int \left\langle \nabla^2 u \nabla u - (Lu)\nabla u, \nabla f \right\rangle d\mu.$$

Remark 7.2. *When $f \equiv 1$, this is Bochner's formula. Notice that for nonconstant f there are two terms on the right; the term involving Lu cannot be discarded.*

Non-standard proof. We use the same idea as in our differentiation proof of integration by parts. Set

$$\alpha(t) = \int f(y + t\nabla u(y)) e^{-V(y+t\nabla u(y))} \det(\mathrm{Id} + t\nabla^2 u(y))\, dy.$$

For sufficiently small t, the change of variables $x = y + t\nabla u(y)$ shows that $\alpha(t) = \int f\, d\mu$ is independent of t. Its first derivative at zero gives

$$0 = \int \Big(\langle \nabla f, \nabla u\rangle + fLu\Big) d\mu.$$

For the second derivative, use

$$\det(\mathrm{Id} + t\nabla^2 u) = 1 + t\Delta u + \frac{t^2}{2}\Big((\Delta u)^2 - \|\nabla^2 u\|_{HS}^2\Big) + o(t^2).$$

Expanding all three factors yields

$$0 = \alpha''(0) = \int \Big[\langle \nabla^2 f \nabla u, \nabla u\rangle + 2(Lu)\langle \nabla f, \nabla u\rangle$$
$$+ f\Big((Lu)^2 - \|\nabla^2 u\|_{HS}^2 - \langle \nabla^2 V \nabla u, \nabla u\rangle\Big)\Big] d\mu.$$

Finally, one integration by parts gives

$$\int \langle \nabla^2 f \nabla u, \nabla u\rangle d\mu = -\int \left\langle \nabla f, (Lu)\nabla u + \nabla^2 u \nabla u \right\rangle d\mu.$$

Substitution proves the formula. The determinant expansion also follows from $\frac{d}{dt}\det A(t) = \det A(t)\,\mathrm{tr}(A(t)^{-1}A'(t))$. □

7.2 Sobolev spaces

The space $W^{1,2}(\mu)$ consists of the functions $f \in L^2(\mu)$ whose weak first derivatives belong to $L^2(\mu)$, with norm

$$\|f\|_{W^{1,2}(\mu)}^2 = \int f^2\, d\mu + \int |\nabla f|^2\, d\mu.$$

The quantity

$$\|f\|_{H^1(\mu)} = \left(\int |\nabla f|^2 \, d\mu \right)^{1/2}$$

is only a seminorm on the whole space, because it vanishes on constants. It is a norm on the mean-zero subspace, or on the quotient by constants. The Poincaré inequality makes it equivalent to the usual Sobolev norm on that subspace.

More generally, $W^{k,p}(\mu)$ consists of functions whose weak derivatives of orders up to k belong to $L^p(\mu)$. These are weak derivatives, rather than necessarily classical derivatives.

It is convenient to define the domain of L by integration by parts: $u \in D(L)$ if $u \in W^{1,2}(\mu)$ and there is an $h \in L^2(\mu)$ such that

$$\int h\varphi \, d\mu = -\int \langle \nabla u, \nabla \varphi \rangle d\mu \quad \text{for all } \varphi \in W^{1,2}(\mu).$$

We then write $Lu = h$. In the smooth case this agrees with the differential expression above. A general $W^{2,2}$ function need not automatically lie in $D(L)$, since the drift $\langle \nabla V, \nabla u \rangle$ must also be controlled.

The upshot is that we can work with weak solutions and use integration by parts without assuming classical differentiability everywhere. Bochner's identity extends on its natural operator domain by approximation, with the energies on the right understood accordingly. See Evans [64] for the general Sobolev and elliptic-regularity background.

7.3 Density of the image of L in $W^{1,2}(\mu)$

We now explain why a mean-zero function can be represented, or approximated, by Lu. This is the analytic input needed for the L^2 method.

Theorem 7.3. *Suppose μ satisfies a Poincaré inequality. For every $g \in L^2(\mu)$ with $\int g \, d\mu = 0$, there is a unique mean-zero weak solution $u \in D(L)$ of $Lu = g$. In particular, mean-zero $g \in W^{1,2}(\mu)$ can be treated by this representation; if smooth solutions are desired for a calculation, one works first with suitable regularized data and then passes to the limit in the relevant energies.*

Remark 7.3. *The condition $\int g \, d\mu = 0$ is essential: integration by parts with the constant test function gives $\int Lu \, d\mu = 0$.*

Lemma 7.1 (Lax–Milgram)**.** *Let H be a Hilbert space, Q a symmetric bilinear form, and ℓ a linear functional. Suppose that*

$$|Q(f,g)| \le C_1 \|f\| \|g\|, \qquad Q(f,f) \ge C_2 \|f\|^2, \qquad |\ell(f)| \le C_3 \|f\|,$$

where $C_2 > 0$. Then there is a unique $u \in H$ such that $Q(f,u) = \ell(f)$ for every $f \in H$.

Proof of theorem 7.3. Take

$$H = \left\{ f \in W^{1,2}(\mu) : \int f \, d\mu = 0 \right\}, \qquad Q(f,u) = \int \langle \nabla f, \nabla u \rangle d\mu,$$

with the H^1 norm. Poincaré makes H complete and gives

$$|\ell(f)| := \left| \int f g \, d\mu \right| \le \sqrt{C_{\mathrm{Poin}}(\mu)} \, \|g\|_{L^2(\mu)} \|f\|_{H^1(\mu)}.$$

Apply Lax–Milgram with $\ell(f) = -\int f g \, d\mu$. The resulting identity

$$\int \langle \nabla f, \nabla u \rangle d\mu = -\int f g \, d\mu$$

is exactly $Lu = g$ in the weak sense. Constants can be added to the test space because g has mean zero. Smoothness for smooth data follows locally from elliptic regularity; global energy estimates and approximation justify the subsequent uses of Bochner's identity. □

An alternative treatment of the range and approximation issues appears in Cordero-Erausquin, Fradelizi, and Maurey [52].

7.4 Review of dual norms

Definition 7.1. For a Banach space $(B, \|\cdot\|)$ and a continuous linear functional $\ell \in B^*$, its dual norm is

$$\|\ell\|_* := \sup_{\|y\| \le 1} |\ell(y)|.$$

When a pairing identifies functionals with vectors or functions, we use that identification in the notation.

Some examples:

1. In $\mathbb{R}^n$, if $\|\cdot\| = \|\cdot\|_K$ for an origin-symmetric convex body, then the dual norm is $\|\cdot\|_{K^\circ}$.

2. For $1 < p < \infty$ and $1/p + 1/q = 1$, Hölder's inequality gives

$$\sup_{\|g\|_p \le 1} \left| \int f g \, d\mu \right| = \|f\|_q.$$

   For nonzero f, equality is attained by a normalized multiple of $\operatorname{sgn}(f)|f|^{q-1}$. This identifies $(L^p(\mu))^*$ with $L^q(\mu)$. The usual L^1–L^∞ duality holds as well; the full dual of L^∞ is in general larger than L^1.

3. For vector fields $F = (f_1, \dots, f_n)$ and a symmetric positive definite matrix field $A(x)$, the norm
$$\|F\|^2 = \int \langle AF, F\rangle d\mu$$
on the corresponding weighted L^2 space has dual, with respect to the pairing $\int \langle F, G\rangle d\mu$,
$$\|G\|_*^2 = \int \langle A^{-1}G, G\rangle d\mu.$$
The pointwise matrix Cauchy–Schwarz inequality proves the upper bound, and F proportional to $A^{-1}G$ gives equality whenever the energy is finite.

