On the ℓ_0 Isoperimetry of Measurable Sets

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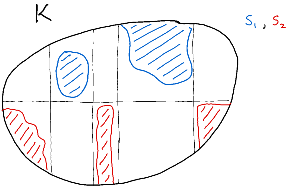
Axis-disjointness

Definition (Axis-Disjointness)

Let $S_1, S_2 \subseteq \mathbb{R}^n$. We say S_1, S_2 are axis-disjoint if there does not exist a line parallel to some coordinate axis that intersects both S_1 and S_2.



Prior results: ℓ_0-isoperimetry

ℓ_0 Isoperimetric inequalities

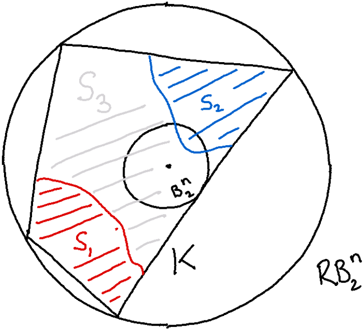
Suppose $K \subseteq \mathbb{R}^n$ is convex and $S_1, S_2 \subseteq K$ are axis-disjoint. and $S_3 = K \setminus (S_1 \cup S_2)$

- Laddha, Vempala [2]: If $B_2 \subseteq K \subseteq RB_2$ and $\varepsilon > 0$ then

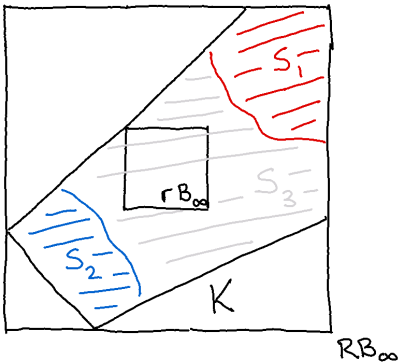
$$\mathbf{vol}(S_3) \geq \Omega\left(\frac{\varepsilon}{n^{3.5}R} \cdot (\min\{\mathbf{vol}(S_1), \mathbf{vol}(S_2)\} - \varepsilon \cdot \mathbf{vol}(K)\right)$$

- Narayanam, Rajaraman, Srivastava [3]: If $rB_\infty \subseteq K \subseteq RB_\infty$ then

$$\mathbf{vol}(S_3) \geq \Omega\left(\frac{(r/R)}{n^{3.5}} \min\{\mathbf{vol}(S_1), \mathbf{vol}(S_2)\}\right)$$



S3
S2
S1
B2n
K
RB2n



S1
S2
S3
rB∞
K
RB∞

ℓ_0 Isoperimetric coefficient

Definition (ℓ_0 Isoperimetric coefficient)

Let $K \subseteq \mathbb{R}^n$ be measurable. The ℓ_0 isoperimetric coefficient of K, ψ_K, is defined as

$$\psi_K := \inf_{\substack{S_1, S_2 \subseteq K \\ S_1, S_2 \text{ are measurable} \\ S_1, S_2 \text{ are axis disjoint}}} \frac{\textbf{vol}(K \setminus (S_1 \cup S_2))}{\min\{\textbf{vol}(S_1), \textbf{vol}(S_2)\}}$$

Theorem (Laddha, Vempala [2])

Let $\mathcal{C}$ denote any axis-aligned cube. Then

$$\psi_{\mathcal{C}} \geq \frac{\log 2}{n}.$$

Proof is via the Loomis-Whitney inequality.

Main result: improved ℓ_0 isoperimetry

Theorem (F.)

Let $\mathcal{C}$ denote any axis-aligned cube. Then there exists absolute constants $c_1, c_2 > 0$ such that

$$\frac{c_1}{\sqrt{n}} \leq \psi_{\mathcal{C}} \leq \frac{c_2}{\sqrt{n}}.$$

Corollary (F. Improved ℓ_0 isoperimetric inequalities)

- *If $B_2 \subseteq K \subseteq RB_2$ and $\varepsilon > 0$ then*

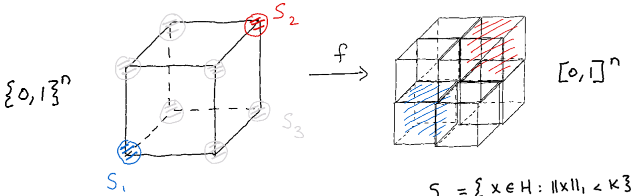
$$\mathbf{vol}(S_3) \geq \Omega\left(\frac{\varepsilon}{n^3 R} \cdot (\min\{\mathbf{vol}(S_1), \mathbf{vol}(S_2)\} - \varepsilon \cdot \mathbf{vol}(K)\right)$$