7.5 The H^{-1} norm

Recall the homogeneous norm $\|u\|_{H^1(\mu)}^2 = \int |\nabla u|^2 d\mu$. Its dual is
$$\|f\|_{H^{-1}(\mu)} := \sup\left\{ \int u f \, d\mu : \int |\nabla u|^2 d\mu \le 1 \right\}.$$

Remark 7.4. *This is finite only if $\int f \, d\mu = 0$. Otherwise constant test functions of arbitrarily large magnitude make the supremum infinite. We therefore consider mean-zero f, or explicitly allow the value $+\infty$.*

Remark 7.5. *If $Lv = f$, then*
$$\int f g \, d\mu = -\int \langle \nabla v, \nabla g\rangle d\mu \le \|v\|_{H^1(\mu)} \|g\|_{H^1(\mu)}.$$
The supremum is attained by $g = -v/\|v\|_{H^1(\mu)}$ when the denominator is nonzero. Thus
$$\|f\|_{H^{-1}(\mu)} = \|v\|_{H^1(\mu)} \qquad \text{when } Lv = f.$$
For a log-concave probability measure, the preceding weak-solution argument supplies this representation for mean-zero L^2 data.

The H^{-1} norm also has a geometric meaning. The following relation to mass transport can be found in Villani [175]; the proof below follows the appendix of Klartag [105].

Theorem 7.4 (Relation of the H^{-1} norm to mass transport)**.** *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$, and let h be bounded with $\int h \, d\mu = 0$. For sufficiently small $\varepsilon > 0$, define $d\mu_\varepsilon = (1 + \varepsilon h) d\mu$. With the H^{-1} seminorm defined using smooth test functions,*
$$\|h\|_{H^{-1}(\mu)} = \liminf_{\varepsilon \downarrow 0} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon},$$
with both sides allowed to be infinite.

Recall that $W_2(\mu,\nu)^2 = \inf_{\pi\in\Gamma(\mu,\nu)} \int |x-y|^2 d\pi(x,y)$.

Proof. We outline the $\le$ direction; the opposite direction is left as homework. Fix a smooth compactly supported φ. Taylor's theorem gives

$$\varphi(y) - \varphi(x) \le \langle \nabla\varphi(x), y-x\rangle + R|x-y|^2.$$

For every coupling π of μ and μ_ε,

$$\begin{aligned}\int h\varphi \, d\mu &= \frac{1}{\varepsilon}\int \Big(\varphi(y)-\varphi(x)\Big) d\pi(x,y) \\ &\le \frac{1}{\varepsilon}\left(\int |\nabla\varphi|^2 d\mu\right)^{1/2}\left(\int |x-y|^2 d\pi\right)^{1/2} + \frac{R}{\varepsilon}\int |x-y|^2 d\pi.\end{aligned}$$

Taking an optimal coupling gives

$$\int h\varphi \, d\mu \le \|\varphi\|_{H^1(\mu)} \frac{W_2(\mu,\mu_\varepsilon)}{\varepsilon} + R\frac{W_2(\mu,\mu_\varepsilon)^2}{\varepsilon}.$$

If the liminf in the theorem is infinite, there is nothing to prove. Otherwise, along a sequence realizing a finite liminf, $W_2(\mu,\mu_\varepsilon) = O(\varepsilon)$, so the last term tends to zero. Taking the supremum over φ with H^1 norm at most one proves the assertion. This argument does not require the remainder to vanish for every measure along every sequence. □

7.6 The Hörmander duality lemma

We say that a norm N_1 dominates a norm N_2 if convergence to zero in N_1 implies convergence to zero in N_2. This condition is useful when passing from a dense class to the desired class of functions.

Lemma 7.2. *Let μ be a log-concave probability measure, and let N be a norm on vector fields, with dual norm N_* for the pairing $\int\langle F,G\rangle d\mu$. Suppose*

$$\int (Lu)^2 d\mu \ge N(\nabla u)^2$$

for all functions in a suitable operator core. Assume that the approximation needed to solve $Lu = f - \int f\,d\mu$ is valid in the corresponding energies. Then

$$\mathrm{Var}_\mu(f) \le N_*(\nabla f)^2. \tag{42}$$

For example, approximation in $W^{1,2}$ suffices if $N_*(\nabla f)$ is continuous in that topology.

Proof. Subtract the mean of f and first suppose $Lu = f$. Integration by parts and the assumption give

$$\begin{aligned}\int f^2 d\mu &= 2\int f Lu\, d\mu - \int (Lu)^2 d\mu \\ &= -2\int \langle \nabla f, \nabla u\rangle d\mu - \int (Lu)^2 d\mu \\ &\le 2N_*(\nabla f) N(\nabla u) - N(\nabla u)^2 \\ &\le N_*(\nabla f)^2.\end{aligned}$$

The last step is $2ab - b^2 \le a^2$. The general case follows by the stated approximation. Notice that the dual pairing here is the integral pairing of vector fields, not a pointwise scalar product. □

Remark 7.6. *The same proof works for any pair of nonnegative functionals satisfying*

$$N(F)^2 + N_*(G)^2 \ge 2\left|\int \langle F, G\rangle d\mu\right|. \tag{43}$$

This is useful for the homogeneous Sobolev seminorm, whose dual may be infinite on fields with nonzero mean.

Remark 7.7 (When does the converse hold?)**.** *For a converse, the space on which we take the dual matters. Let $\mathcal{G}$ be the closure of the smooth gradient fields in $L^2(\mu;\mathbb{R}^n)$, and suppose that N is a norm on $\mathcal{G}$ equivalent to its L^2 norm. Define its dual* on the same space *by*

$$N^\dagger(G) = \sup_{F\in\mathcal{G},\, N(F)\le 1} \left|\int \langle F, G\rangle d\mu\right|, \qquad G \in \mathcal{G}.$$

With this duality, the two inequalities

$$\int (Lu)^2 d\mu \ge N(\nabla u)^2, \qquad \mathrm{Var}_\mu(f) \le N^\dagger(\nabla f)^2$$

are equivalent. For the reverse direction, use $|\int \langle \nabla u, \nabla f\rangle d\mu| \le \|Lu\|_2 \sqrt{\mathrm{Var}_\mu(f)}$ and then dualize on $\mathcal{G}$; the details are left as homework. The converse need not hold for an arbitrary norm whose dual is taken over all *vector fields: fields orthogonal to gradients can change that dual norm without changing the integration-by-parts pairing. The forward implication in the lemma has no such restriction.*

Bochner's formula gives two immediate choices. Convexity of V implies

$$\int (Lu)^2 d\mu \ge \int \|\nabla^2 u\|_{HS}^2 d\mu = \sum_{i=1}^n \|\partial_i u\|_{H^1(\mu)}^2, \tag{44}$$

and also

$$\int (Lu)^2 d\mu \ge \int \langle \nabla^2 V \nabla u, \nabla u\rangle d\mu. \tag{45}$$

We discuss the consequences of these two estimates in turn.

7.7 Klartag's H^{-1} inequality

Combining (44) with Hörmander's argument gives the following result.

Corollary 7.1 (Klartag [105]; see also Barthe–Klartag [19])**.** *For $f \in W^{1,2}(\mu)$,*

$$\mathrm{Var}_\mu(f) \le \sum_{i=1}^n \|\partial_i f\|^2_{H^{-1}(\mu)} =: \|\nabla f\|^2_{H^{-1}(\mu)}.$$

The right-hand side is informative when $\int \partial_i f \, d\mu = 0$ for each i; otherwise it is infinite. Even functions under even measures satisfy these centering conditions.

Corollary 7.2. *Under the hypotheses of the transport formula, and assuming that each $\partial_i f$ is bounded and has mean zero,*

$$\mathrm{Var}_\mu(f) \le \liminf_{\varepsilon \downarrow 0} \sum_{i=1}^n \left(\frac{W_2(\mu, (1+\varepsilon\partial_i f)\mu)}{\varepsilon} \right)^2.$$

Extensions beyond bounded derivatives and compact support require the corresponding integrability and approximation assumptions.

Suppose one can construct, for the relevant mean-zero perturbations ψ, maps $T_{\varepsilon\#}\mu = (1+\varepsilon\psi)\mu$ such that

$$\int |T_\varepsilon x - x|^2 d\mu(x) \le C(\mu)\varepsilon^2 \int \psi^2 d\mu.$$

Then the preceding corollary gives

$$\mathrm{Var}_\mu(f) \le C(\mu) \int |\nabla f|^2 d\mu$$

for the functions whose derivatives are covered by that construction. The factor ε^2 is essential.

Using these ideas and the unconditional symmetries, Klartag [105] obtained the following bound.

Theorem 7.5 (Klartag)**.** *If μ is the uniform probability measure on an* isotropic *unconditional convex body in $\mathbb{R}^n$, then*

$$\mathrm{Var}_\mu(f) \le C \log^2(en) \int |\nabla f|^2 d\mu$$

for all $f \in W^{1,2}(\mu)$, with an absolute constant C.

The same estimate holds for isotropic unconditional log-concave densities; see Barthe and Cordero-Erausquin [17, Corollary 14]. The later logarithmic variance bound discussed in section 5.7 applies to all isotropic log-concave measures. Throughout, our Poincaré constant is the coefficient multiplying $\int |\nabla f|^2$, not its square root.

7.8 The L^2 proof of the Brascamp–Lieb inequality, which gives another proof of the Prékopa–Leindler and Brunn–Minkowski inequalities

Combining (45) with Hörmander's lemma and the matrix-weighted dual norm gives another proof of theorem 3.15.

Corollary 7.3 (Brascamp–Lieb, 1976)**.** *If $\nabla^2 V$ is positive definite, then*

$$\mathrm{Var}_\mu(f) \le \int \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu.$$

Recall the linearization discussed earlier: the same inequality expresses the concavity of $\log \int e^{-(F+tG)^*} dx$ along admissible Legendre interpolations. Integrating this local inequality gives Prékopa's theorem, and hence the Brunn–Minkowski inequality for convex bodies. The arbitrary-function form of Prékopa–Leindler is more general; its proof was given earlier.

Kolesnikov and Milman [118] gave a direct L^2 proof of Brunn–Minkowski, without passing through Prékopa–Leindler, in a more general Riemannian setting. On spaces with boundary, the method includes boundary terms, and these create additional geometric information. We refer to Kolesnikov and Milman [116, 117, 118, 119, 120], Kolesnikov and Livshyts [121, 122], and Klartag [105].

The weighted Bochner identity also leads to the following extension, whose proof is left as homework.

Theorem 7.6 (Weighted Brascamp–Lieb, Cordero-Erausquin–Livshyts–Rotem [141])**.** *Let K be convex with $\mu(K) > 0$, let $d\mu = e^{-V} dx$ be a log-concave probability measure with $\nabla^2 V$ positive definite, and let $g \ge 0$ be a finite concave function on K. Then every sufficiently regular f satisfying $\int_K f \, d\mu = 0$ obeys*

$$\int_K g f^2 d\mu \le \int_K g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu.$$

The centering is with respect to $\mu|_K$, not $g\mu|_K$. Thus for an arbitrary f, the left-hand side is $\int_K g(f - \fint_K f \, d\mu)^2 d\mu$. When $0 < \int_K g \, d\mu < \infty$, this implies the weighted-variance estimate

$$\int_K g f^2 d\mu - \frac{(\int_K g f \, d\mu)^2}{\int_K g \, d\mu} \le \int_K g \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\mu,$$

which follows directly by applying ordinary Brascamp–Lieb to the measure proportional to $g \mathbf{1}_K \, d\mu$. Indeed, the weighted mean minimizes $\int_K g(f-a)^2 d\mu$ over constants a. For the proof of the stronger statement, work first on a smooth bounded convex domain with smooth g and solve $Lu = f$ with zero normal derivative on the boundary. The mean-zero condition is precisely the compatibility condition for this Neumann problem. Concavity of g and convexity of the boundary give the required signs in the weighted Bochner formula. Approximation and exhaustion extend the result to the stated setting; a degenerate Hessian can be treated by a strictly convex regularization.

7.9 The L^2 proof of the symmetric Gaussian Poincaré inequality restricted to a symmetric convex set

Earlier, we obtained the symmetric Gaussian Poincaré inequality by linearizing the Blaschke–Santaló inequality, and extended it to symmetric convex sets using Caffarelli's contraction theorem; see section 4.11. We now give a proof using the L^2 method, following Cordero-Erausquin, Fradelizi and Maurey [52].

First, we need the uniqueness of solutions to our equation.

Lemma 7.3. *If $Lu = f$ has a weak solution in $W^{1,2}(\mu)$, it is unique up to addition of a constant.*

Proof. If $Lu = Lv$, let $h = u - v$. Testing $Lh = 0$ against h gives

$$0 = \int h Lh \, d\mu = -\int |\nabla h|^2 d\mu.$$

The Poincaré inequality then gives $\mathrm{Var}_\mu(h) = 0$, so h is constant almost everywhere. □

Corollary 7.4. *If μ and f are even and $Lu = f$, then u is even.*

Proof. The function $v(x) = u(-x)$ also solves $Lv = f$. By uniqueness, $u(x) - u(-x) = c$ almost everywhere. Replacing x by $-x$ shows that $c = -c$, hence $c = 0$. □

Corollary 7.5. *Let K be a symmetric convex set of positive Gaussian measure. For every even $f \in W^{1,2}(\gamma|_K)$,*

$$\fint_K f^2 d\gamma - \left(\fint_K f \, d\gamma\right)^2 \le \frac{1}{2} \fint_K |\nabla f|^2 d\gamma, \qquad \fint_K F \, d\gamma := \frac{1}{\gamma(K)} \int_K F \, d\gamma.$$

Proof. First consider $d\mu = e^{-w} d\gamma$, where w is smooth, even and convex, and μ is a probability measure. Bochner's formula gives

$$\begin{aligned} \int (Lu)^2 d\mu &= \int \|\nabla^2 u\|_{HS}^2 + \langle (\nabla^2 w + \mathrm{Id}) \nabla u, \nabla u \rangle d\mu \\ &\ge \int \|\nabla^2 u\|_{HS}^2 d\mu + \int |\nabla u|^2 d\mu. \end{aligned}$$

For even f, we may solve $Lu = f - \int f \, d\mu$ with u even. Its partial derivatives are odd, so $\int \partial_i u \, d\mu = 0$. The ordinary Poincaré inequality for μ, whose potential has Hessian at least Id, gives

$$\int \|\nabla^2 u\|_{HS}^2 d\mu = \sum_i \int |\nabla \partial_i u|^2 d\mu \ge \sum_i \int (\partial_i u)^2 d\mu = \int |\nabla u|^2 d\mu.$$

Consequently,

$$\int (Lu)^2 d\mu \ge 2 \int |\nabla u|^2 d\mu.$$

Apply the proof of Hörmander's lemma on the even subspace, with $\|F\| = \sqrt{2} \|F\|_{L^2(\mu)}$. Its dual is $\|F\|_* = \|F\|_{L^2(\mu)}/\sqrt{2}$, giving $\mathrm{Var}_\mu(f) \le \frac{1}{2} \int |\nabla f|^2 d\mu$. Finally approximate the normalized restriction $d\mu = \mathbf{1}_K d\gamma / \gamma(K)$ by smooth even log-concave perturbations of γ. □

7.10 Connections to the B-conjecture

Recall the B-conjecture from conjecture 4.7: for every even log-concave probability measure μ and every symmetric convex set K, the function $t \mapsto \log \mu(e^t K)$ is concave on $\mathbb{R}$. As discussed in section 4.12, its Gaussian case follows from corollary 7.5 and was proved by Cordero-Erausquin, Fradelizi and Maurey.