- *If $rB_\infty \subseteq K \subseteq RB_\infty$ then*

$$\mathbf{vol}(S_3) \geq \Omega\left(\frac{(r/R)}{n^3} \min\{\mathbf{vol}(S_1), \mathbf{vol}(S_2)\}\right)$$

Lemma

Let $K \subseteq \mathbb{R}^n$ be an axis-aligned cube. Then $\psi_K = O(n^{-1/2})$



$I_0 = [0, 1/2)$ $\quad$ $f : H \longrightarrow \mathcal{P}(Q)$

$I_1 = [1/2, 1]$ $\quad$ $f(x) = I_{x_1} \times I_{x_2} \times \cdots \times I_{x_n}$

$S_1 = \{ x \in H : \|x\|_1 < k \}$

$S_2 = \{ x \in H : \|x\|_1 > k \}$

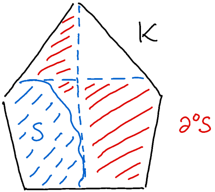
$S_3 = \{ x \in H : \|x\|_1 = k \}$

observation: $f(S_1)$, $f(S_2)$ are axis-disjoint!

ℓ_0 boundary

Definition (ℓ_0 boundary)

Let $K \subseteq \mathbb{R}^n$ be measurable and $S \subseteq K$ be measurable. The ℓ_0 boundary of S, $\partial^0 S$ is defined as the set of all points in $K \setminus S$ that are not axis-disjoint from S.



Remark

$$\psi_K = \inf_{\substack{S_1, S_2 \subseteq K \\ S_1, S_2 \text{ are measurable} \\ S_1, S_2 \text{ are axis disjoint}}} \frac{\mathbf{vol}(\partial^0 S_1)}{\min\{\mathbf{vol}(S_1), \mathbf{vol}(S_2)\}}.$$

Structured subsets of $[0,1]^n$

Definition

Let $A \subseteq [0,1]^n$.

1. We say that A is anchored if for every $x \in A$ the rectangle $[0,x_1] \times \cdots \times [0,x_n]$ is contained in A.
2. We say A has a grid structure if there exists a grid such that A can be written as a union of grid blocks.

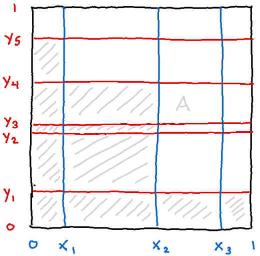
Definition

Given $r \in \{0,1,\cdots,n\}$ and $p \in [0,1]$ we define the p-weighted hamming ball of radius r, $H(p,r)$, as

$$H(p,r) := \{x \in [0,1]^n \ : \ \#\{i \ : \ x_i \leq p\} \geq n-r\}.$$

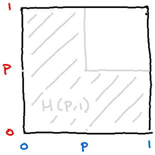
Harper's Theorem for structured subsets [1]

$Q := [0,1]^n$



$\mathcal{A} := \{ A \subseteq Q$ such that
① A has a grid structure
② A is anchored $\}$

$H(p,r) := \{ x \in Q : \#\{ i : 0 \le x_i \le p \} \ge n-r \}$



Thm: Harper

Fix $0 < \lambda < 1$. Then $\min\limits_{\substack{A \in \mathcal{A} \\ vol(A) = \lambda}} vol(\partial^\circ A)$ is achieved by $H(p,r)$ for some $0 < p < 1$, $0 < r < n$

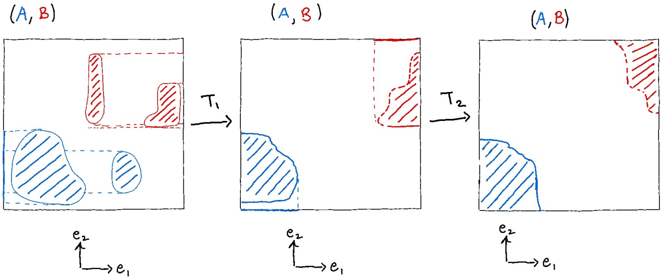
Goal: Reduce to Harper

Continuous compression via shaking

$$(A,B) \underset{T_i}{\mapsto} (\hat{A},\hat{B})$$

$$\hat{A} := \bigcup_{x \in \mathbf{Proj}_{e_i^\top} A} \{x\} \times \left[0, \mathbf{vol}_1(\{y \in A \mid \mathbf{Proj}_{e_i^\top} y = x\})\right]$$

$$\hat{B} := \bigcup_{x \in \mathbf{Proj}_{e_i^\top} A} \{x\} \times \left(1 - \mathbf{vol}_1(\{y \in B \mid \mathbf{Proj}_{e_i^\top} y = x\}), 1\right)$$



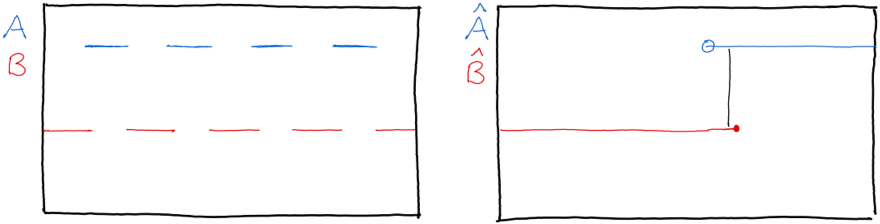
Compression guarantees

Lemma

If $A, B \subseteq [0,1]^n$ then $(\hat{A}, \hat{B})$ satisfies the following:

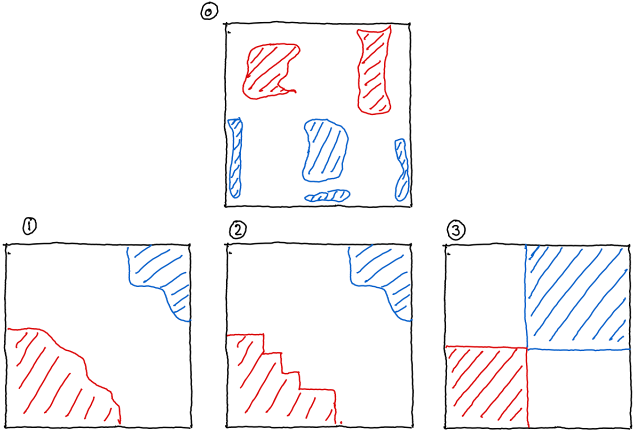
1. $\mathbf{vol}(A) = \mathbf{vol}(\hat{A}), \mathbf{vol}(B) = \mathbf{vol}(\hat{B})$.
2. *If A, B are axis-disjoint then $\hat{A}, \hat{B}$ are axis-disjoint.*
3. *The boundary of $\hat{A}$ has measure 0.*

Illustration of (2)



General proof strategy

Overview: 1) Compress 2) Under approximate 3) Harper



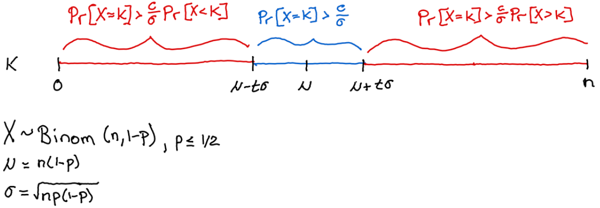
Binomial inequality

Observation: The volume of $B(p, r)$ is equal to the probability that a binomial random variable, $Bin(n, 1-p)$, is at most r.

Lemma

There exists a universal constant $c > 0$ such that the following is true: Let $X \sim Bin(n, 1-p)$ be a binomial random variable. Let $k \in \{0, 1, \cdots, n\}$ Then the following inequality holds:

$$\mathbb{P}[X = k] \geq \frac{c}{\sqrt{np(1-p)}} \min\{\mathbb{P}[X < k], \mathbb{P}[X > k]\}$$



General upper bound

Theorem (F. General upper bound)

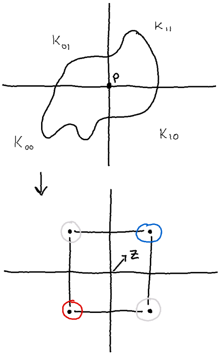
There exists a universal constant $c > 0$ such that the following holds: Let $K \subseteq \mathbb{R}^n$ be a measurable set. Then

$$\psi_K \leq cn^{-1/2}$$

- To prove the upper bound we develop a general framework for constructing axis-disjoint subsets $S_1, S_2 \subseteq K$ for which $\frac{\mathbf{vol}(K \setminus (S_1 \cup S_2))}{\min\{\mathbf{vol}(S_1), \mathbf{vol}(S_2)\}}$ is small.
- Our proof is an application of the Probabilistic Method.