The L^2 method also suggests an approach to the general conjecture. We first write its local form.

Proposition 7.1. *Let $d\mu = e^{-V} dx$. At $t = 0$, the inequality $\frac{d^2}{dt^2} \log \mu(e^t K) \le 0$ is equivalent to*

$$\mathrm{Var}_{\mu|_K}(\langle \nabla V, x \rangle) \le \fint_K \Big(\langle \nabla^2 V x, x \rangle + \langle \nabla V, x \rangle \Big) d\mu. \tag{46}$$

Here and below, $\mu|_K$ denotes the normalized restriction of μ to K.

Proof. Write

$$\mu(e^t K) = e^{nt} \int_K e^{-V(e^t x)} dx.$$

The first derivative of $V(e^t x)$ at zero is $\langle \nabla V, x \rangle$, and the second is $\langle \nabla^2 V x, x \rangle + \langle \nabla V, x \rangle$. Differentiating the logarithm gives the stated variance inequality. The same formula at any t is obtained by replacing K with $e^t K$. □

The next proposed strengthening of Brascamp–Lieb would imply (46).

Conjecture 7.1. *For every smooth even convex potential V, there is a symmetric nonnegative matrix field $T = T(x)$, with $Tx = \nabla V(x)$, such that for every symmetric convex set K and every even f,*

$$\mathrm{Var}_{\mu|_K}(f) \le \fint_K \langle (\nabla^2 V + T)^{-1} \nabla f, \nabla f \rangle d\mu. \tag{47}$$

For the displayed inverse, assume first that $\nabla^2 V + T$ is positive definite; degenerate cases are understood by regularization.

Remark 7.8. *This is stronger than Brascamp–Lieb, since $(\nabla^2 V + T)^{-1} \preceq (\nabla^2 V)^{-1}$ whenever both matrices are positive definite. Nonnegative definiteness of T is the natural formulation: a strictly positive T with $Tx = \nabla V(x)$ need not exist when V has flat directions.*

Proposition 7.2. *Conjecture 7.1 implies the B-conjecture.*

Proof. Take the even function $f(x) = \langle \nabla V(x), x \rangle$. Since $Tx = \nabla V$,

$$\nabla f = \nabla^2 V x + \nabla V = (\nabla^2 V + T) x.$$

Thus (47) gives

$$\begin{aligned} \mathrm{Var}_{\mu|_K}(\langle \nabla V, x \rangle) &\le \fint_K \langle x, (\nabla^2 V + T) x \rangle d\mu \\ &= \fint_K \Big(\langle \nabla^2 V x, x \rangle + \langle \nabla V, x \rangle \Big) d\mu. \end{aligned}$$

This is (46). Applying it to every dilate $e^t K$ proves concavity on all of $\mathbb{R}$. □

Proposition 7.3 (A consequence of Hörmander's lemma)**.** *Suppose that an even log-concave measure $d\mu = e^{-V}dx$ admits a symmetric nonnegative matrix field T, with $Tx = \nabla V$, such that for every symmetric convex set K and every even smooth u,*

$$\int_K \|\nabla^2 u\|_{HS}^2 d\mu \geq \int_K \langle T\nabla u, \nabla u\rangle d\mu.$$

Then (47) *holds.*

Proof. Let $d\nu = Z^{-1}e^{-w}d\mu$, with w smooth, even and convex. The assumed Hessian estimate also holds for ν: apply it to the symmetric convex sublevel sets of w and integrate using the layer-cake formula for e^{-w}. Bochner's formula therefore yields, for even u,

$$\begin{aligned}\int (L_\nu u)^2 d\nu &= \int \|\nabla^2 u\|_{HS}^2 + \langle (\nabla^2 V + \nabla^2 w)\nabla u, \nabla u\rangle d\nu \\ &\geq \int \langle (T + \nabla^2 V)\nabla u, \nabla u\rangle d\nu.\end{aligned}$$

The solution for even data is even. Hörmander's argument and the matrix-weighted dual norm give

$$\mathrm{Var}_\nu(f) \leq \int \langle (T + \nabla^2 V)^{-1}\nabla f, \nabla f\rangle d\nu$$

for even f. Approximate the normalized restriction to K by such perturbations to conclude. □

Corollary 7.6. *Under the Hessian estimate in proposition 7.3, the B-conjecture holds for μ.*

7.11 The B-conjecture for rotation-invariant measures

We now outline the proof of Cordero-Erausquin and Rotem [51] for rotation-invariant log-concave measures.

Definition 7.2. A measure μ is rotation-invariant if $\mu(RA) = \mu(A)$ for every measurable set A and every orthogonal transformation R.

Examples include Gaussian measure, the probability measures with densities proportional to $e^{-|x|}$ or $e^{-|x|^p/p}$ for $p \geq 1$, and the uniform measure on B_2^n.

The key estimate is the following.

Theorem 7.7 (Cordero-Erausquin and Rotem, 2021)**.** *Let $d\mu = e^{-V}dx$ be a rotation-invariant log-concave probability measure, with $V(x) = v(|x|)$. For every odd g and every symmetric convex set K,*

$$\int_K |\nabla g|^2 d\mu \geq \int_K \frac{v'(|x|)}{|x|} g^2 d\mu.$$

We first interpret the formula for smooth potentials; the quotient at zero is understood by continuity when available, and its value at that point does not affect the integral.

Corollary 7.7. *The B-conjecture holds for every rotation-invariant log-concave probability measure.*

Proof. Take

$$T(x) = \frac{v'(|x|)}{|x|} \operatorname{Id}.$$

The function v is nondecreasing, so T is nonnegative, and $Tx = v'(|x|)x/|x| = \nabla V(x)$. For even u, each $\partial_i u$ is odd. Applying theorem 7.7 and summing gives

$$\int_K \|\nabla^2 u\|_{HS}^2 d\mu \geq \int_K \frac{v'(|x|)}{|x|} |\nabla u|^2 d\mu = \int_K \langle T\nabla u, \nabla u\rangle d\mu. \tag{48}$$

Now use corollary 7.6. □

We prove the key estimate using two lemmas.

Lemma 7.4 (A one-dimensional inequality)**.** *Let w, v be continuous on $[0,\infty)$ and C^1 on $(0,\infty)$. Let $f \in C^1([0,\infty))$ be compactly supported, with $f(0) = 0$. For $\alpha \geq 0$,*

$$\int_0^\infty \frac{v'(t)}{t} f(t)^2 t^\alpha e^{-w(t)-v(t)} dt \leq \int_0^\infty \left((f')^2 + \alpha \frac{f^2}{t^2} - \frac{w'}{t} f^2 \right) t^\alpha e^{-w-v} dt,$$

provided the displayed terms are integrable.

Proof. For clarity assume first that f is C^2, and write $f(t) = tg(t)$. Integration by parts gives

$$\begin{aligned} \int_0^\infty v' g^2 t^{\alpha+1} e^{-w-v} dt &= -\int_0^\infty g^2 t^{\alpha+1} e^{-w} (e^{-v})' dt \\ &= \int_0^\infty \Big(2tgg' + (\alpha+1)g^2 - tw'g^2 \Big) t^\alpha e^{-w-v} dt. \end{aligned}$$

On the other hand,

$$(f')^2 + \alpha(f/t)^2 - w'f^2/t = 2tgg' + (\alpha+1)g^2 - tw'g^2 + t^2(g')^2.$$

The difference between the right and left sides is therefore $\int_0^\infty t^2 (g')^2 t^\alpha e^{-w-v} dt \geq 0$. The stated regularity follows by approximation. □

Remark 7.9. *In dimension one, take $\alpha = 0$. If w is the restriction of an even convex function, then $w' \geq 0$ on $(0,\infty)$, so the right side is at most $\int (f')^2 e^{-w-v} dt$. Applying this to an odd function on the two half-lines proves the one-dimensional case of theorem 7.7, including even convex perturbations of the measure.*

The second lemma is a spherical consequence of log-concavity. It does not require w to be even.

Lemma 7.5. *Let $n \geq 2$ and let $w : \mathbb{R}^n \to \mathbb{R}$ be convex and C^1. Write $d\sigma_w(\theta) = e^{-w(\theta)} d\theta$ on $\mathbb{S}^{n-1}$, where $d\theta$ is surface measure. If $\int_{\mathbb{S}^{n-1}} g \, d\sigma_w = 0$, then*

$$\int_{\mathbb{S}^{n-1}} \Big(n - 1 - \langle \nabla w(\theta), \theta \rangle\Big) g^2 d\sigma_w \leq \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} g|^2 d\sigma_w.$$

For a smooth ambient extension of g, $\nabla_{\mathbb{S}^{n-1}} g = \nabla g - \langle \nabla g, \theta \rangle \theta$.

Proof sketch. First take smooth g. For sufficiently small $|t|$, the function $h_t = 1 + tg$ is the support function of a convex body K_t. Let $d\nu = e^{-w} dx$ on $\mathbb{R}^n$. Prékopa–Leindler implies that $t \mapsto \log \nu(K_t)$ is concave. Hence

$$\nu(B_2^n) \, \frac{d^2}{dt^2} \nu(K_t) \bigg|_{t=0} \leq \left(\frac{d}{dt} \nu(K_t) \bigg|_{t=0} \right)^2. \tag{49}$$

The first and second variation formulas at the ball are

$$\begin{aligned} \frac{d}{dt} \nu(K_t) \bigg|_{t=0} &= \int_{\mathbb{S}^{n-1}} g \, d\sigma_w, \\ \frac{d^2}{dt^2} \nu(K_t) \bigg|_{t=0} &= \int_{\mathbb{S}^{n-1}} \Big[\Big(n - 1 - \langle \nabla w, \theta \rangle\Big) g^2 - |\nabla_{\mathbb{S}^{n-1}} g|^2 \Big] \, d\sigma_w. \end{aligned}$$

See Kolesnikov and Milman [118] for the general variation formulas. The first variation vanishes by hypothesis, so the second is nonpositive, which is the desired inequality. Approximation gives the C^1 case. □

Remark 7.10. *An alternative route would be to obtain lemma 7.5 from Brascamp–Lieb for a suitable radial auxiliary potential, followed by polar integration or a limiting argument. Finding such a derivation is left as homework.*

Proof of theorem 7.7. We prove the stronger statement with $d\nu = e^{-v(|x|) - w(x)} dx$, where w is even and convex, and then approximate the restriction to K. Work first with an odd smooth compactly supported function h. In polar coordinates,

$$\int_{\mathbb{R}^n} \frac{v'(|x|)}{|x|} h^2 d\nu = \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{v'(t)}{t} h(t\theta)^2 t^{n-1} e^{-v(t) - w(t\theta)} dt \, d\theta.$$

Apply lemma 7.4 with $f_\theta(t) = h(t\theta)$, $w_\theta(t) = w(t\theta)$ and $\alpha = n - 1$. Since $f_\theta' = \langle \nabla h(t\theta), \theta \rangle$ and $w_\theta' = \langle \nabla w(t\theta), \theta \rangle$, the preceding integral is at most

$$\int_{\mathbb{S}^{n-1}} \int_0^\infty \left[\langle \nabla h(t\theta), \theta \rangle^2 + \left(\frac{n-1}{t^2} - \frac{\langle \nabla w(t\theta), \theta \rangle}{t} \right) h(t\theta)^2 \right] t^{n-1} e^{-v(t) - w(t\theta)} dt \, d\theta.$$

For fixed t, the function $g_t(\theta) = h(t\theta)$ has mean zero under $e^{-w(t\theta)} d\theta$, by oddness of h and evenness of w. Apply lemma 7.5 to $x \mapsto w(tx)$ and g_t. Its conclusion, divided by t^2, bounds the second term above by

$$\int_{\mathbb{S}^{n-1}} \frac{|\nabla_{\mathbb{S}^{n-1}} g_t(\theta)|^2}{t^2} e^{-w(t\theta)} d\theta.$$

The vector $t^{-1}\nabla_{\mathbb{S}^{n-1}} g_t$ is the tangential component of the ambient gradient $\nabla h(t\theta)$. Its squared norm plus $\langle \nabla h(t\theta), \theta\rangle^2$ equals $|\nabla h(t\theta)|^2$. Integrating in t proves

$$\int \frac{v'(|x|)}{|x|} h^2 d\nu \le \int |\nabla h|^2 d\nu.$$

The one-dimensional case was handled after lemma 7.4. Cutoffs, smooth approximation and approximation of $\mathbf{1}_K$ complete the proof. □

Remark 7.11. *The following equivalence, understood as a statement in all dimensions, is left as homework: the B-conjecture for all rotation-invariant log-concave measures is equivalent to*

$$\nu(tB_2^n)\nu(t^{-1}B_2^n) \le \nu(B_2^n)^2$$

for every even *log-concave measure ν and every $t > 0$. Thus the result of Cordero-Erausquin and Rotem implies a special case of the log-concave Blaschke–Santaló conjecture from conjecture 4.6.*

Remark 7.12. *Cordero-Erausquin and Rotem [51] in fact prove the B-property for a larger class of rotation-invariant measures, beyond the log-concave case.*

7.12 Klartag's improvement of Lichnerowicz's inequality

Recall that $C_{\mathrm{Poin}}(\mu)$ is the smallest constant such that

$$\mathrm{Var}_\mu(f) \le C_{\mathrm{Poin}}(\mu) \int |\nabla f|^2 d\mu.$$

The KLS conjecture conjecture 4.5 asserts that this constant is bounded by an absolute constant for isotropic log-concave probability measures. Equivalently, for arbitrary log-concave probability measures,

$$C_{\mathrm{Poin}}(\mu) \le C \| \mathrm{Cov}(\mu)\|_{op}. \tag{50}$$

Here $\mathrm{Cov}(\mu)_{ij} = \mathbb{E}[(X_i - \mathbb{E}X_i)(X_j - \mathbb{E}X_j)]$.