Random splitting plane

Construction



1) Pick $x \in \mathbb{R}^n$ such that

$$\mathrm{vol}(K \cap \{x_i \le p_i\}) = \mathrm{vol}(K \cap \{x_i > p_i\})$$

2) Decompose K into orthants induced by P

3) Pick z uniformly from $\{0,1\}^n$

4) Define

$$S_1 := \bigcup_{\|s-z\|_1 < \lfloor n/2 \rfloor} K_s \qquad S_2 := \bigcup_{\|s-z\|_1 > \lceil n/2 \rceil} K_s$$

$$S_3 := \bigcup_{\|s-z\|_1 \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}} K_s$$

Goal: Show that with non-zero probability $\mathbf{vol}(S_3) = O(n^{-1/2}), \mathbf{vol}(S_1), \mathbf{vol}(S_2) = \Omega(1)$.

Symmetric case

Let $K \subseteq \mathbb{R}^n$ be symmetric:

- First moment argument: With any constant probability $\mathbf{vol}(S_3) = O(n^{-1/2})$
- symmetric property: $\mathbf{vol}(S_1) = \mathbf{vol}(S_2)$

Lemma

Let $K \subseteq \mathbb{R}^n$ be symmetric. Then for every $z \in \{0,1\}^n$ we have

$$\sum_{\|s-z\|_1 < \lfloor n/2 \rfloor} \mathbf{vol}(K_s) = \sum_{\|s-z\|_1 > \lceil n/2 \rceil} \mathbf{vol}(K_s).$$

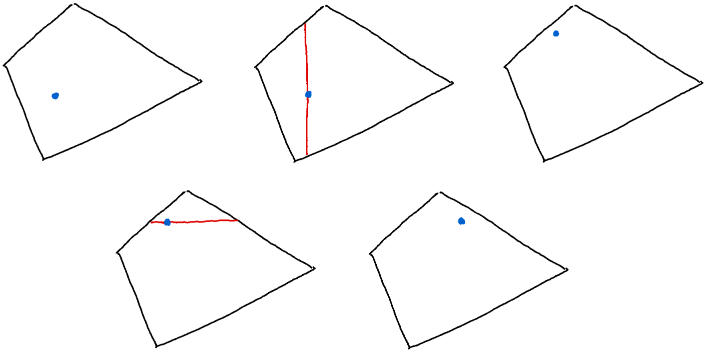
Consequence: $\mathbf{vol}(S_1) = \mathbf{vol}(S_2) = \Omega(1)$.

Non-symmetric case

- Problem: If K is non-symmetric then $\mathbf{vol}(S_1), \mathbf{vol}(S_2)$ need not be equal.
- Workaround: second moment method
- Need to show: $\mathbb{E}[\mathbf{vol}(S_1)^2] < 1/2 - c$ for some absolute constant c.
- Idea 1: If s_1, s_2 are far apart then K_{s_1}, K_{s_2} are more likely to be on opposite sides.
- Idea 2: The volume of K is not too concentrated amongst a cluster of orthants.

Random Walk: Coordinate Hit-and-Run

- Introduced by Turchin (1971) [4]



- Simple to implement, good in practice.
- No strong theoretical guarantees until recently.

Problem: lowerbounding conductance

Theorem (Lovasz, Simonivits)

1) $\chi^2(Q, Q_t) \le \chi^2(Q_0, Q)(1 - \phi^2/2)^t$

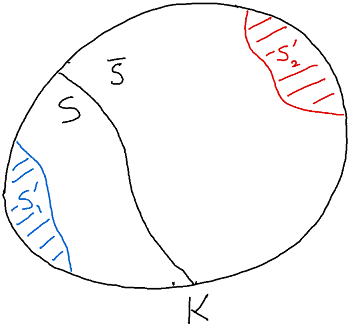
2) $d_{TV}(Q, Q_t) \le sM + M(1 - Q_s/2)^t$

$$\phi(S) := \frac{\int_{x \in S} P_x(\bar{S})\, dQ}{\min(Q(S), Q(\bar{S}))}$$

$$\phi := \inf_{S \subseteq K} \phi(S)$$

$$\phi_s := \inf_{\substack{S \subseteq K \\ s < vol(S) \le 1/2}} \phi(S)$$

$$M = \sup_{S \subseteq K} \frac{Q(S)}{Q(S_0)}$$



$S_1' = \{x \in S : P_x(\bar{S}) < \frac{1}{2n}\}$

$S_2' = \{x \in \bar{S} : P_x(S) < \frac{1}{2n}\}$

Lemma (Laddha, Vempala)

S_1', S_2' are axis disjoint

Open questions

- What is the best lowerbound on the isoperimetric coefficient for convex bodies K? Is it also $\Omega(n^{-1/2})$?
- Can the upper bound on ψ_K be improved by depending on the regularity of K?

Conclusion

Thank you!

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