Lichnerowicz's inequality, which follows from Brascamp–Lieb or from Caffarelli's contraction theorem, says that if $d\mu = e^{-V}dx$ and $\nabla^2 V \succeq t\,\mathrm{Id}$ for $t > 0$, then

$$\mathrm{Var}_\mu(f) \le \frac{1}{t} \int |\nabla f|^2 d\mu. \tag{51}$$

Testing linear functions also gives

$$\| \mathrm{Cov}(\mu)\|_{op} \le \frac{1}{t}. \tag{52}$$

Thus the conjectured KLS bound would strengthen (51), up to an absolute constant, when the covariance is small.

Recall also that Gaussian measure minimizes the Poincaré constant among isotropic probability measures. In fact, if $\nabla^2 V \succeq \mathrm{Id}$ and μ is isotropic, then μ must be standard Gaussian. To see the rigidity, linear functions attain equality in the Poincaré inequality; the corresponding weak eigenvalue equations imply $\partial_i V = x_i$. Therefore a uniformly convex perturbation of Gaussian measure will generally lose isotropicity. This distinction is relevant in stochastic localization, as developed by Eldan, Lee–Vempala, Chen, Klartag–Lehec and others [60, 128, 46, 110].

Klartag obtained a bound which is the geometric mean of the Lichnerowicz bound and the conjectured covariance bound.

Theorem 7.8 (Klartag [107])**.** *If $d\mu = e^{-V}dx$ is a log-concave probability measure and $\nabla^2 V \succeq t\,\mathrm{Id}$ for $t > 0$, then*

$$C_{\mathrm{Poin}}(\mu) \le \sqrt{\frac{\|\operatorname{Cov}(\mu)\|_{op}}{t}}.$$

Translate so that $\int x\,d\mu = 0$. Then

$$\|\operatorname{Cov}(\mu)\|_{op} = \sup_{\theta \in \mathbb{S}^{n-1}} \int \langle x, \theta\rangle^2 d\mu. \tag{53}$$

Write $\lambda = 1/C_{\mathrm{Poin}}(\mu)$ for the spectral gap. Its variational characterization is

$$\lambda = \inf_{\substack{f \in W^{1,2}(\mu),\ \int f\,d\mu = 0 \\ f \neq 0}} \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu} = \inf_{\mathrm{Var}_\mu(f) > 0} \frac{\int |\nabla f|^2 d\mu}{\mathrm{Var}_\mu(f)}.$$

If the infimum is attained, an extremizer satisfies $Lf = -\lambda f$ and is called a first eigenfunction. Such an eigenfunction need not exist for every log-concave measure; the two-sided exponential distribution on $\mathbb{R}$ is an example.

For the proof, we work in the smooth setting where a sufficiently regular first eigenfunction exists. The approximation and decay arguments needed for the general statement are given in [107]. This lets us focus on the following key computation.

Lemma 7.6. *Suppose $Lf = -\lambda f$, $\int f\,d\mu = 0$ and $\int f^2 d\mu = 1$, where λ is the spectral gap. Then*

(i) $\left| \int \nabla f\,d\mu \right|^2 \ge \frac{1}{\lambda} \int \langle \nabla^2 V \nabla f, \nabla f\rangle d\mu;$

(ii) $\left| \int \nabla f\,d\mu \right|^2 \le \lambda^2 \|\operatorname{Cov}(\mu)\|_{op}.$

Proof. By Bochner's formula,

$$\int (Lf)^2 d\mu = \int \|\nabla^2 f\|_{HS}^2 d\mu + \int \langle \nabla^2 V \nabla f, \nabla f\rangle d\mu. \tag{54}$$

Apply the Poincaré inequality to each partial derivative:

$$\int |\nabla \partial_i f|^2 d\mu \geq \lambda \left[\int (\partial_i f)^2 d\mu - \left(\int \partial_i f \, d\mu \right)^2 \right]. \tag{55}$$

Summing gives

$$\int \|\nabla^2 f\|_{HS}^2 d\mu \geq \lambda \left(\int |\nabla f|^2 d\mu - \left| \int \nabla f \, d\mu \right|^2 \right). \tag{56}$$

Consequently,

$$\int (Lf)^2 d\mu \geq \lambda \left(\int |\nabla f|^2 d\mu - \left| \int \nabla f \, d\mu \right|^2 \right) + \int \langle \nabla^2 V \nabla f, \nabla f \rangle d\mu. \tag{57}$$

The eigenfunction equation and integration by parts give $\int (Lf)^2 d\mu = \lambda^2$ and $\int |\nabla f|^2 d\mu = \lambda$. Thus

$$\lambda^2 \geq \lambda^2 - \lambda \left| \int \nabla f \, d\mu \right|^2 + \int \langle \nabla^2 V \nabla f, \nabla f \rangle d\mu, \tag{58}$$

which proves (i).

For (ii), let $\theta \in \mathbb{S}^{n-1}$. Integration by parts and Cauchy–Schwarz give

$$\begin{aligned} \left| \int \langle \nabla f, \theta \rangle d\mu \right| &= \left| - \int Lf \, \langle x, \theta \rangle d\mu \right| \\ &= \lambda \left| \int f \langle x, \theta \rangle d\mu \right| \leq \lambda \left(\int \langle x, \theta \rangle^2 d\mu \right)^{1/2}. \end{aligned}$$

Square and take the supremum over θ, using (53). □

Proof of theorem 7.8. Combine the two conclusions of the lemma:

$$\lambda^2 \| \operatorname{Cov}(\mu) \|_{op} \geq \frac{1}{\lambda} \int \langle \nabla^2 V \nabla f, \nabla f \rangle d\mu \geq \frac{t}{\lambda} \int |\nabla f|^2 d\mu = t.$$

Since $C_{\mathrm{Poin}} = 1/\lambda$, this is exactly the desired bound. □

Remark 7.13. *Combining this estimate with stochastic localization, Klartag [107] proved in 2023 that $C_{\mathrm{Poin}}(\mu) \leq C \log(en)$ for isotropic log-concave probability measures. The corresponding bound for the square root of the Poincaré constant, or for the comparable inverse Cheeger constant, is $C\sqrt{\log(en)}$. These are different conventions for the constant; throughout these notes, C_{Poin} denotes the variance coefficient.*

For Gaussian measure, the first eigenspace consists of linear functions. On a centered coordinate box, coordinate sine functions in the longest directions provide first eigenfunctions, with the frequency determined by the side length and Neumann boundary condition. These

examples are odd and have a preferred direction. When the first eigenvalue has multiplicity, linear combinations need not depend on just one coordinate.

The following preferred-direction condition would be sufficient for the KLS bound:

$$\left| \int \nabla f \, d\mu \right|^2 \geq c_0 \int |\nabla f|^2 d\mu$$

for normalized first eigenfunctions, with an absolute $c_0 > 0$. The conditional implication is discussed in [107] and follows from part (ii) of lemma 7.6. It is a useful exercise to deduce a Poincaré bound for measures whose first eigenfunctions satisfy the displayed preferred-direction condition.

7.13 The dimensional Brunn–Minkowski conjecture

We close with the conjecture proposed in the Gaussian case by Gardner and Zvavitch [71], and formulated more generally by Colesanti, Livshyts and Marsiglietti [49].

Conjecture 7.2 (Dimensional Brunn–Minkowski)**.** *If μ is an even log-concave probability measure on $\mathbb{R}^n$, K, L are symmetric convex sets and $\lambda \in [0,1]$, then*

$$\mu(\lambda K + (1-\lambda)L)^{1/n} \geq \lambda \mu(K)^{1/n} + (1-\lambda)\mu(L)^{1/n}. \tag{59}$$

Here are some remarks and historical context.

- Prékopa–Leindler gives the weaker statement

  $$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}. \tag{60}$$

  Indeed, weighted power means increase with the exponent. Thus

  $$\mu(\lambda K + (1-\lambda)L)^p \geq \lambda \mu(K)^p + (1-\lambda)\mu(L)^p \tag{61}$$

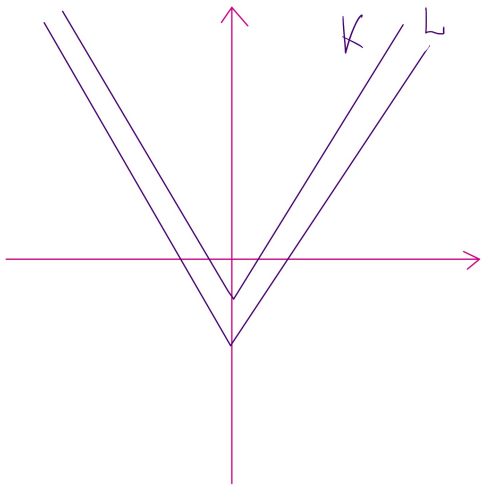
  is a stronger assertion when $p > 0$ is larger, and its limit as $p \to 0$ is (60).

- For Lebesgue measure, homogeneity makes (59) and (60) equivalent when they are assumed for all sets in the relevant class.

- A structural restriction on the sets is essential. Choose a bounded convex K with $\mu(K) > 0$, and let $L = K + Re_1$. As $R \to \infty$, the measure of $(K+L)/2$ tends to zero, whereas $\mu(K)$ stays positive. Hence no fixed positive power can work for all translates.

- The power $1/n$ is the largest possible uniformly over symmetric convex sets for a nondegenerate even log-concave measure. Near the origin its density is positive and continuous, so small dilates have measure asymptotic to a constant times their volume. The sharp obstruction for Lebesgue measure therefore persists.

- For sets with measure at least a, one may obtain stronger powers $p(a)$, sometimes with $p(a) \to \infty$ as $a \to 1$. Ehrhard's inequality gives this for Gaussian measure. Livshyts [140] proved the corresponding conclusion for symmetric convex sets and *uniformly* strictly log-concave even measures, meaning $\nabla^2 V \succeq c\,\mathrm{Id}$ for some $c > 0$.
- Gardner and Zvavitch [71] verified the Gaussian conjecture in dimension one.
- They also asked whether containing the origin could replace symmetry in the Gaussian case. Nayar and Tkocz [151] constructed a planar counterexample. Thus the condition $0 \in K \cap L$ does not suffice for the power $1/n$.



- Livshyts, Marsiglietti, Nayar and Zvavitch [138], and subsequently Hosle, Kolesnikov and Livshyts [90], showed that the dimensional conjecture follows from the log-Brunn–Minkowski conjecture conjecture 4.9. In particular, it holds in dimension two, and also when the measure and both sets are unconditional.

A first application of the L^2 method gave the following.

Theorem 7.9 (Kolesnikov–Livshyts [121])**.** *If K, L are convex sets containing the origin and $\lambda \in [0,1]$, then*

$$\gamma(\lambda K + (1-\lambda)L)^{1/(2n)} \geq \lambda \gamma(K)^{1/(2n)} + (1-\lambda)\gamma(L)^{1/(2n)}.$$

Symmetry is not required, but the power is $1/(2n)$ rather than $1/n$. Aishwarya and Rotem [1] extended this type of result to measures with convex p-homogeneous potentials, $p > 1$. Their theorem permits sets that are merely star-shaped about the origin, with power $(p-1)/(pn)$; for Gaussian measure this recovers $1/(2n)$. Thus convexity can be relaxed here, while the star-shapedness assumption remains.

Theorem 7.10 (Cordero-Erausquin–Rotem [51]). *If μ is rotation-invariant and log-concave and K, L are symmetric convex sets, then*

$$\mu(\lambda K + (1-\lambda)L)^{1/n} \geq \lambda \mu(K)^{1/n} + (1-\lambda)\mu(L)^{1/n}.$$

Eskenazis and Moschidis [63] had proved the Gaussian case. An idea from their proof also enters the argument of Cordero-Erausquin and Rotem, whose new ingredient is the Hessian estimate (48).

More generally, the same type of estimate has the following consequence.

Proposition 7.4. *Let $d\mu = e^{-V}dx$ be even and log-concave. Suppose there is a symmetric nonnegative matrix field T, with $Tx = \nabla V$, such that for every symmetric convex set K and every even smooth u,*

$$\int_K \|\nabla^2 u\|_{HS}^2 d\mu \geq \int_K \langle T\nabla u, \nabla u\rangle d\mu.$$

Then conjecture 7.2 holds for μ.

The hypothesis is the same as in proposition 7.3. To see the key algebra, apply it to $u - s|x|^2/2$ and use $Tx = \nabla V$. Writing $\mathbb{E}_K$ for normalized integration over K gives

$$\mathbb{E}_K\Big[\|\nabla^2 u\|_{HS}^2 + \langle \nabla^2 V \nabla u, \nabla u\rangle\Big] \geq \mathbb{E}_K\langle (T + \nabla^2 V)\nabla u, \nabla u\rangle + 2s\,\mathbb{E}_K Lu - s^2\Big(n - \mathbb{E}_K\langle \nabla V, x\rangle\Big).$$

Taking $s = (\mathbb{E}_K Lu)/n$, and using evenness and convexity of V, bounds the right side from below by $(\mathbb{E}_K Lu)^2/n$. The boundary L^2 criterion, applied to solutions with constant Lu and prescribed Neumann data, then yields the claimed $1/n$-concavity. The geometric boundary argument is not developed further here; see [118, 121, 51].

Finally, one can improve log-concavity to a positive power for all even log-concave measures.

Theorem 7.11 (Livshyts [140]). *There is a positive dimensional constant $c_n = n^{-4-o(1)}$ such that, for every even log-concave probability measure and symmetric convex K, L,*

$$\mu(\lambda K + (1-\lambda)L)^{c_n} \geq \lambda \mu(K)^{c_n} + (1-\lambda)\mu(L)^{c_n}.$$

Remark 7.14 (A refinement using the 2023 Poincaré bound). *The argument in [140], combined with Klartag's bound [107], permits the explicit choice*

$$c_n = \frac{c}{n^4 \log(en)}.$$

Here is how the constants enter. Let κ_n bound the variance-form Poincaré constant of every isotropic log-concave probability measure in dimension n. The local argument first puts $\nu = \mu|_K$ in isotropic position, then finds a symmetric convex subset A with $\nu(A) \geq c_0 > 0$ and $|\nabla V| \leq Cn^2$ on A. Since $\mathrm{Cov}(\nu|_A) \leq c_0^{-1}\,\mathrm{Id}$, we have $C_{\mathrm{Poin}}(\nu|_A) \leq c_0^{-1}\kappa_n$. Applying the Hessian completion-of-the-square argument on A, and then using nonnegativity of the Hessian energy outside A, gives a denominator bounded by $C\kappa_n n^4$ in the local concavity estimate. Thus $c_n \geq c/(\kappa_n n^4)$, and $\kappa_n \leq C\log(en)$ gives the displayed choice. This is a consequence of the two results; the theorem in [140] is stated as $n^{-4-o(1)}$. In particular, the logarithmic factor enters through the variance-form constant, rather than its square root.

The proofs of the remaining results in this subsection were not included in the original notes.

7.14 Homework

Question 7.1 (1 point)**.** *Give the change-of-variables proof of Bochner's identity: set $x = y + t\nabla u(y)$, differentiate the resulting integral twice at $t = 0$, and justify the cancellations. You may first assume smooth compactly supported data.*

Question 7.2 (1 point)**.** *Let $A(x)$ be a measurable symmetric positive definite matrix.*

(a) On the space of vector fields for which the integral is finite, show that

$$\|F\|_A = \left(\int \langle AF, F \rangle d\mu \right)^{1/2}$$

is a norm modulo equality almost everywhere.

(b) With the pairing $\int \langle F, G \rangle d\mu$, show that the dual norm is

$$\|G\|_{A,*} = \left(\int \langle A^{-1} G, G \rangle d\mu \right)^{1/2}.$$

Question 7.3 (2 points)**.** *Under the hypotheses of theorem 7.4, let $d\mu_\varepsilon = (1 + \varepsilon h) d\mu$, where h is bounded and $\int h \, d\mu = 0$. We proved*

$$\|h\|_{H^{-1}(\mu)} \leq \liminf_{\varepsilon \downarrow 0} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon}.$$

Prove the reverse inequality. One possible approach uses the continuity equation for the path $(1 + t\varepsilon h)\mu$, together with the dynamic formulation of W_2.

Question 7.4 (5 points)**.** *Recover the historical $C \log^2(en)$ Poincaré bound for isotropic unconditional log-concave measures using transport. Work first with smooth functions and bounded derivatives, so that $1 + \varepsilon \partial_i \psi \geq 1/2$ for sufficiently small ε.*

(a) Let ν be an unconditional log-concave probability measure supported in $[-R, R]^n$, and let ψ be unconditional. Construct a map

$$(T_\varepsilon)_\# \nu = (1 + \varepsilon \partial_i \psi) \nu, \qquad \int |T_\varepsilon x - x|^2 d\nu \leq C R^2 \varepsilon^2 \int |\partial_i \psi|^2 d\nu.$$

Hint: keep the other coordinates fixed and use monotone transport on each coordinate fiber. Its conditional density q is even and nonincreasing on $[0, R]$. For $h = \partial_i \psi$, the primitive $H(s) = \int_{-R}^{s} h(t) q(t) dt$ vanishes at both endpoints and satisfies

$$\int_{-R}^{R} \frac{H(s)^2}{q(s)} ds \leq \frac{R^2}{2} \int_{-R}^{R} h(s)^2 q(s) ds.$$

Integrals are taken over the support of q.

(b) *Use the H^{-1} inequality to obtain the Poincaré bound CR^2 for unconditional test functions. Extend it to all test functions by averaging over coordinate reflections. Hint: condition on $(|X_1|, \dots, |X_n|)$ and use the product structure of the signs; on each coordinate fiber, an odd function a satisfies $\int a^2 q \le CR^2 \int (a')^2 q$.*

(c) *Now let μ be isotropic and unconditional, without bounded support. Choose $R = C\log(en)$ so that $\mu([-R,R]^n) \ge 0.99$, using the exponential tails of its one-dimensional marginals. You may quote the following comparison consequence of E. Milman's transference theorem [148, Theorem 2.1] and his comparison of isoperimetric and Poincaré constants [147]: for log-concave μ and convex A with $\mu(A) \ge 0.99$,*

$$C_{\mathrm{Poin}}(\mu) \le C C_{\mathrm{Poin}}(\mu|_A).$$

Deduce $C_{\mathrm{Poin}}(\mu) \le C\log^2(en)$. Finally, use the continuity-equation argument from the preceding transport problem to obtain maps for the original measure satisfying

$$(T_\varepsilon)_{\#}\mu = (1 + \varepsilon\partial_i\psi)\mu, \qquad \int |T_\varepsilon x - x|^2 d\mu \le C\log^2(en)\varepsilon^2 \int |\partial_i\psi|^2 d\mu.$$

The target measure has mass one by symmetry. The logarithm squared here belongs to the historical unconditional argument; the later general logarithmic bound uses additional ideas.

Question 7.5 (1 point)**.** *Explain why the improved Brascamp–Lieb estimate in Question 3.46 implies the B-property when both the measure and the convex set are unconditional.*

Question 7.6 (1 point)**.** *Prove the reverse implication in Hörmander's lemma with the gradient-space duality of remark 7.7. Explain where the fact that $N^\dagger$ is the dual norm on $\mathcal{G}$ is used. In particular, deduce the equivalence*

$$\mathrm{Var}_\mu(f) \le c^{-1} \int |\nabla f|^2 d\mu \quad \Longleftrightarrow \quad \int (Lu)^2 d\mu \ge c \int |\nabla u|^2 d\mu.$$

Question 7.7 (2 points)**.** *Use the weighted Bochner formula to prove theorem 7.6: if $g \ge 0$ is concave on a convex set K and $\int_K f\, d\mu = 0$, then*

$$\int_K g f^2 d\mu \le \int_K g\langle (\nabla^2 V)^{-1}\nabla f, \nabla f\rangle d\mu.$$

Hint: for smooth data, solve $Lu = f$ with $\partial_\nu u = 0$. Combine integration by parts with the weighted Bochner identity to show

$$\int_K g f^2 d\mu \le -2\int_K g\langle \nabla f, \nabla u\rangle d\mu - \int_K g\langle \nabla^2 V \nabla u, \nabla u\rangle d\mu,$$

and complete the square. The terms discarded in this step are nonpositive because $\nabla^2 g \le 0$, $g \ge 0$, and the second fundamental form of the boundary is nonnegative.

Question 7.8 (1 point)**.** *For an even log-concave measure $d\mu = e^{-V}dx$, prove that the B-property for every symmetric convex K is equivalent to*

$$\mathrm{Var}_{\mu|_K}(\langle \nabla V, x\rangle) \leq \fint_K \Big(\langle \nabla^2 V x, x\rangle + \langle \nabla V, x\rangle\Big) d\mu.$$

Explain why checking the second derivative at zero for every K is enough.

Question 7.9 (2 points)**.** *Show the equivalence described in remark 7.11: the B-property for all rotation-invariant log-concave measures and symmetric convex bodies, in all dimensions, is equivalent to*

$$\nu(tB_2^n)\nu(t^{-1}B_2^n) \leq \nu(B_2^n)^2$$

for every even log-concave measure ν and $t > 0$.

Question 7.10 (4 points)**.** *Derive the weighted spherical inequality in lemma 7.5 from the Brascamp–Lieb inequality.*

Question 7.11 (1 point)**.** *Let $d\mu = e^{-V}dx$ be a log-concave probability measure with $\nabla^2 V \succeq t\,\mathrm{Id}$ for $t > 0$. Prove that $\|\mathrm{Cov}(\mu)\|_{op} \leq 1/t$. Hint: translate to mean zero and test the Poincaré inequality on linear functions.*

Question 7.12 (2 points)**.** *Show that a log-concave probability measure need not have a first eigenfunction. Consider $d\mu = \frac{1}{2}e^{-|x|}dx$ on $\mathbb{R}$: identify its spectral gap and show that the Rayleigh quotient has no nonconstant minimizer.*

Question 7.13 (1 point)**.** *Assume that a first eigenfunction of an isotropic log-concave probability measure satisfies the preferred-direction condition*

$$\left|\int \nabla f \, d\mu\right|^2 \geq c \int |\nabla f|^2 d\mu$$

with $c > 0$. Show that $C_{\mathrm{Poin}}(\mu) \leq 1/c$. Thus an absolute preferred-direction bound on an appropriate approximating class would imply KLS.

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